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On the construction of Miura type transformations by others of this kind

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Miura type transformations and nonlinear evolution scalar and vector equations related by these transformations are considered. It is shown how to construct other such transformations and equations, using the given ones.

As is known (see ref. [1]) the Miura transformation is

$$w=u_x+\alpha-u^2 \tag{1}$$

and links together the following well-known equations,

$$u_t=u_{xxx}+6(\alpha-u^2)u_x\,, \tag{2}$$

$$w_t=w_{xxx}+6ww_x\,, \tag{3}$$

where α is an arbitrary constant. We shall be interested in analogous objects in the more general case and consider evolution equations of the form

$$u_t=f(x,u,u_x,u_{xx},...,\partial^n u/\partial x^n) \tag{4}$$

and transformations of the form

$$w=p(x,u,u_x)\,, \tag{5}$$

where u, f, w, p are N-vectors. This Letter consists of a theorem on the construction of such transformations and equations related by them, and of examples.

Let us denote the vector functions $r(x, u, u_x, ..., \partial^m u/\partial x^m)$ by $r[u]$ and write down eqs. (4) in the following way,

$$u_t=f[u]\,. \tag{6}$$

Equation (6) is said to be reduced to an equation

$$w_t=h[w] \tag{7}$$

via transformation (5) if formula (5) yields a solution of (7) for any solution of (6). In other words, the condition

$$\partial_f(p)=h[p] \tag{8}$$

has to hold. Here ∂_f is the differentiation with respect to t by virtue of (6),

$$\partial_f(p)=p_u f+p_{u_x}D_x(f)\,,$$

D_x is the differentiation with respect to x,

$$D_x(f)=f_x+f_u u_x+f_{u_x}u_{xx}+\dots\,,$$

and p_u, p_{u_x}, f_u, f_{u_x}, ... are matrices of partial derivatives. For example, $p_u=(\partial p^i/\partial u^j)$, where p^i, u^j are coefficients of the vectors p, u.

Theorem. Let eq. (6) and the equation

$$U_t=F[U] \tag{9}$$

be reduced to eq. (7) via the transformations

$$w=D_x(a(x,u))+\varphi(x,u)\,, \tag{10}$$

$$w=D_x(b(x,U))+\psi(x,U) \tag{11}$$

respectively, and let the following matrix be nondegenerate,

$$\begin{vmatrix} -a_u(x,u) & b_U(x,U) \\ \varphi_u(x,u) & -\psi_U(x,U) \end{vmatrix}\neq 0\,. \tag{12}$$

Then there exists an equation

$$v_t=g[v] \tag{13}$$

that will be reduced to eqs. (6), (9) by transformations of the form

$$u=q(x, v, v_x)\ , \tag{14}$$

$$U=Q(x, v, v_x) \tag{15}$$

(see diagram (19)). The formulae

$$b(x, Q)-a(x, q)=v\ , \tag{16}$$

$$\varphi(x, q)-\psi(x, Q)=v_x\ , \tag{17}$$

$$g[v]=b_U(x, Q)F[Q]-a_u(x, q)f[q] \tag{18}$$

enable one to construct the vector functions q, Q, g:

$$\begin{array}{ccc} \langle f\rangle & \longleftarrow & \langle g\rangle \\ \downarrow & & \downarrow \\ \langle h\rangle & \longleftarrow & \langle F\rangle \end{array} \tag{19}$$

It will be noted that if

$$|a_u|\neq 0\ , \quad |b_U|\neq 0\ , \tag{20}$$

then one can apply point transformations $\tilde{u}=a(x, u)$, $\tilde{U}=b(x, U)$ to eqs. (6), (9) and bring (10), (11) into a form such that $a=u$, $b=U$. In this case, instead of (12), (16)–(18) we have the more simple nondegeneracy condition

$$|\varphi_u(x, u)-\psi_U(x, U)|\neq 0 \tag{21}$$

and the following formulae for q, Q, g:

$$\varphi(x, q)-\psi(x, q+v)=v_x\ , \tag{22}$$

$$Q=q+v\ , \quad g[v]=F[q+v]-f[q]\ . \tag{23}$$

Proof. We shall consider pairs of solutions (u, U) of eqs. (6), (9) which satisfy the constraint

$$\begin{aligned} &D_x(a(x, u))+\varphi(x, u) \\ &\quad =D_x(b(x, U))+\psi(x, U)\ . \end{aligned} \tag{24}$$

