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# CLASSIFICATION OF TODA TYPE SCALAR LATTICES

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**Abstract.** Toda type scalar lattices are considered. All lattices of this kind possessing local conservation laws of the high enough order are enumerated. Essentially new interesting examples are obtained.

The symmetry approach [1,2] to the investigation of integrable differential equations first was applied to scalar evolution partial differential equations analogous to the Korteweg-de Vries equation [3]. In 1983 this approach was developed for discrete equations similar to the Volterra equation $du_n/dx = u_n(u_{n+1} - u_{n-1})$, where $n$ is an arbitrary integer [4]. A complete list was given of discrete equations of the form $(u_n)_x = f(u_{n+1}, u_n, u_{n-1})$ possessing an infinite set of local conservation laws. It is demonstrated here how to apply the symmetry approach to scalar discrete equations generalizing the Toda lattice

$$(u_n)_{xx} = \exp(u_{n+1} - u_n) - \exp(u_n - u_{n-1}). \qquad (1)$$

This work has been published only in a local preprint [5].

## 1. Main results

All lattices of the form

$$u_{nxx} = f(u_{nx}, u_{n+1}, u_n, u_{n-1}), \qquad (2)$$

where $\partial f/\partial u_{n\pm1} \neq 0$, having an infinite set of local conservation laws will be enumerated. A local conservation law of the lattice (2) is a relationship of the form

$$(p_n)_x = q_{n+1} - q_n, \qquad (3)$$

where the functions $p_n, q_n$ depend on a finite number of the variables $u_k, u_{kx}$. For example for the Toda lattice (1) there is the relationship (3) with $p_n = u_{nx}$, $q_n = \exp(u_n - u_{n-1})$. Using a local conservation law (3) of Eq.(2)

one can construct a constant of the motion for this equation. In the case of periodic solutions $u_{n+N} = u_n$, $u_{n+N,x} = u_{nx}$ such a constant will be of the form $H = \sum_1^N p_n$, since $dH/dx = 0$.

At first we shall obtain necessary conditions of the existence of the local conservation laws. Three of them are of the form of the local conservation laws:

$$(p_n^k)_x = q_{n+1}^k - q_n^k \qquad (k = 1,2,3),$$

$$p_n^1 = \ln(\partial f/\partial u_{n+1}), \qquad p_n^2 = 2q_n^1 + \partial f/\partial u_{nx},$$

$$p_n^3 = 2q_n^2 - 1/2\ (\partial f/\partial u_{nx})_x + 1/4\ [(\partial f/\partial u_{nx})^2 + (p_n^2)^2] + \partial f/\partial u_n. \tag{4}$$

Another two have the form:

$$r_n^k = s_{n+1}^k - s_n^k \qquad (k = 1,2),$$

$$r_n^1 = \ln(\partial f/\partial u_{n+1}) - \ln(\partial f/\partial u_{n-1}), \qquad r_n^2 = (s_n^1)_x + \partial f/\partial u_{nx}. \tag{5}$$

The conditions (4),(5) mean that there must exist functions $q_n^k, s_n^k$ such that the relationships (4),(5) hold. It is easy to see that the Toda lattice satisfies the necessary conditions.

The symmetry approach enables one to get an infinite number of the necessary conditions. However, one usually uses in the classification problems only several simplest conditions such that equations are determined up to arbitrary constants. It turns out that these conditions are not only necessary but also sufficient.

In the second place we shall describe Eqs.(2) satisfying the conditions (4),(5). The following lattices are of greatest interest:

$$u_{nxx} = R(u_{nx})[g(u_{n+1} - u_n) - g(u_n - u_{n-1})], \tag{6}$$

where $R(a) = \epsilon a^2 + \alpha a + \beta$, the function $g(a)$ satisfies the ordinary differential equation $g' = \epsilon g^2 + \gamma g + \delta$, and $\epsilon, \alpha, \beta, \gamma, \delta$ are arbitrary constants;

$$u_{nxx} = [u_{nx}^2 + P(u_n)][(u_n - u_{n-1})^{-1} - (u_{n+1} - u_n)^{-1}] - 1/2\ P'(u_n), \tag{7}$$

where $P$ is an arbitrary polynomial of the fourth degree. Another lattices are reduced to Eqs.(6),(7) by rather simple transformations.

