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REVIEW ARTICLE

Symmetries as integrability criteria for differential difference equations

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Abstract
In this paper we review the results obtained by the generalized symmetry method in the case of differential difference equations during the last 20 years. Together with general theory of the method, classification results are discussed for classes of equations which include the Volterra, Toda and relativistic Toda lattice equations.

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Contents

1. Introduction

The generalized symmetry approach to the classification of integrable equations has mainly been developed by a group of researchers belonging to the scientific school of A B Shabat in Ufa, Russia (see, e.g., review articles [10, 25, 38, 39, 41, 55, 56]). Its discrete version, considered in this review, is discussed in the papers [7, 32, 33, 53, 70, 71, 74, 79] and partly in the surveys [10, 25, 56].

The purpose of the review is to provide an overview of the progress made in this field during the last 20 years, and mainly to discuss the discrete version of the generalized symmetry method and the corresponding classification results. We will consider discrete-differential equations which belong to the three most important classes of equations: Volterra, Toda and relativistic Toda-type equations.

The generalized symmetry approach is the only method which enables one not only to test equations for integrability but also to classify integrable equations in classes characterized by arbitrary functions of many variables. Using this method, the classification problem has been solved for classes which include such well-known and important equations as the Burgers, Korteweg–de Vries and nonlinear Schrödinger equations or, in the discrete-differential case, the Volterra, Toda and relativistic Toda lattice equations. Together with exhaustive lists of integrable equations, a number of essentially new examples have been obtained as a result of this classification.

As is known, equations integrable by the inverse scattering method have infinitely many generalized symmetries and conservation laws. The generalized symmetry approach enables one to recognize equations possessing this property. The existence of infinite hierarchies of generalized symmetries and/or conservation laws is used by this method as integrability criteria.

Let us first briefly review the situation for partial differential equations, as presented in the surveys [10, 38, 39, 41, 55, 56]. We discuss equations of the form

$$u_t = f(u, u_1, u_2, u_3), \tag{1.1}$$

where $u_t = \frac{\partial u}{\partial t}$ and $u_j = \frac{\partial^j u}{\partial x^j}$ for any $j > 0$. The Korteweg–de Vries equation

$$u_t = u_3 + 6uu_1 \tag{1.2}$$

belongs to this class. All functions (right-hand sides of generalized symmetries, conserved densities, coefficients of formal series) will have the form

$$\phi = \phi(u, u_1, u_2, \ldots, u_k) \tag{1.3}$$

(here $k \geqslant 0$, $u_0 = u$), and the number k is called the order of the function.

A generalized symmetry of equation (1.1) is an equation of the form

$$u_\tau = g(u, u_1, u_2, \ldots, u_m) \tag{1.4}$$

compatible with (1.1). The compatibility condition between equations (1.1) and (1.4) implies for the functions f and g:

$$\frac{\partial^2 u}{\partial t \partial \tau} - \frac{\partial^2 u}{\partial \tau \partial t} = D_t g - D_\tau f = 0, \tag{1.5}$$

where D_t, D_τ are the operators of total differentiation corresponding to equations (1.1), (1.4), defined, together with the operator of total x-derivative D, by

$$D = \frac{\partial}{\partial x} + \sum_{i\geqslant 0} u_{i+1}\frac{\partial}{\partial u_i}, \qquad D_t = \frac{\partial}{\partial t} + \sum_{i\geqslant 0} D^i f \frac{\partial}{\partial u_i}, \qquad D_\tau = \frac{\partial}{\partial \tau} + \sum_{i\geqslant 0} D^i g \frac{\partial}{\partial u_i}.$$

The generalized symmetry of equation (1.1) of the order m is defined as an equation of the form (1.4) with the right-hand side g satisfying the last of equations (1.5).

A conservation law of equation (1.1) is given by the equation

$$D_t p = Dq \tag{1.6}$$

for some local functions p and q of the form (1.3). The function p is called a conserved density, and it is such that $D_t p \in \operatorname{Im} D$, i.e is represented as the total x-derivative of a function q.