Let us show that (24) is compatible with (6), (9). Differentiating (24) with respect to t we obtain the compatibility condition

$$\partial_f(D_x(a)+\varphi)=\partial_F(D_x(b)+\psi)\ . \tag{25}$$

Now let us use that eqs. (6), (9) are reduced to eq. (7) via (10), (11). This allows us to write down the condition (25) as follows,

$$h[D_x(a)+\varphi]=h[D_x(b)+\psi] \tag{26}$$

(see (8)). It is obvious that (24) implies (26). Thus system (6), (9), (24) is compatible, and relationship (25) holds for the solutions of this system. It is easy to see that (25) can be expressed in the form

$$D_x(a_u f)+\varphi_u f=D_x(b_U F)+\psi_U F\ . \tag{27}$$

Let us introduce a new vector function in the following way,

$$v=b(x, U)-a(x, u)\ . \tag{28}$$

In virtue of (24) the equality

$$v_x=\varphi(x, u)-\psi(x, U) \tag{29}$$

holds. It follows from (12) that the change of variables (28), (29) between u, U and v, v_x is invertible. The inverse change of variables has the form (14), (15) where q, Q satisfy (16), (17). It is clear that

$$\begin{pmatrix} -a_u & b_U \\ \varphi_u & -\psi_U \end{pmatrix}\begin{pmatrix} q_v & q_{vx} \\ Q_v & Q_{vx} \end{pmatrix}=E\ . \tag{30}$$

Differentiating (28) with respect to t we obtain (13), (18). Hence, v satisfies (13), (18) for any solutions of (6), (9), (24). We have to prove that u, U are solutions of (6), (9) for any one of (13), (18). That is to say, we have to establish that

$$\partial_g(q)-f[q]=\partial_g(Q)-F[Q]=0 \tag{31}$$

(see (8)). To do this, let us transform (31) into

$$\begin{pmatrix} q_v & q_{vx} \\ Q_v & Q_{vx} \end{pmatrix}\begin{pmatrix} g \\ D_x(g) \end{pmatrix}=\begin{pmatrix} f[q] \\ F[Q] \end{pmatrix} \tag{32}$$

and note that (32) is equivalent to

$$g=b_U F[U]-a_u f[u]\ , \tag{33}$$

$$D_x(g)=\varphi_u f[u]-\psi_U F[U] \tag{34}$$

(see (30)). It is seen that (33) follows from (18) as well as (34) follows from (27), (33).

So we see that in order to obtain (13)–(15) from (6), (7), (9)–(11), we consider the constraint (24) and choose a new variable v in a special way. The result can be generalized if one can choose such a variable in the case of more complicated constraints. Note that the question of the choice of new variables in constraints analogous to (24) was investigated in ref. [2].

There are many instances in which the theorem can be applied. We shall discuss here some of them. All scalar equations and transformations of the form (4) and (5) we shall deal with have been well-known for a long time.

The functions f, F, h as well as the transformations (10), (11) may coincide with each other. This means in particular that (10), (11) may be auto-transformations. On the other hand, if we are given only one transformation, we can take advantage of the theorem too (see below).

In the first instance we shall assume that (6), (10) coincide with (9), (11) respectively. The diagram (19) will become such that

$$\langle h\rangle \leftarrow \langle f\rangle \leftrightarrows \langle g\rangle .$$

Let us consider scalar equations of the form

$$w_t=(D_x^3+4wD_x+2w_x)H[w] , \tag{35}$$

$$u_t=(D_x^2+2uD_x+2u_x)H[u_x-u^2] , \tag{36}$$

where H is an arbitrary function. They are related by means of the Miura transformation $w=u_x-u^2$ for any H. If $H=w$, then eq. (35) is the Korteweg–de Vries equation (3). We have other integrable equations as $H=w_{xx}+\gamma w^2$, $\gamma\in\{\frac{1}{2},3,8\}$. With the help of (22), (23) one can easily construct the equation

$$V_t=2(D_x+V_x)H[\tfrac{1}{4}(2V_{xx}-V_x^2-e^{2V})] \tag{37}$$

being reduced to eq. (36) by

$$2u=V_x+\sigma e^V , \quad \sigma^2=1 .$$

It is clear that one may apply the theorem once more.