Eqs.(6),(7) are the infinite Hamiltonian systems, for they can be expressed in the form:

$$u_{nx} = r(u_n, v_n)\delta h_n/\delta v_n, \qquad v_{nx} = -r(u_n, v_n)\delta h_n/\delta u_n, \tag{8}$$



where $v_n = u_{nx}$, $\delta h_n/\delta v_n = \sum_k \partial h_k/\partial v_n$, $\delta h_n/\delta u_n$ is defined by the similar way. In the case of Eq.(6) $r = R(v_n)$, $h_n = G(u_{n+1} - u_n) + A(v_n)$, where $G'(a) = g(a)$, $A'(a) = a/R(a)$. For Eq.(7) $r = v_n^2 + P(u_n)$, $h_n = \frac{1}{2}\ln r - \ln(u_{n+1} - u_n)$. The Hamiltonian form (8) permits us to obtain a compatible with Eq.(8) discrete equation from any conservation law (3) of Eq.(8). The formula for the construction of symmetries is $u_{nt} = r\delta p_n/\delta v_n$, $v_{nt} = -r\delta p_n/\delta u_n$. The formulae (4),(5) yield for Eqs.(6),(7) several simple conservation laws.

Note that by means of transformations of the form $\tilde{u}_n = \tilde{u}_n(u_n, v_n)$, $\tilde{v}_n = \tilde{v}_n(u_n, v_n)$ we can reduce Eq.(8) to the form (8) with $r = 1$. On the other hand, in the periodicity case $u_{n+N} = u_n$, $v_{n+N} = v_n$ the functions $\delta h_n/\delta v_n$, $\delta h_n/\delta u_n$ are the partial derivatives of the Hamiltonian $H = \sum_1^N h_n$ with respect to $v_n, u_n$, respectively. Thus, in this case the system (8) is expressible as the classical Hamiltonian one: $u_{nx} = \partial H/\partial v_n$, $v_{nx} = -\partial H/\partial u_n$, where $n = 1, 2, ..., N$.

As it is shown in the article [6] the lattices (6),(7) are closely connected with the systems of partial differential equations:

$$u_t = u_{xx} + 2R(u_x)g(u+v) + \gamma u_x^2,$$
$$-v_t = v_{xx} + 2R(-v_x)g(u+v) + \gamma v_x^2, \tag{9}$$

$$u_t = u_{xx} - 2[u_x^2 + P(u)]/(u-v) + 1/2\ P'(u),$$
$$-v_t = v_{xx} - 2[v_x^2 + P(v)]/(v-u) + 1/2\ P'(v), \tag{10}$$

respectively. In particular, the lattices provide corresponding systems with invertible differential substitutions which do not change these systems. The system (9) is invariant under the following transformation: $\tilde{v} = -u$, $\tilde{u} = u + g^{-1}[g(u+v) + u_{xx}/R(u_x)]$. Such a transformation for (10) is $\tilde{v} = u$, $\tilde{u} = u + \{(u-v)^{-1} - [u_{xx} + P'(u)/2]/[u_x^2 + P(u)]\}^{-1}$. All the systems (9),(10) are well-known integrable Hamiltonian ones (see, e.g., [1]). There is, in particular, the nonlinear Schrodinger equation among the systems of the form (9). The system (10) is related to the classical Landau-Lifshits model by the stereographic projection.

The connection of the lattices (6),(7) with the integrable systems (9),(10) means that these lattices are integrable too. In particular, infinite sets of local conservation laws and symmetries can be constructed.

## 2. Necessary conditions

Here the necessary conditions of the existence of the local conservation laws will be derived.

To have more simple formulae, we shall not write the subscript $n$. So the lattice (2) will be of the form:

$$u_{xx} = f(u_x, u_1, u, u_{-1}), \qquad f_{u_1} f_{u_{-1}} \neq 0 \tag{11}$$

$(f_{u_k} = \partial f/\partial u_k)$. One may think that all relationships are written down at $n = 0$. All relationships are invariant under the shift $D$:

$$D[F(u_k, u_{k+1}, \ldots, u_m)] = F(u_{k+1}, u_{k+2}, \ldots, u_{m+1}).$$

Therefore a relationship at $n = 0$ determines one at all another points.

