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INTEGRABILITY CONDITIONS FOR AN ANALOGUE OF THE RELATIVISTIC TODA CHAIN

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We consider a class of discrete-differential equations that contains the relativistic Toda chain and is characterized by one arbitrary function of six variables. We derive three conditions that allow testing the integrability of any given equation in this class. In deriving these conditions, we use higher symmetries distinguishing the equations that are integrable via the inverse scattering method.

Keywords: relativistic Toda chain, higher symmetry, integrability condition

1. Introduction

We consider discrete-differential equations of the form

$$\ddot{u}_n = f(u_{n+1}, u_n, u_{n-1}, \dot{u}_{n+1}, \dot{u}_n, \dot{u}_{n-1}) \equiv f_n, \tag{1}$$

where $u_n = u_n(t)$, $n \in \mathbb{Z}$, $\dot{u}_n = du_n/dt$, and $\ddot{u}_n = d^2u_n/dt^2$. We additionally assume that $\partial f_n/\partial \dot{u}_{n+1} \neq 0$ or $\partial f_n/\partial \dot{u}_{n-1} \neq 0$. A known representative of this equation class is given by the relativistic Toda chain [1]

$$\ddot{u}_n = \frac{\dot{u}_{n+1}\dot{u}_n}{1 + e^{u_n - u_{n+1}}} - \frac{\dot{u}_n\dot{u}_{n-1}}{1 + e^{u_{n-1} - u_n}}. \tag{2}$$

Other examples of integrable equations of form (1) were obtained in [2], [3]; the symmetry method used in [3] allowed obtaining a list of such equations. The symmetry method involves a distinguishing signature of the equations integrable via the inverse scattering method, namely, the existence of higher symmetries and conservation laws. This method allows finding all the integrable equations of a certain form or testing the integrability of a given equation. The method was used to obtain the integrability conditions and full lists of integrable equations of classes including, for example, the Korteweg–De Vries equation, the nonlinear Schrödinger equation, and the Toda and Volterra chains. These results and the description of the method can be found, for example, in [4]–[6].

We briefly recall how the list of equations was obtained in [3]. The equations considered had the form

$$\ddot{u}_n = f(u_{n+1}, u_n, \dot{u}_{n+1}, \dot{u}_n) - g(u_n, u_{n-1}, \dot{u}_n, \dot{u}_{n-1}) \equiv h_n \tag{3}$$

and were assumed to have the symmetry

$$u_{n,\tau} = f(u_{n+1}, u_n, \dot{u}_{n+1}, \dot{u}_n) + g(u_n, u_{n-1}, \dot{u}_n, \dot{u}_{n-1}) \equiv s_n \tag{4}$$

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(where the subscript τ denotes the derivative with respect to τ); in other words, Eqs. (3) and (4) were assumed to be compatible. The compatibility means that these equations have common solutions $u_n(t,\tau)$. For such solutions, the relation $\partial^3 u_n/\partial t^2\partial\tau = \partial^3 u_n/\partial\tau\partial t^2$ holds, which can be written as

$$D_t^2 s_n = D_\tau h_n, \tag{5}$$

$$D_t = \sum_i \dot u_{n+i}\frac{\partial}{\partial u_{n+i}} + \sum_i h_{n+i}\frac{\partial}{\partial \dot u_{n+i}}, \qquad D_\tau = \sum_i s_{n+i}\frac{\partial}{\partial u_{n+i}} + \sum_i D_t(s_{n+i})\frac{\partial}{\partial \dot u_{n+i}}.$$

The compatibility condition for Eqs. (3) and (4) thus leads to Eq. (5) for the functions f and g; just this equation was used in [3] to seek integrable equations of form (3). This approach allows finding new integrable examples but does not give natural conditions for testing the integrability of a given equation. The underlying reason is that the form of the symmetry is rigidly fixed here. But integrable equations have infinitely many higher symmetries, and choosing another form of the symmetry may lead to different results in general.

In this work, we follow a more general scheme of the symmetry method, which was used in [7] to investigate the class of equations of the form

$$\dot u_n = a(u_{n+1}, u_n, u_{n-1}) \equiv a_n, \tag{6}$$

containing the Volterra equation, and the Toda chain class,

$$\ddot u_n = b(u_{n+1}, u_n, u_{n-1}, \dot u_n) \equiv b_n. \tag{7}$$

In accordance with this scheme, we use the assumption that the equation has one or two higher symmetries (or conservation laws) of a sufficiently high order. The form and the order of the symmetry are not fixed. Under this assumption, we derive several conditions for the right-hand side of the equation, i.e., for the functions a_n and b_n. These conditions are independent of the form and order of the symmetry and are easily tested for any given equation.

We give an example of the simplest integrability condition and the corresponding statement for the equation of form (6). If Eq. (6) with $\partial a_n/\partial u_{n+1} \neq 0$ has a higher symmetry of order $m \geq 2$, then there exists a function $q_n = q(u_{n+1}, u_n, u_{n-1}, u_{n-2})$ satisfying the relation

$$D_t \log\frac{\partial a_n}{\partial u_{n+1}} = q_{n+1} - q_n, \tag{8}$$

where $D_t = \sum_i a_{n+i}\partial/\partial u_{n+i}$. What such a higher symmetry and its order are and how condition (8) can be tested for any given function a_n is discussed below in the example of equations of form (1).

Our aim in this work is to derive three integrability conditions, similar to relation (8), for chains of form (1). We follow the symmetry method scheme developed for equation classes (6) and (7) in [7] and detailed in [8], [9]. In deriving these conditions, we encounter an essentially new theoretical obstacle, which is overcome using a lemma formulated in Sec. 3.