As a next example consider the Korteweg–de Vries equation (3). To obtain more results than in the previous example, we should use the existence of the parameter α (see (1), (2)). Let eqs. (6), (7) and the transformation (10) be (2), (3), (1) respectively. As (9), (11) we take (2), (1) with a parameter β in place of α. Then eq. (13) is the Calogero–Degasperis equation

$$V_t=V_{xxx}-\tfrac{1}{2}V_x^3-\tfrac{3}{2}(e^{2V}+\gamma^2e^{-2V})V_x+3\delta V_x , \tag{38}$$

where $\gamma=\beta-\alpha$, $\delta=\beta+\alpha$ (we have performed the point transformation $v=e^V$). The transformation

$$2u=V_x+\gamma e^{-V}+\kappa e^V , \quad \kappa^2=1 , \tag{39}$$

reduces (38) to (2) with $2\alpha=\delta+\kappa\gamma$. In a similar way, one can construct an equation that will be reduced to (38).

It is possible to use the fact that the right hand side of (3) is expressed in the form $D_x(w_{xx}+3w^2)$. This means that there is the equation

$$U_t=U_{xxx}+3U_x^2+\epsilon \tag{40}$$

linked with (3) via

$$w=U_x . \tag{41}$$

If (6), (7), (9), (10), (11) are (2), (3), (40), (1), (41) respectively, eq. (13) will be

$$v_t=v_{xxx}+3v_{xx}^2/4(\alpha-v_x)+3v_x^2+\epsilon . \tag{42}$$

The transformations (14), (15) are

$$u=(\alpha-v_x)^{1/2} , \quad U=(\alpha-v_x)^{1/2}+v .$$

In a similar way one can use the fact that (2) is expressed in the form

$$u_t=D_x(u_{xx}+6\alpha u-2u^3) .$$

In ref. [3] eqs. (38), (42) and other equations related to (3) were obtained from (3) in another way.

Let us discuss the well-known (see refs. [4–6]) integrable system

$$U_t=U_{xxx}+UU_x+VV_x ,$$
$$-V_t=2V_{xxx}+UV_x . \tag{43}$$

The transformation

$$U=u_x+\epsilon(v^2-2u^2) , \quad V=v_x-4\epsilon uv , \tag{44}$$

where $\epsilon=\frac{1}{12}$, links together (43) and

$$u_t=u_{xxx}+D_x(\tfrac{1}{2}vv_x-\epsilon uv^2-\tfrac{2}{3}\epsilon u^3) ,$$
$$v_t=-2v_{xxx}+D_x(vu_x+2\epsilon vu^2+\tfrac{1}{3}\epsilon v^3) \tag{45}$$

(see refs. [5,6]). This allows one to obtain the system

$$p_t=p_{xxx}-3q_xq_{xx}+\tfrac{1}{2}p_x[3q_x^2-p_x^2-\epsilon e^{2p}\,\mathrm{ch}(2q)] ,$$
$$q_t=-2q_{xxx}+3q_xp_{xx}+\epsilon p_x\,e^{2p}\,\mathrm{sh}(2q)$$
$$+\tfrac{1}{2}q_x[3p_x^2-q_x^2+\epsilon e^{2p}\,\mathrm{ch}(2q)] . \tag{46}$$

Note that formulae (22), (23) give a system $\tilde{u}_t = A[\tilde{u}, \tilde{v}]$, $\tilde{v}_t = B[\tilde{u}, \tilde{v}]$ which does not coincide with (46). However, setting $\tilde{u} = e^p \operatorname{ch}(q)$, $\tilde{v} = \lambda e^p \operatorname{sh}(q)$, $\lambda^2 + 2 = 0$ we have (46) and the transformation

$$u = 3p_x + \sigma e^p \operatorname{ch}(q)\,, \quad v/\lambda = 3q_x + \sigma e^p \operatorname{sh}(q)\,, \tag{47}$$

where $4\sigma^2 = 1$. System (46) appears to be new.

There are several possibilities to construct other integrable systems. In the first place, one can take advantage of (47). In the second place, one can act as in the case of eq. (42), for (45) has a corresponding form. Another possibility is explained by the fact that there are the system

$$u_t = u_{xxx} + \tfrac{1}{2}(u_x^2 + R^2) + \alpha\,,$$

$$v_t = -2v_{xxx} + 3\lambda u_{xxx} + D_x(v u_x - \tfrac{1}{18} v^3) \tag{48}$$

and the link between (43), (48)

$$U = u_x\,, \quad V = R\,, \quad R = v_x - \lambda u_x + \tfrac{1}{12}\lambda v^2\,, \tag{49}$$

where $\lambda^2 + 2 = 0$, α is an arbitrary constant. We can use (49) together with (44).