The local conservation laws are closely connected with two important equations for formal series $L, S$ in powers of the shift operator $D^{-1}$:

$$L = \sum_{k=m}^{-\infty} L_k D^k, \qquad S = \sum_{k=n}^{-\infty} S_k D^k, \tag{12}$$

where $L_m \neq 0$, $S_n \neq 0$. The coefficients

$$L_k = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix}, \qquad S_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \tag{13}$$

depend on $u_i, u_{ix}$. The equations have the form:

$$B(L) = (L)_x + Lf_* - f_* L = 0, \tag{14}$$

$$A(S) = (S)_x + Sf_* + f_*^T S = 0. \tag{15}$$

Note that the first one determines the recursion operator. Here $f_*$ is the Frechet derivative:

$$f_* = f_1 D + f_0 + f_{-1} D^{-1}, \quad f_{\pm 1} = \begin{bmatrix} 0 & 0 \\ f_{u_{\pm 1}} & 0 \end{bmatrix}, \quad f_0 = \begin{bmatrix} 0 & 1 \\ f_u & f_{u_x} \end{bmatrix},$$

where $f_{u_x} = \partial f/\partial u_x$. $f_*^T$ is the transposed operator:

$$f_*^T = D(f_{-1}^T) D + f_0^T + D^{-1}(f_1^T) D^{-1},$$

where $f_k^T$ are the transposed matrices.

We shall be interested in approximate solutions of these equations. The series $L$ from (12) is called the approximate solution of Eq.(14) of the length $M$ if the coefficients $B_k$ of the series

$$B(L) = B_{m+1} D^{m+1} + B_m D^m + \ldots \tag{16}$$



vanish for $m+1 \geq k \geq m - M + 2$. By the order of $L$ we mean the number $m$. Definitions for Eq.(15) are analogous. It will be shown that if a lattice of the form (11) admits a long enough approximate solution of Eq.(14), then the conditions (4) hold. If there is an approximate solution of Eq.(15), one also can derive the conditions (5).

Let us consider Eq.(14). Let the series $L$ be non-degenerate (i.e. the first coefficient of $L$ be non-degenerate) and $m > M \geq 4$, where $m$ is the order and $M$ is the length of $L$. The equality $B_{m+1} = 0$ (see (16)) implies $\beta_m = 0$,

$$f_{u_1} D(\alpha_m) = \delta_m D^m(f_{u_1}), \tag{17}$$

where $\alpha_m, \beta_m, \delta_m$ are the elements of the matrix $L_m$ (see (13)). It follows from $B_m = 0$ that

$$\alpha_m = \delta_m, \tag{18}$$

$$\gamma_m = (\alpha_m)_x + \beta_{m-1} D^{m-1}(f_{u_1}), \tag{19}$$

$$\gamma_m = -(\delta_m)_x + \delta_m (1 - D^m)(f_{u_x}) + f_{u_1} D(\beta_{m-1}). \tag{20}$$

By virtue of (17),(18) we may consider that

$$\alpha_m = f_{u_1} D(f_{u_1}) \ldots D^{m-1}(f_{u_1}) \tag{21}$$

because $L$ may be multiplied by a constant. Let us subtract (20) from (19) and divide by $\alpha_m$:

$$2(\ln \alpha_m)_x = (1 - D^m)(f_{u_x}) + (D - 1)[\beta_{m-1} \alpha_m^{-1} D^{m-1}(f_{u_1})]. \tag{22}$$

The equalities (21),(22) mean that the condition (4) with $k = 1$ is satisfied. By the similar way, using $B_{m-1} = B_{m-2} = 0$ we can derive the conditions (4) with $k = 2, 3$.

In the case when the approximate solution $L$ with $m > M \geq 5$ is degenerate it follows from (18),(19) that $\beta_m = \alpha_m = \delta_m = 0$, $\gamma_m \beta_{m-1} \neq 0$. It means that $L^2$ has the order $2m - 1$ and is non-degenerate.By virtue of the formula $B(L^2) = B(L)L + LB(L)$ (see (14)) the series $L^2$ is an approximate solution too, but its length equals $M - 1$. So this case is reduced to the previous one, and we have the following Proposition.

**Proposition 1.** Let for a lattice of the form (2) there be an approximate solution $L$ of Eq.(14) and $m > M \geq 5$, where $m$ is the order and $M$ is the length of $L$. Then the lattice satisfies the conditions (4).

Calculating the coefficients of the series $S$ from (15) we have to do it twice:in the non-degenerate case and in the degenerate one. In the first of them, for example, we get the equality: $D^n(f_{u_1})/f_{u_{-1}} = D(c_n)/c_n$ (see notations in (12),(13)). It means that the condition (5) with $k=1$ holds. Hence the function $s^1$ from (5) with $k=1$ is determined. The formula $c_n = D(f_{u_1}) \ldots D^{n-1}(f_{u_1})D(\exp s^1)$ and the relationship $2(\ln c_n)_x + (D^n + 1)(f_{u_x}) = (D-1)[d_{n-1}c_n^{-1}D^{n-1}(f_{u_1})]$ enables us to obtain (5) with $k=2$. Note that it is necessary to use (4) with $k=1$.