2. Preliminaries

We first recall some facts from the general theory of the symmetry method following [8] and [9]; these facts are needed for formulating the results in this paper. It is convenient to introduce the function $v_n = \dot u_n$ and pass from Eq. (1) to the equivalent system of equations

$$\dot u_n = v_n, \qquad \dot v_n = f_n = f(u_{n+1}, u_n, u_{n-1}, v_{n+1}, v_n, v_{n-1}). \tag{9}$$

The vector form of the system is

$$\dot{U}_n = F_n = F(U_{n+1}, U_n, U_{n-1}), \qquad U_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \qquad F_n = \begin{pmatrix} v_n \\ f_n \end{pmatrix}. \tag{10}$$

In the vector form, higher symmetries of system (9) are written as

$$U_{n,\tau} = G_n = G(U_{n+m}, U_{n+m-1}, \ldots, U_{n+m'}), \qquad G_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}, \tag{11}$$

where $m > m'$.

Symmetry (11) is an equation compatible with (10), and these equations therefore have a common solution $U_n(t,\tau)$. The equality $\partial^2 U_n/\partial t\partial\tau = \partial^2 U_n/\partial\tau\partial t$ is written for such solutions as $D_t G_n = D_\tau F_n$, where D_t and D_τ are the differentiation operators in accordance with (10) and (11). The obtained relation can be represented as

$$\sum_{i=m'}^{m} \frac{\partial G_n}{\partial U_{n+i}} F_{n+i} = \sum_{i=-1}^{1} \frac{\partial F_n}{\partial U_{n+i}} G_{n+i}, \tag{12}$$

where $\partial G_n/\partial U_{n+i}$ are matrices of the form

$$\frac{\partial G_n}{\partial U_{n+i}} = \begin{pmatrix} \dfrac{\partial \varphi_n}{\partial u_{n+i}} & \dfrac{\partial \varphi_n}{\partial v_{n+i}} \\ \dfrac{\partial \psi_n}{\partial u_{n+i}} & \dfrac{\partial \psi_n}{\partial v_{n+i}} \end{pmatrix}$$

and the matrices $\partial F_n/\partial U_{n+i}$ are defined similarly. The functions u_{n+i} and v_{n+i} are considered independent variables in (12), and relation (12), which is the compatibility condition for Eqs. (10) and (11), therefore imposes strong constraints on the vector functions F_n and G_n.

In what follows, we say that the higher symmetry of Eq. (10) (or system (9)) is an equation of form (11) with $m > m'$ whose right-hand side G_n satisfies compatibility condition (12). The number m is called the order of the symmetry, which is nondegenerate if $\det(\partial G_n/\partial U_{n+m}) \neq 0$. Using higher symmetries allows constructing approximate solutions L_n of the equation

$$\dot{L}_n = [F_n^*, L_n], \tag{13}$$

which plays an important role in what follows. We now pass to a detailed discussion of this equation.

We define Fréchet derivatives of the vector functions F_n and G_n as

$$F_n^* = \sum_{i=-1}^{1} \frac{\partial F_n}{\partial U_{n+i}} T^i, \qquad G_n^* = \sum_{i=m'}^{m} \frac{\partial G_n}{\partial U_{n+i}} T^i,$$

where T^i are the powers of the shift operator T: $n \to n+1$, for example,

$$T^i F_n = F_{n+i} = F(U_{n+i+1}, U_{n+i}, U_{n+i-1}).$$

With the notation

$$f_n^{(i)} = \frac{\partial f_n}{\partial u_{n+i}}, \qquad g_n^{(i)} = \frac{\partial f_n}{\partial v_{n+i}} \tag{14}$$

for the derivative of f_n in system (9), we obtain a more detailed formula,

$$F_n^* = \begin{pmatrix} 0 & 0 \\ f_n^{(1)} & g_n^{(1)} \end{pmatrix} T + \begin{pmatrix} 0 & 1 \\ f_n^{(0)} & g_n^{(0)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ f_n^{(-1)} & g_n^{(-1)} \end{pmatrix} T^{-1}. \tag{15}$$

Solutions of Eq. (13) are formal series of the form

$$L_n = \sum_{i=-\infty}^{k} l_n^{(i)} T^i \tag{16}$$

with their coefficients $l_n^{(i)}$ being $2{\times}2$ matrices. The series are multiplied in accordance with the rule $(l_n T^i)(\hat{l}_n T^j) = l_n \hat{l}_{n+i} T^{i+j}$, and $[F_n^*, L_n] = F_n^* L_n - L_n F_n^*$ is the standard commutator. Finally, $\dot{L}_n = \sum_{i=-\infty}^{k} \dot{l}_n^{(i)} T^i$, i.e., the coefficients are differentiated with respect to time in accordance with Eq. (10).

If $\det l_n^{(k)} \neq 0$, then formal series (16) can be inverted using the standard definition $L_n^{-1} L_n = L_n L_n^{-1} = E$, where $E = ET^0$ is the operator of multiplication by the identity matrix E. The inverse series is given by $L_n^{-1} = \sum_{i=-\infty}^{-k} \hat{l}_n^{(i)} T^i$, and the first coefficients are found in accordance with the formulas

$$\hat{l}_n^{(-k)} = \bigl(l_{n-k}^{(k)}\bigr)^{-1}, \qquad \hat{l}_n^{(-k-1)} = -\bigl(l_{n-k}^{(k)}\bigr)^{-1} l_{n-k}^{(k-1)} \bigl(l_{n-k-1}^{(k)}\bigr)^{-1}.$$

The operator G_n^* can be regarded as a special case of series (16) for $k = m$ and $l_n^{(i)} = 0$ with $i < m'$. If symmetry (11) is nondegenerate, then the formal series $(G_n^*)^{-1}$ can be defined, to be used below.