A conserved density of the form $p = c + D\tilde{p}$, where c is a constant, is trivial, as in such a case one can always find a function q: $q = \tilde{c} + D_t \tilde{p}$, where $\tilde{c}$ is another constant. One can easily check if p is trivial by using a formal variational derivative operator. A formal variational derivative of a function (1.3) of order k is defined as

$$\frac{\delta\phi}{\delta u} = \sum_{i=0}^{k} (-D)^i \frac{\partial\phi}{\partial u_i}. \tag{1.7}$$

This notion is closely related to the notion of the standard variational derivative (cf e.g. [49]). The following result is very useful:

$$\frac{\delta\phi}{\delta u} = 0 \tag{1.8}$$

iff ϕ can be written as

$$\phi = c + D\psi, \tag{1.9}$$

where c is a constant and ψ is a function of the form (1.3). As a consequence of this result, a conserved density p is trivial if $\frac{\delta p}{\delta u} = 0$. The order of a nontrivial conserved density is defined as the order of the function $\frac{\delta p}{\delta u}$.

Generalized symmetries and conservation laws lead to the so-called integrability conditions. Before introducing them we need to define formal series in powers of D. A formal series of the order k has the form

$$A_k = a_k D^k + a_{k-1} D^{k-1} + \cdots + a_0 + a_{-1} D^{-1} + \cdots, \tag{1.10}$$

where a_j are functions of the form (1.3). The product of two formal series is uniquely defined by

$$\begin{aligned} A_k \circ B_m &= \sum_{i \leqslant k, j \leqslant m} a_i D^i \circ b_j D^j, \\ a_i D^i \circ b_j D^j &= a_i b_j D^{i+j} + \sum_{n \geqslant 1} \binom{i}{n} a_i D^n(b_j) D^{i+j-n}, \end{aligned} \tag{1.11}$$

where

$$\binom{i}{n} = \frac{i(i-1)(i-2)\cdots(i-n+1)}{n!}$$

is the standard binomial coefficient, and by $\circ$ we mean the multiplication of operators. For any function ϕ (1.3), we define the Frechét derivative ϕ_* and its corresponding adjoint operator $\phi_*^\dagger$:

$$\phi_* = \sum_{i=0}^{k} \frac{\partial\phi}{\partial u_i} D^i, \qquad \phi_*^\dagger = \sum_{i=0}^{k} (-D^i) \circ \frac{\partial\phi}{\partial u_i}, \tag{1.12}$$

which are the particular cases of the formal series (1.10).

One can rewrite the compatibility condition (1.5) as

$$(D_t - f_*)g = 0 \tag{1.13}$$

and the relation $\frac{\delta}{\delta u} D_t p = 0$, which follows from equation (1.6), as

$$(D_t + f_*^\dagger)\varrho = 0, \qquad \varrho = \frac{\delta p}{\delta u}. \tag{1.14}$$

One can obtain from equation (1.4) the formal series of the order m,

$$L = g_* + 0D^{-1} + 0D^{-2} + \cdots, \tag{1.15}$$

which will be an approximate solution of the length m of the equation

$$L_{,t} = [f_*, L] = f_* L - L f_*, \tag{1.16}$$

where $L_{,t}$ is obtained from L by differentiating its coefficients with respect to t. The series $L_{,t} - [f_*, L]$ has the form

$$L_{,t} - [f_*, L] = b_{m+3}D^{m+3} + b_{m+2}D^{m+2} + b_{m+1}D^{m+1} + \cdots, \tag{1.17}$$

and L is called an approximate solution of equation (1.16) of the length l if first l coefficients of (1.17) vanish, i.e.

$$L_{,t} - [f_*, L] = b_{m+3-l}D^{m+3-l} + b_{m+2-l}D^{m+2-l} + \cdots.$$

The fact that $l = m$ is proved by applying the Frechét derivative to both sides of equation (1.13). One obtains

$$g_{*,t} - [f_*, g_*] = f_{*,\tau}.$$

In a similar way, if a conservation law has order $m > 3$, we can apply the Frechét derivative to equation (1.14) and prove the following result: the formal series

$$S = \varrho_* + 0D^{-1} + 0D^{-2} + \cdots \tag{1.18}$$

of the order m is an approximate solution of the length $m - 3$ of the equation

$$S_{,t} + Sf_* + f_*^\dagger S = 0. \tag{1.19}$$

Approximate solutions of the length l are defined in this case in a quite similar way. Such approximate solutions of equations (1.16) and (1.19) are called the formal symmetry and the formal conserved density, respectively.