**Proposition 2.** Let for a lattice of the form (2) there exist an approximate solution $S$ of Eq.(15) and $n > N \geq 3$, where $n$ is the order and $N$ is the length of $S$. Besides, let the condition (4) with $k=1$ hold. Then the lattice satisfies the conditions (5).

If there are conservation laws of the high enough order, there exist solutions of Eqs.(14),(15). Let us denote

$$H_{w_k} = \begin{bmatrix} F_{u_k} & F_{u_{kx}} \\ G_{u_k} & G_{u_{kx}} \end{bmatrix},$$

where $H = (F,G)^T$, $w_k = (u_k, u_{kx})^T$ are the column vectors. Let us construct the vector $P = (\delta p/\delta u, \delta p/\delta u_x)^T$ for any conserved density $P$ from (3) (see notations in (8) and below). By the order of the local conservation law we shall mean the maximum of numbers $n$ such that $P_{w_n} \neq 0$. The condition $P_{w_n} \neq 0$ means that one of the functions $p_{uu_n}, p_{uu_{nx}}, p_{u_xu_n}, p_{u_xu_{nx}}$ does not vanish.

The vector $P$ satisfies the equation

$$(P)_x + f_*^T(P) = 0. \tag{23}$$

The equation (23) is equivalent to the equalities $\delta(p)_x/\delta u = \delta(p)_x/\delta u_x = 0$, which follow from (3). The Frechet derivative $S = \sum P_{w_k} D^k$ of $P$ is an approximate solution of Eq.(15). To verify this, one should calculate the Frechet derivative of the left hand side of (23). A relationship of the form $A(S) = \sum_{k=-2}^{2} H_k D^k$ will be obtained.

**Proposition 3.** If a lattice of the form (2) possessis a local conservation law of the order $n \geq 2$, then there exists an approximate solution of Eq.(15) with the order $n$ and the length $n-1$.

Let $S,T$ be formal series of the form (12) and $S$ be invertible. Then the formula $B(S^{-1}T) = S^{-1}A(T) - S^{-1}A(S)S^{-1}T$ takes place. Therefore one can construct solutions of Eq.(14), using solutions of Eq.(15). It is important that a solution of Eq.(15) is invertible, even though it is degenerate. For any degenerate solution $S$ of the length $N \geq 2$ and of the order $n$ we have $b_n = c_n = d_n = 0$, $a_n d_{n-1} \neq 0$. It is not very hard to verify that in this case the series $S^{-1}$ exists and is unique. Its order is $n-1$ and



$$S^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & l \end{bmatrix} D^{1-n} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} D^{-n} + \ldots, \quad la \neq 0.$$

From what has been said above one can get, in particular, that if a solution $T$ of Eq.(15) with the length $M \geq 2$ and the order $m$ also is degenerate, then the solution $S^{-1}T$ of Eq.(14) is non-degenerate and has the order $m-n$.

**Proposition 4.** If a lattice of the form (2) possessis two local conservation laws of the orders $n_1, n_2$ such that $n_1 \geq 4$, $n_2 > 2n_1 - 3$, then an approximate solution of Eq.(14) exists and has the length $n_1 - 3$ and the order $m > n_1 - 3$.

It follows from the Propositions 1-4 that the Theorem 1 takes place.

**Theorem 1.** The conditions (4),(5) are satisfied for any lattice (2) possessing two local conservation laws with the orders $n_1 \geq 8$ and $n_2 > 2n_1 - 3$.

If a lattice has conservation laws of the arbitrarily high order, the hypotheses of the Theorem 1 remain valid. Note that the conditions (4),(5) can be obtained in a more simple way. The computation of the coefficients of $L, S$ from (14),(15) is essentially simplified if $L$ has the order 1 and $S$ is the series of the order 0. To provide Eqs.(14),(15) with solutions of this kind, we need additional assumptions about the orders of conservation laws.

## 3. Complete list of lattices

Lattices of the form (2) are classified up to the point transformations:

$$\tilde{u}_n = \sigma(u_n), \qquad \tilde{x} = cx, \tag{24}$$

where $c$ is a constant.