We deal with series (16) of the order $k = 1$ with a nondegenerate leading coefficient,

$$L_n = \sum_{i \le 1} l_n^{(i)} T^i, \qquad \det l_n^{(1)} \neq 0, \tag{17}$$

which is an approximate length-m solution of Eq. (13). In general, the formula

$$\dot{L}_n - [F_n^*, L_n] = \sum_{i \le 2} \theta_n^{(i)} T^i \tag{18}$$

holds for series (17). Formal series (17) is called an approximate length-($m \ge 1$) solution of Eq. (13) if the first m coefficients of (18) are equal to zero: $\theta_n^{(i)} = 0$ for $2 \ge i \ge 3 - m$. The following statement can be proved.

Theorem 1. *If an equation of form* (10) *has two nondegenerate higher symmetries* $U_{n,\tau} = G_n$ *and* $U_{n,\hat{\tau}} = \widehat{G}_n$ *of the orders* $m \ge 1$ *and* $m+1$*, then the corresponding Eq.* (13) *admits an approximate length-*m *solution, which has form* (17) *and is constructed as* $L_n = \widehat{G}_n^* (G_n^*)^{-1}$.

We comment on the motivation underlying the assumptions in this theorem. The assumption regarding the orders $m \ge 1$ and $m + 1$ of the higher symmetries was used previously in the case of analogues (6) of the Volterra equation to obtain first-order solution (17). The validity of this assumption was justified as follows. We consider chains similar to the Volterra equation with a symmetry of any order $m \ge 1$. The notion of the nondegeneracy of a symmetry has no meaning in this scalar case. It can be verified that for any order $m \ge 1$, relativistic Toda chain (2) has both a degenerate and a nondegenerate higher symmetry (see, e.g., [9]). Therefore, in considering equations of this type, we can use the additional assumption regarding the nondegeneracy of the symmetries to invert the operator G_n^*. In the case of the classical Toda chain and

Eqs. (7), the situation is similar, but the nondegeneracy condition can be avoided because the operator G_n^* can be inverted for symmetries of any form.

Instead of the solutions G_n of Eq. (12), we next use approximate solutions (17) of Eq. (13). Evaluating the coefficients $l_n^{(i)}$, we obtain integrability conditions for the function f_n determining system (9). It is important that because a formal series L_n is constructed from the right-hand side of Eqs. (11) in accordance with Theorem 1, the entries of the matrices $l_n^{(i)}$ are functions of the form

$$\phi_n = \phi(u_{n+k_1}, v_{n+k_2}, u_{n+k_1-1}, v_{n+k_2-1}, \dots, u_{n+k_1'}, v_{n+k_2'}) \tag{19}$$

with any integers k_i and k_i' such that $k_1 \geq k_1'$ and $k_2 \geq k_2'$.

We note the advantages of Eq. (13) compared with Eq. (12). Because we consider solutions of form (17), the calculations become simpler, and the obtained integrability conditions are independent of the order m of the higher symmetry. Approximate solutions of Eq. (13) not only constitute a linear space but can also be multiplied. Finally, using solutions of this equation allows constructing local conservation laws for system (9) in the form of relations $D_t p_n = (T-1)q_n$, where p_n and q_n are functions of form (19) (p_n is called the conservation law density). Indeed, the formula

$$D_t \operatorname{tr} \operatorname{res} L_n^j \in \operatorname{Im}(T-1) \tag{20}$$

holds for solutions (17) of Eq. (13) (it is understood that the above function belongs to the image of $T-1$). Here, $L_n^j = \sum_{i \leq j} \tilde{l}_n^{(i)} T^i$, and the residue of this formal series is defined as the coefficient at T^0: $\operatorname{res} L_n^j = \tilde{l}_n^{(0)}$. Formula (20) implies that the trace of the matrix $\tilde{l}_n^{(0)}$ is a local conservation law density for system (9) (the function q_n can be easily constructed from the known density). If (17) is an approximate solution of length $m \geq 3$, then formula (20) can be used for the powers $1 \leq j \leq m-2$.

3. The main complication

From Eqs. (13) and (17), we here derive two integrability conditions for systems of form (9) to identify the main complication, which we then handle using the lemma. Only one restriction is taken into account in what follows:

$$g_n^{(1)} = \frac{\partial f_n}{\partial v_{n+1}} \neq 0 \tag{21}$$

(see (14)). The symmetric case $g_n^{(-1)} \neq 0$ reduces to inequality (21) because the change of variables $\tilde{u}_n = u_{-n}$, $\tilde{v}_n = v_{-n}$ takes symmetries of system (9) into symmetries, and the system integrability is therefore preserved.

In addition to formulas (15) and (17), we introduce the notation

$$F_n^* = F_n^{(1)} T + F_n^{(0)} + F_n^{(-1)} T^{-1}, \qquad l_n^{(i)} = \begin{pmatrix} a_n^{(i)} & b_n^{(i)} \\ c_n^{(i)} & d_n^{(i)} \end{pmatrix}. \tag{22}$$

We suppose that (17) is an approximate length-($m{\geq}2$) solution of Eq. (13). We can then use the relations $\theta_n^{(2)} = 0$ and $\theta_n^{(1)} = 0$ (see (18)), which are written as

$$F_n^{(1)} l_{n+1}^{(1)} = l_n^{(1)} F_{n+1}^{(1)}, \tag{23}$$

$$\dot{l}_n^{(1)} = F_n^{(1)} l_{n+1}^{(0)} + F_n^{(0)} l_n^{(1)} - l_n^{(1)} F_{n+1}^{(0)} - l_n^{(0)} F_n^{(1)}. \tag{24}$$

Relation (23) gives three equations for the entries of the matrix $l_n^{(1)}$, one of which has the form $b_n^{(1)} g_{n+1}^{(1)} = 0$. It follows from condition (21) that $b_n^{(1)} = 0$, and because $\det l_n^{(1)} = a_n^{(1)} d_n^{(1)}$, taking (17) into account, we obtain

$$b_n^{(1)} = 0, \qquad a_n^{(1)} \neq 0, \qquad d_n^{(1)} \neq 0. \tag{25}$$

Another equation in (23) is written as $g_n^{(1)} d_{n+1}^{(1)} = d_n^{(1)} g_{n+1}^{(1)}$. Representing it in the form $(T-1)\big(d_n^{(1)}/g_n^{(1)}\big) = 0$ and recalling that the kernel of $T-1$ consists of constant functions, we obtain the formula $d_n^{(1)} = \alpha g_n^{(1)}$ with a constant $\alpha \neq 0$. Finally, dividing relation (18) by α, we show that an approximate length-($m \geq 2$) solution of Eq. (13) exists in form (17):

$$d_n^{(1)} = g_n^{(1)}. \tag{26}$$

Precisely this solution is used in what follows.