Unlike all the above transformations of the form (10) and (11), the following ones,

$$U = u_x - u^2 v\,, \quad V = v\,, \tag{50}$$

$$U = u\,, \quad V = -v_x - v^2 u\,, \tag{51}$$

do not satisfy condition (20). Both (50) and (51) relate (see refs. [7,8])

$$u_t = u_{xx} - 2u^2 v_x - 2u^3 v^2\,,$$

$$v_t = -v_{xx} - 2v^2 u_x + 2v^3 u^2 \tag{52}$$

to the nonlinear Schrödinger equation

$$U_t = U_{xx} + 2U^2 V\,, \quad -V_t = V_{xx} + 2V^2 U\,. \tag{53}$$

Let us make use of formulae (16)–(18) in the case when (6), (9) are (52), and (10), (11) are (51), (50) respectively. To set in the obtained system

$$p = 2\tilde{u}/(\tilde{u}\tilde{v} - 1)\,, \quad q = 2\tilde{v}/(\tilde{u}\tilde{v} - 1)\,,$$

they will be

$$p_t = p_{xx} - \frac{q p_x^2 + 2p p_x q_x}{2(pq+1)}\,,$$

$$q_t = -q_{xx} + \frac{p q_x^2 + 2q q_x p_x}{2(pq+1)}\,. \tag{54}$$

The links between (52) and (54) are

$$u = p_x/2R\,, \quad v = (R+1)/p\,, \tag{55}$$

$$u = (R+1)/q\,, \quad v = -q_x/2R\,, \tag{56}$$

where $R = (pq+1)^{1/2}$. System (54) is closely connected with the Heisenberg model. The transformations (55), (56) arose in ref. [9] in connection with discrete models.

It will be observed that there exist examples of vector equations and transformations of arbitrarily high dimension the theorem can be applied to. For instance, there are multi-field generalizations of (50)–(53) in ref. [10].

It is interesting that the described scheme of the construction of Miura type transformations and nonlinear evolution equations yields results in the case of the Kadomtsev–Petviashvili equation (59). We start with the following transformation (57) (see ref. [11]) and eqs. (58), (59),

$$w = u_x + \alpha - u^2 + \epsilon D_x^{-1} u_y\,, \tag{57}$$

$$u_t = u_{xxx} + 6(\alpha - u^2) u_x + 3\epsilon^2 D_x^{-1} u_{yy}$$

$$+ 6\epsilon u_x D_x^{-1} u_y\,, \tag{58}$$

$$w_t = w_{xxx} + 6 w w_x + 3\epsilon^2 D_x^{-1} w_{yy}\,, \tag{59}$$

where α, ϵ are arbitrary constants, D_x^{-1} is an inverse of D_x. An analog of the relationship (24) has the form

$$(U-u)_x = \alpha - \beta + U^2 - u^2 - \epsilon D_x^{-1}(U-u)_y\,, \tag{60}$$

where U satisfies (58) with β instead of α. A new variable is chosen to be

$$e^V = U - u\,. \tag{61}$$

Representing (60) in the form

$$U + u = V_x + R e^{-V}\,,$$

$$R = \gamma + \epsilon D_x^{-1}(e^V)_y\,, \quad \gamma = \beta - \alpha\,, \tag{62}$$

we can express U, u via V. Now, differentiating (61) with respect to t and using (58), (61), (62), we obtain

$$V_t = V_{xxx} - \tfrac{1}{2} V_x^3 - \tfrac{3}{2}(e^{2V} + R^2 e^{-2V}) V_x$$

$$+ 3[\delta + \epsilon D_x^{-1}(R e^{-V})_y] V_x + 3\epsilon R_y e^{-V}\,, \tag{63}$$

where $\delta = \beta + \alpha$. It is not hard to verify that eq. (63) is reduced to eq. (58) with $2\alpha = \delta + \kappa\gamma$ by means of

$$2u = V_x + R e^{-V} + \kappa e^V\,, \quad \kappa^2 = 1\,. \tag{64}$$

It is readily seen that if $\epsilon=0$, then (57)–(59), (63), (64) coincide with (1)–(3), (38), (39). In particular, eq. (63) generalizes the Calogero–Degasperis equation (38).

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