**Theorem 2.** A lattice (2) satisfies the conditions (4),(5) if and only if it is reduced by a transformation (24) to one of the following discrete equations: (6),(7),

$$u_{xx} = [u_x^2 + Q(u)][(u_1 + u)^{-1} + (u + u_{-1})^{-1}] - 1/2\, Q'(u), \tag{25}$$

$$u_{xx} = \exp(u_1 - 2u + u_{-1}) + c, \tag{26}$$

$$u_{xx} = u_1 + u_{-1} + \delta u + \epsilon, \tag{27}$$

where $Q(u) = \alpha u^4 + \beta u^2 + \gamma$, and $\alpha, \beta, \gamma, \delta, \epsilon, c$ are constants.

Using another transformations one can decrease the number of equations. The transformation $\tilde{u}_n = u_{n+1} - u_n$ reduces Eq.(26) to Eq.(6) (more precisely to the Toda lattice (1)). The change of variables $\tilde{u}_n = (-1)^n u_n$ takes Eq.(25) into Eq.(7).

There is not any possibility to go into detail of the proof of the Theorem 2. We only can essentially specify the form of the function $f$ from (11). Besides, it will be shown how to use the necessary conditions (4),(5). It is helpfull to introduce the following equivalence relation: $F \sim G$ if $F - G = (D-1)(H)$ ($F, G, H$ are functions of the finite number of variables $u_k, u_{kx}$). Now the condition (4) with $k = 1$, for example, is expressible as follows: $(\ln f_{u_1})_x \sim 0$. It is necessary to remark that the variables $u_k, u_{kx}$ are assumed to be independent.

**Proposition 5.** (a) A lattice (11) satisfies the condition (5) with $k = 1$ if and only if $f = F(a, b, c)$, where $a = u_x$, $b = u$, $c = \delta z(u, u_{-1})/\delta u = z_u + D(z_{u_{-1}})$.

(b) The condition (5) with $k = 2$ is satisfied if and only if $F = 1/2\ Aa^2 + Ba + C$, the functions $A, B, C$ depend on $b, c$, and

$$A(u, \delta z/\delta u) + \delta(\ln z_{uu_{-1}})/\delta u = 0, \tag{28}$$

$$B(u, \delta z/\delta u) \sim 0. \tag{29}$$

**Proof.** The condition (5) with $k = 1$ is the equality $r^1 = D(s^1) - s^1$. The function $s^1$ depends only on $u, u_{-1}$, for $r^1$ is a function of $u_x, u, u_{\pm 1}$. Let us express this equality in the form of $f_{u_1} \exp s^1 = f_{u_{-1}} D(\exp s^1)$. It is easy to see that we have a first order linear partial differential equation for $f$. Its general solution has been given above, where $z_{uu_{-1}} = \exp s^1$.

Let us consider the next condition (5) with $k = 2$: $(\ln z_{uu_{-1}})_x + f_{u_x} \sim 0$. One may change such a relationship in the followig way: if $F + G \sim 0$, then $D^k(F) + G \sim 0$ for any integer $k$. Therefore we have $u_x \delta(\ln z_{uu_{-1}})/\delta u + f_{u_x} \sim 0$. Here the function being equivalent to 0 depends on $u_x, u, u_{\pm 1}$ and consequently has the form: $(D-1)\phi(u, u_{-1})$. So $\partial^3 f/\partial u_x^3 = 0$, and we have

$$[A(u, \delta z/\delta u) + \delta(\ln z_{uu_{-1}})/\delta u] u_x + B(u, \delta z/\delta u) \sim 0$$



instead of (5) with $k = 2$. The conditions (28),(29) have been obtained, and the Proposition 5 has been proved.

Now we shall restrict ourselves to the scheme of the proof. Making use of (4) with $k = 1$ it is possible to prove that $\partial^2 F/\partial c^2 = 0$ or

$$F = 1/2\ A(b)a^2 + D(b) + \exp[p(b)c + q(b)]. \tag{30}$$

If $F$ is of the form (30), the lattice is reduced by (24) to the form $u_{xx} = \exp(u_1 - 2u + u_{-1}) + r(u)$. This follows from (4) with $k = 2$. It is easy to verify with the help of (4) with $k = 3$ that $r' = 0$.

Let us consider the case $F_{cc} = 0$. If $F_a = 0$ (i.e. the right side of (11) does not depend on $u_x$), one can reduce the lattice to Eq.(6) or to $u_{xx} = u_1 + u_{-1} + s(u)$. To show that $s'' = 0$, we use the condition (4) with $k = 3$. In all the other cases this condition is not used. Now $F_a \neq 0$. In the case $B \neq 0$ the lattice can be reduced to Eq.(6). The case $B = A_c = 0$ is trivial. The last case contains the most complicated equations. Here $B = 0$, $A_c \neq 0$ and we get Eqs.(6),(7),(25).

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