The lower left entries in relation (23) and the upper right entries in (24) yield formulas for the functions $c_n^{(1)}$ and $b_n^{(0)}$:

$$b_n^{(0)} = 1 - \frac{a_n^{(1)}}{g_n^{(1)}}, \qquad c_n^{(1)} = f_n^{(1)} - a_n^{(1)} \varrho_n, \qquad \varrho_n = \frac{f_{n-1}^{(1)}}{g_{n-1}^{(1)}}. \tag{27}$$

Finally, from the diagonal part of (24), we obtain equations for $a_n^{(1)}$ and $d_n^{(0)}$ as

$$D_t \log a_n^{(1)} = (T-1)\varrho_n, \tag{28}$$

$$D_t \log g_n^{(1)} = (T-1)\left(d_n^{(0)} - g_n^{(0)} - \frac{a_n^{(1)} \varrho_n}{g_n^{(1)}}\right), \tag{29}$$

where D_t acts on functions (19) in accordance with

$$D_t = \sum_i v_{n+i} \frac{\partial}{\partial u_{n+i}} + \sum_i f_{n+i} \frac{\partial}{\partial v_{n+i}}.$$

We have thus obtained two conditions for the function f_n in (9), which can be formulated as follows: there exist functions $a_n^{(1)}$ and $d_n^{(0)}$ of form (19) satisfying relations (28) and (29). These conditions are meaningful if we recall that the functions u_{n+i} and v_{n+i} are here considered independent variables. The integrability conditions have the form of local conservation laws with (29) being similar to condition (8). For a given system (9), we know the left-hand side of (29) and can test whether it belongs to $\operatorname{Im}(T-1)$ (we explain how this can be done in the next section). If it does, then we can find a linear combination $d_n^{(0)} - a_n^{(1)} \varrho_n / g_n^{(1)}$ of the unknown functions $d_n^{(0)}$ and $a_n^{(1)}$. In the case of equations of form (6) and (7), all the integrability conditions are similar to (29), and if the equation is integrable, then these conditions allow finding all the coefficients of formal series (17).

In Eq. (28), in contrast, the right-hand side is known but the conservation law density $\log a_n^{(1)}$ is an unknown function. We cannot test such integrability conditions for a given function f_n and cannot find $a_n^{(1)}$. The relations for the other functions $a_n^{(i)}$ and $d_n^{(i)}$ are similar to (28) and (29). We cannot find the functions $a_n^{(i)}$, and because they enter all the relations, all these integrability conditions become useless. This is precisely the complication that occurs in applying the standard symmetry method scheme to systems (9) and therefore to Eqs. (1).

This complication can be overcome using an approximate solution of Eq. (13) simpler than (17). In constructing it, we use a certain exact solution Λ_n, which exists for any function f_n and is in this sense trivial. The operator of multiplication by the identity matrix E is the obvious trivial solution of Eq. (13). The formal series Λ_n turns out to be a square root of this operator.

Lemma. *There exists a unique solution* Λ_n *of Eq.* (13) *such that*

$$\Lambda_n = \sum_{i\le 0} \lambda_n^{(i)} T^i, \qquad \lambda_n^{(i)} = \begin{pmatrix} \alpha_n^{(i)} & \beta_n^{(i)} \\ \gamma_n^{(i)} & \delta_n^{(i)} \end{pmatrix}, \qquad \lambda_n^{(0)} = \begin{pmatrix} 1 & 0 \\ \gamma_n^{(0)} & -1 \end{pmatrix}, \qquad \Lambda_n^2 = E. \tag{30}$$

Proof (outline)**.** We decompose Λ_n into diagonal and antidiagonal parts:

$$\Lambda_n = \sigma + R_n + S_n, \qquad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{31}$$

$$R_n = \sum_{i\le -1} r_n^{(i)} T^i = \sum_{i\le -1} \begin{pmatrix} \alpha_n^{(i)} & 0 \\ 0 & \delta_n^{(i)} \end{pmatrix} T^i, \qquad S_n = \sum_{i\le 0} s_n^{(i)} T^i = \sum_{i\le 0} \begin{pmatrix} 0 & \beta_n^{(i)} \\ \gamma_n^{(i)} & 0 \end{pmatrix} T^i, \tag{32}$$

where $\beta_n^{(0)} = 0$, and first consider the condition $\Lambda_n^2 = E$. It is equivalent to the two relations

$$2\sigma R_n + R_n^2 + S_n^2 = 0, \qquad R_n S_n + S_n R_n = 0. \tag{33}$$

We consider the first of these as an equation for the formal series R_n. Introducing the notation

$$\chi_n^{(-1)} = 0, \qquad \chi_n^{(k)} = \sum_{k+1\le i\le -1} \left(r_n^{(i)} r_{n+i}^{(k-i)} + s_n^{(i)} s_{n+i}^{(k-i)} \right), \quad k \le -2,$$

and collecting the coefficients of like powers of T, we obtain

$$2\sigma r_n^{(k)} + \chi_n^{(k)} + s_n^{(0)} s_n^{(k)} + s_n^{(k)} s_{n+k}^{(0)} = 0, \quad k \le -1. \tag{34}$$

Hence, for any $k \le -1$, the functions $\alpha_n^{(k)}$ and $\delta_n^{(k)}$ are explicitly expressed in terms of $\beta_n^{(-1)}, \beta_n^{(-2)}, \dots, \beta_n^{(k)}$ and $\gamma_n^{(0)}, \gamma_n^{(-1)}, \dots, \gamma_n^{(k+1)}$, and R_n is uniquely determined. The same equation allows obtaining a representation of the form $R_n = \sigma \sum_{i\ge 1} c_i S_n^{2i}$ with constant coefficients c_i. Because $(s_n^{(0)})^2 = 0$, it follows that $S_n^{2i} = \sum_{j\le -i} \tilde{s}_n^{(j)} T^j$, and such a representation is hence well defined. Collecting coefficients of like powers of S_n in the first equation in (33), we obtain formulas for the constants c_i:

$$2c_1 + 1 = 0, \qquad 2c_k + \sum_{1\le i\le k-1} c_i c_{k-i} = 0, \quad k \ge 2.$$

We now have the relation

$$R_n S_n + S_n R_n = \sigma \left[\sum_{i\ge 1} c_i S_n^{2i}, S_n \right],$$

which shows that R_n also satisfies the second equation in (33).

It remains to use Eq. (13), i.e., $\Omega_n = \dot{\Lambda}_n - [F_n^*, \Lambda_n] = 0$. We introduce the notation $\Omega_n^{\parallel} = 0$ and $\Omega_n^{\perp} = 0$ for the diagonal and antidiagonal parts of this equation and also the operators and formal series

$$f_n^{*,u} = f_n^{(1)} T + f_n^{(0)} + f_n^{(-1)} T^{-1}, \qquad f_n^{*,v} = g_n^{(1)} T + g_n^{(0)} + g_n^{(-1)} T^{-1},$$

$$A_n = \sum_{i\le -1} \alpha_n^{(i)} T^i, \qquad B_n = \sum_{i\le -1} \beta_n^{(i)} T^i, \qquad C_n = \sum_{i\le 0} \gamma_n^{(i)} T^i, \qquad D_n = \sum_{i\le -1} \delta_n^{(i)} T^i.$$

The equation $\Omega_n^{\perp} = 0$ is then written as the system

$$B_n f_n^{*,v} + \dot{B}_n + A_n - D_n + 2 = 0,$$
$$f_n^{*,v} C_n - \dot{C}_n + f_n^{*,u} A_n - D_n f_n^{*,u} + 2 f_n^{*,u} = 0. \tag{35}$$

Collecting the coefficients of like powers of T and using inequality (21), we can easily obtain explicit recurrence formulas for the coefficients B_n and C_n, which uniquely define the formal series S_n.

We now only need to show that with the formal series R_n and S_n thus defined, the last condition $\Omega_n^{\|} = 0$ is satisfied automatically. Because E is a solution of Eq. (13), $\Lambda_n^2 = E$, and $\Omega_n^{\perp} = 0$, it follows that

$$\dot{E} - [F_n^*, E] = D_t(\Lambda_n^2) - [F_n^*, \Lambda_n^2] = \Lambda_n \Omega_n + \Omega_n \Lambda_n = \Lambda_n \Omega_n^{\|} + \Omega_n^{\|} \Lambda_n = 0. \tag{36}$$

If $\Omega_n^{\|} \neq 0$, then there is the representation $\Omega_n^{\|} = \sum_{i \le l} \omega_n^{(i)} T^i$, where $\omega_n^{(i)}$ are diagonal matrices and $\omega_n^{(l)} \neq 0$. But collecting the coefficients of T^l in the last equality in (36) and considering the diagonal part of the resulting relation, we obtain a contradiction: $\sigma \omega_n^{(l)} + \omega_n^{(l)} \sigma = 2\sigma \omega_n^{(l)} = 0$. The lemma is proved.

Thus, the solution coefficients Λ_n can be found as follows. Relations (34) allow expressing the functions $\alpha_n^{(i)}$ and $\delta_n^{(i)}$ in terms of $\beta_n^{(i)}$ and $\gamma_n^{(i)}$, for example,

$$\alpha_n^{(-1)} = -\frac{1}{2} \beta_n^{(-1)} \gamma_{n-1}^{(0)}, \qquad \delta_n^{(-1)} = \frac{1}{2} \gamma_n^{(0)} \beta_n^{(-1)},$$
$$\alpha_n^{(-2)} = -\frac{1}{2} \bigl(\alpha_n^{(-1)} \alpha_{n-1}^{(-1)} + \beta_n^{(-1)} \gamma_{n-1}^{(-1)} + \beta_n^{(-2)} \gamma_{n-2}^{(0)} \bigr), \tag{37}$$
$$\delta_n^{(-2)} = \frac{1}{2} \bigl(\delta_n^{(-1)} \delta_{n-1}^{(-1)} + \gamma_n^{(-1)} \beta_{n-1}^{(-1)} + \gamma_n^{(0)} \beta_n^{(-2)} \bigr).$$

From system (35), we next obtain recurrence formulas for $\beta_n^{(i)}$ and $\gamma_n^{(i)}$, for example,

$$\beta_n^{(-1)} = -\frac{2}{g_{n-1}^{(1)}}, \qquad \gamma_n^{(0)} = -\frac{2 f_{n-1}^{(1)}}{g_{n-1}^{(1)}},$$
$$\beta_n^{(-2)} = -\frac{\beta_n^{(-1)} g_{n-1}^{(0)} + \dot{\beta}_n^{(-1)} + \alpha_n^{(-1)} - \delta_n^{(-1)}}{g_{n-2}^{(1)}}, \tag{38}$$
$$\gamma_n^{(-1)} = -\frac{g_{n-1}^{(0)} \gamma_{n-1}^{(0)} - \dot{\gamma}_{n-1}^{(0)} + f_{n-1}^{(1)} \alpha_n^{(-1)} - \delta_{n-1}^{(-1)} f_{n-2}^{(1)} + 2 f_{n-1}^{(0)}}{g_{n-1}^{(1)}}.$$

Hence, we express the functions $\beta_n^{(i)}$ and $\gamma_n^{(i)}$ in terms of the $f_n^{(i)}$ and $g_n^{(i)}$ specified by (14), i.e., in terms of the function f_n that determines system (9).

Using the formal series Λ_n in the lemma, we can now construct an exact solution of Eq. (13) as

$$\Lambda_n^+ = \frac{1}{2}(E + \Lambda_n), \qquad \Lambda_n^- = \frac{1}{2}(E - \Lambda_n). \tag{39}$$

Using the formal series L_n in Theorem 1, we can next introduce two approximate solutions of Eq. (13),

$$L_n^+ = \Lambda_n^+ L_n \Lambda_n^+, \qquad L_n^- = \Lambda_n^- L_n \Lambda_n^-, \tag{40}$$

which have the same order 1 and the same length m. Solutions (40) allow overcoming the above complication and separating the most useful integrability conditions from useless ones.

4. Integrability conditions

We first consider the more important case of an approximate solution L_n^- of Eq. (13) of length $m \geq 4$. It turns out that in contrast to solution (17), we obtain only the integrability conditions similar to (8), which are useful.

From formulas (30) and (39), we have

$$\Lambda_n^- = \begin{pmatrix} 0 & 0 \\ -\gamma_n^{(0)}/2 & 1 \end{pmatrix} - \frac{1}{2}\lambda_n^{(-1)}T^{-1} - \frac{1}{2}\lambda_n^{(-2)}T^{-2} - \dots .$$

We keep the same notation $L_n^- = \sum_{i\leq 1} l_n^{(i)}T^i$ and (22) as for formal series (17). We see from expressions (40) that the functions $b_n^{(1)}$ and $d_n^{(1)}$ for L_n and L_n^- coincide, i.e., $b_n^{(1)} = 0$ and $d_n^{(1)} = g_n^{(1)} \neq 0$ (see (21), (25), and (26)). Other coefficients of the formal series L_n^- can be conveniently found using the equation $\Lambda_n^- L_n^- \Lambda_n^- = L_n^-$, which follows from the property $(\Lambda_n^-)^2 = \Lambda_n^-$. For example, it yields

$$a_n^{(1)} = 0, \qquad c_n^{(1)} = f_n^{(1)}, \qquad b_n^{(0)} = 1, \qquad a_n^{(0)} = \varrho_n$$

with ϱ_n given by (27). We can similarly express any of the functions $a_n^{(i)}$, $b_n^{(i)}$, and $c_n^{(i)}$ in terms of f_n and $d_n^{(i)}$ with $i \leq 0$.

Equation (13) is used only to find the functions $d_n^{(i)}$. Because L_n^- is a length-($m{\geq}4$) solution, we obtain relations for $d_n^{(0)}$, $d_n^{(-1)}$, and $d_n^{(-2)}$. All of them are written as local conservation laws also using formula (20) with $j = 1, 2$ for $d_n^{(-1)}$ and $d_n^{(-2)}$. As a result, we obtain equations of the form

$$D_t p_n^{(i)} = (T-1)q_n^{(i)}, \quad i = 1, 2, 3, \tag{41}$$

where $p_n^{(i)}$ and $q_n^{(i)}$ are functions of form (19) that are expressed in terms of f_n. In addition, $q_n^{(1)}$ and $p_n^{(2)}$ are expressed in terms of $d_n^{(0)}$; $q_n^{(2)}$ and $p_n^{(3)}$ are expressed in terms of $d_n^{(0)}$ and $d_n^{(-1)}$; and $q_n^{(3)}$ are expressed in terms of $d_n^{(0)}$, $d_n^{(-1)}$, and $d_n^{(-2)}$. The simplest such formulas are given by

$$p_n^{(1)} = \log g_n^{(1)}, \qquad q_n^{(1)} = d_n^{(0)} - g_n^{(0)}, \qquad p_n^{(2)} = \operatorname{tr}\operatorname{res} L_n^- = d_n^{(0)} + \varrho_n.$$

In the final formulas, it is convenient to pass from $d_n^{(i)}$ to $q_n^{(i)}$. In addition, we need integrability conditions for Eqs. (1), and we therefore replace v_n with $\dot{u}_n$. Now f_n is the function in (1), $D_t = \sum_i \dot{u}_{n+i}\partial/\partial u_{n+i} + \sum_i f_{n+i}\partial/\partial \dot{u}_{n+i}$, and we use the notation

$$\varrho_n = \frac{\partial f_{n-1}}{\partial u_n}\left(\frac{\partial f_{n-1}}{\partial \dot{u}_n}\right)^{-1}, \qquad \omega_n = \frac{\partial f_n}{\partial u_n} - \frac{\partial f_n}{\partial \dot{u}_n}\varrho_n - \varrho_n^2 + D_t\varrho_n. \tag{42}$$

The formulas for the conservation law densities then become

$$p_n^{(1)} = \log\frac{\partial f_n}{\partial \dot{u}_{n+1}}, \qquad p_n^{(2)} = q_n^{(1)} + \frac{\partial f_n}{\partial \dot{u}_n} + \varrho_n, \qquad p_n^{(3)} = q_n^{(2)} + \frac{1}{2}\big(p_n^{(2)}\big)^2 + \frac{\partial f_n}{\partial \dot{u}_{n-1}}\frac{\partial f_{n-1}}{\partial \dot{u}_n} + \omega_n. \tag{43}$$

Relations (41) and (43) are a corollary of the integrability of the equation of form (1). Therefore, the necessary integrability conditions for Eq. (1) can be formulated as follows: there must exist functions $q_n^{(1)}$, $q_n^{(2)}$, and $q_n^{(3)}$ depending on finitely many variables

$$u_n, \quad \dot{u}_n, \quad u_{n+1}, \quad \dot{u}_{n+1}, \quad u_{n-1}, \quad \dot{u}_{n-1}, \quad u_{n+2}, \quad \dot{u}_{n+2}, \quad u_{n-2}, \quad \dot{u}_{n-2}, \quad \dots \tag{44}$$

such that relations (41) and (43) are satisfied. In other words, the functions $q_n^{(i)}$ have form (19) with $v_n = \dot u_n$. Functions (44) are considered independent variables in relations (41) and (43), and these integrability conditions therefore impose strong restrictions on the function f_n.

For the solution L_n^+ given by formulas (39) and (40), a similar argument yields $b_n^{(1)} = 0$ and $a_n^{(1)} \neq 0$. Other coefficients L_n^+ can be conveniently found using the equation $\Lambda_n^+ L_n^+ \Lambda_n^+ = L_n^+$, for example,

$$d_n^{(1)} = 0, \qquad c_n^{(1)} = -a_n^{(1)} \varrho_n, \qquad b_n^{(0)} = -\frac{a_n^{(1)}}{g_n^{(1)}}, \qquad d_n^{(0)} = \frac{a_n^{(1)} \varrho_n}{g_n^{(1)}}$$

with ϱ_n defined in (27). Using these equations, we express any of the functions $b_n^{(i)}$, $c_n^{(i)}$, and $d_n^{(i)}$ in terms of f_n and $a_n^{(i)}$ whenever necessary.

To obtain equations for the functions $a_n^{(1)}$ and $a_n^{(0)}$ in (13), we use the approximate solution L_n^+ of length $m \geq 3$. As previously, we write these equations as conservation laws,

$$D_t \hat p_n^{(i)} = (T - 1)\hat q_n^{(i)}, \quad i = 1, 2, \tag{45}$$

where formula (20) with $j = 1$ is applied in one case. Finally, passing from the unknown functions $a_n^{(i)}$ to $\hat p_n^{(i)}$ and from the variable v_n to $\dot u_n$, we obtain Eqs. (45) with

$$\hat q_n^{(1)} = \varrho_n, \qquad \hat q_n^{(2)} = \omega_n \left(\frac{\partial f_n}{\partial \dot u_{n+1}} \right)^{-1} e^{\hat p_n^{(1)}}. \tag{46}$$

As in the case of conditions (41) and (43), the function f_n and the operator D_t here correspond to Eq. (1), and notation (42) is used. This time, the necessary integrability conditions require the existence of functions $\hat p_n^{(1)}$ and $\hat p_n^{(2)}$ that depend on a finite number of variables (44) and satisfy Eqs. (45) and (46).

With Theorem 1, the main result in this work can be formulated as the following theorem (we recall that the higher symmetries are found for Eq. (1) written in equivalent form (9)).

Theorem 2. *If an equation of form* (1) *with* $\partial f_n / \partial \dot u_{n+1} \neq 0$ *has two nondegenerate higher symmetries of orders* $m \geq 4$ *and* $m + 1$, *then there exist functions* $q_n^{(i)}$ *with* $i = 1, 2, 3$ *and* $\hat p_n^{(i)}$ *with* $i = 1, 2$ *of finitely many variables* (44) *such that conditions* (41), (43) *and* (45), (46) *are satisfied.*

As previously noted, we cannot test conditions (45) and (46). Written here just to complete the picture, they might prove useful in the future. To test the integrability of Eqs. (1), we propose using the three conditions in (41) and (43). For a given Eq. (1), these integrability conditions are tested in accordance with the general scheme described in [8], [9]. We formulate only the main statement of this scheme for the function of the most general form

$$\varphi_n = \varphi(u_{n+k_1}, \dot u_{n+k_2}, u_{n+k_1-1}, \dot u_{n+k_2-1}, \ldots, u_{n+k_1'}, \dot u_{n+k_2'}), \tag{47}$$

where $k_1 \geq k_1'$ and $k_2 \geq k_2'$, using formal variational derivatives

$$\frac{\delta \varphi_n}{\delta u_n} = \sum_{i=-k_1}^{-k_1'} \frac{\partial \varphi_{n+i}}{\partial u_n}, \qquad \frac{\delta \varphi_n}{\delta \dot u_n} = \sum_{i=-k_2}^{-k_2'} \frac{\partial \varphi_{n+i}}{\partial \dot u_n}.$$

The function φ_n satisfies the condition

$$\frac{\delta \varphi_n}{\delta u_n} = \frac{\delta \varphi_n}{\delta \dot u_n} = 0 \tag{48}$$

if and only if it can be represented in the form

$$\varphi_n = c + (T-1)\psi_n, \tag{49}$$

where c is a constant and ψ_n is a function of form (47) with (possibly) different k_i and k_i'.

For Eq. (41) with $i = 1$, for example, we first test condition (48) with $\varphi_n = D_t p_n^{(1)}$. If the answer is affirmative, we must represent φ_n in form (49) (which is easy to do). If $c = 0$, then the first integrability condition in (41) is satisfied, and we find $q_n^{(1)} = \psi_n$. We now know $p_n^{(2)}$ and can pass to the next integrability condition. It is clear that these conditions are tested consecutively as the functions $q_n^{(i)}$ are found.

We note that the kernel of $T-1$ consists of constant functions, and the functions $q_n^{(i)}$ are therefore defined up to arbitrary constants c_i. From some partial solution $\{q_n^{(i)}\colon i = 1, 2, 3\}$ of system (41), (43), we pass to the general solution: $\{q_n^{(i)} + c_i\}$. Then the corresponding conservation law densities $p_n^{(i)}$ transform as

$$p_n^{(1)} \to p_n^{(1)}, \qquad p_n^{(2)} \to p_n^{(2)} + c_1, \qquad p_n^{(3)} \to p_n^{(3)} + c_1 p_n^{(2)} + \frac{1}{2}c_1^2 + c_2.$$

Because the operators D_t and $T-1$ are linear, we see that the answer to the question whether system (41), (43) has solutions is independent of the choice of the c_i constants. In other words, in testing the integrability conditions, we can choose the functions $q_n^{(i)}$ arbitrarily. If an equation of form (1) satisfies all the three conditions in (41) and (43), then we obtain three local conservation laws for it. The above method also allows performing additional integrability tests for the equation, if necessary. For this, we similarly evaluate the next several coefficients of the formal series L_n^- determined using relations (40). In the case where the result of such a test is affirmative, we also construct additional conservation laws using formula (20). We note that some (or even all) of the conservation laws thus obtained may be trivial (a conservation law is said to be trivial if its density is a function of form (49)).

To conclude, we consider the example of relativistic Toda chain (2). It can be easily verified that all three conditions in (41) and (43) are satisfied. Thus, with the notation $\phi(z) = 1/(1+e^{-z})$ and $w_n = u_{n+1} - u_n$, we have

$$p_n^{(1)} = \log \dot u_n + \log \phi(w_n), \qquad q_n^{(1)} = \dot u_n + \dot u_{n-1}\phi(w_{n-1}). \tag{50}$$

At the second stage, we find

$$p_n^{(2)} = 2\dot u_n + (T-1)r_n, \qquad q_n^{(2)} = 2\dot u_{n-1} r_n + D_t r_n,$$

where $r_n = \dot u_n \phi(w_{n-1})$. It turns out that the function $\hat p_n^{(i)}$ can also be chosen such that relations (45) and (46) are satisfied. We use the function

$$p_n = \frac{1}{\dot u_n}(e^{w_n} + 1)(e^{w_{n-1}} + 1), \tag{51}$$

which is a local conservation law density for Eq. (2): $D_t p_n = -(T-1)e^{w_{n-1}}$. The needed formulas have the forms

$$\hat p_n^{(1)} = \log p_n + \log \phi(w_n), \qquad \hat p_n^{(2)} = 2p_n + (T-1)\bigl(p_n\bigl(2 - \phi(w_n)\bigr)\bigr).$$

We note that if a conservation law density is multiplied by any number and the trivial density of form (49) is added to it, then the conservation law changes insignificantly. This is just the relation between $\hat p_n^{(1)}$ and $p_n^{(1)}$:

$$\hat p_n^{(1)} = -p_n^{(1)} + (T-1)\bigl(u_n + u_{n-1} + \log \phi(w_{n-1})\bigr).$$

Thus, having verified four integrability conditions, we obtain three essentially different and nontrivial conservation laws with the densities $\dot u_n$ and $p_n^{(1)}$, p_n from Eqs. (50) and (51).

5. Conclusions

We have derived integrability conditions for equations of form (1), which are necessary conditions for the existence of higher symmetries. The standard symmetry method scheme, which amounts to investigating approximate solutions L_n of form (17) of Eq. (13), does not give reasonable results in this case: the arising integrability conditions cannot be used. This complication can be overcome using a lemma on the "trivial" exact solution Λ_n of Eq. (13). The lemma allows constructing approximate solutions L_n^- and L_n^+ of Eq. (13) in accordance with formulas (39) and (40) instead of L_n. Next, evaluating the coefficients of these solutions and applying formula (20), we obtain standard integrability conditions (41) and (43) from L_n^-, which are easy to use, and nonstandard conditions (45) and (46) from the solution L_n^+.

Conditions (41) and (43) are similar to those that occurred previously in [7] in investigating analogues of Toda and Volterra chains. They can be easily tested for any given Eq. (1) using the equivalence of relations (48) and (49). These conditions thus give a convenient tool for testing the integrability of equations of form (1). If the equation satisfies all three integrability conditions, then we obtain three local conservation laws for it. Conditions (41) and (43) can also be used to seek new integrable examples, by investigating equations of form (1) with a small functional arbitrariness, or even to fully classify integrable equations of this class.

We note that proceeding further with calculating the coefficients of the solution L_n^- and applying formula (20), we can similarly perform an additional integrability test for a given equation, obtain additional conservation laws for it, and write one or two more integrability conditions in the most general case.

Integrability conditions (45) and (46) differ principally from conditions (41) and (43). They have the form of local conservation laws with an unknown density; they have not been encountered in investigating the classes of Toda and Volterra chains. It is not clear how these conditions can be used to test the integrability of a given equation or to classify them. Learning how such integrability conditions can be used is an unsolved problem.

The exact solution Λ_n of Eq. (13), which is discussed in the lemma and is the square root of the operator of multiplication by the identity matrix, occurs for the first time in the symmetry method literature. There are also other cases where such an exact solution exists that helps overcome similar complications in the theory. These are, first, the class of discrete-differential systems of the form

$$\dot{u}_n = f(u_{n+1}, u_n, v_n), \qquad \dot{v}_n = g(v_{n-1}, v_n, u_n)$$

involving a Hamiltonian form of the known integrable relativistic chains of form (1), including relativistic Toda chain (2) (see [6], [9], [10]). Another class consists of systems of partial differential equations with

$$u_t = u_{xx} + f(u, v, u_x, v_x), \qquad v_t = -v_{xx} + g(u, v, u_x, v_x),$$

which include the split nonlinear Schrödinger equation or the Ablowitz–Kaup–Newell–Segur system (see, e.g., [5], [11] for the applications of the symmetry method to such systems). The technical trick involving such a solution Λ_n can also prove useful in other problems.

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