We can not only multiply formal series (1.10) but also obtain its inverse A_k^{-1} and its k-order root $A_k^{1/k}$, using the standard definitions: $A_k^{-1}A_k = A_k A_k^{-1} = 1$, $\left(A_k^{1/k}\right)^k = A_k$. So, the fractional powers $A_k^{i/k}$, where i is an arbitrary integer, are well defined. Let us note that if one starts from the series (1.15) or (1.18), one can obtain infinitely many nonzero coefficients in a resulting series.

Two formal symmetries of orders m and $\hat{m}$, L and $\hat{L}$, provide new formal symmetries: $L\hat{L}$ and $L^{i/m}$. Two formal conserved densities S and $\hat{S}$ generate a formal symmetry: $L = S^{-1}\hat{S}$. The following formula is valid for constructing a new formal conserved density: $SL = \hat{S}$. Using these properties, we can simplify the problem and consider just formal symmetries and conserved densities of the order 1 and of an arbitrarily big length k.

If equation (1.1) has generalized symmetries and conservation laws of arbitrarily high orders, we can calculate arbitrarily many coefficients of the formal symmetries and conserved densities of the first order, using equations (1.16) and (1.19). In doing so, integrability conditions will appear which will have the form $H \in \operatorname{Im} D$, where the function H does not depend on the form and orders of symmetries and conservation laws, but it is expressed only in terms of the right-hand side f of equation (1.1). As an example, let us write down the integrability conditions in the following particular case:

$$u_t = u_3 + F(u, u_1). \tag{1.20}$$

The integrability conditions, which come from the existence of generalized symmetries, derived from equation (1.16), are of the form

$$D_t p_i = Dq_i, \qquad i \geqslant 1, \tag{1.21}$$

i.e. have the form of conservation law. The first three conserved densities read

$$p_1 = \frac{\partial F}{\partial u_1}, \qquad p_2 = \frac{\partial F}{\partial u}, \qquad p_3 = q_1. \tag{1.22}$$

The conditions which come from the existence of conservation laws, obtained from equation (1.19), are of the form

$$p_{2j} = D\sigma_{2j}, \qquad j \geqslant 1. \tag{1.23}$$

Conditions (1.23) mean that the even conserved densities are trivial. The functions q_{2j} are easily expressed in terms of the functions σ_{2j}: $q_{2j} = c_{2j} + D_t\sigma_{2j}$, where c_{2j} are some constants.

The integrability conditions (1.21)–(1.23) can be formulated in the alternative way:

$$D_t\frac{\partial F}{\partial u_1}, \frac{\partial F}{\partial u}, D_t q_1 \in \operatorname{Im} D,$$

i.e. some functions q_1, σ_2, q_3 must exist which satisfy relations (1.21) with $i = 1, 3$ and relation (1.23) with $j = 1$. The other conserved densities p_i are similar to p_3 and have a dependence on the functions q_i defined by the previous conditions.

One has to check the integrability conditions step by step. At first we check (1.21) with $i = 1$ and find the function q_1. Then we pass to (1.21) with $i = 3$. To check condition (1.21) with $i = 1$, we use the equivalence between equations (1.8) and (1.9). We check equation (1.8) for $\phi = D_t p_1$ and then, if it is satisfied, represent ϕ in the form (1.9) and verify whether $c = 0$.

Integrability conditions allow one to check whether a given equation is integrable. Moreover, in many cases these conditions enable us to classify equations, i.e. to obtain complete lists of integrable equations. As integrability conditions are only necessary conditions for the existence of generalized symmetries and/or conservation laws, we then have to prove that equations of the resulting list really possess generalized symmetries and conservation laws. We mainly construct them using Miura-type transformations and master symmetries.

The use of Miura-type transformations in the classification problems is discussed in [39–42, 54, 55, 61, 62, 73, 75, 76]. The original Miura transformation [44]

$$u = v_1 - v^2 \tag{1.24}$$

brings any solution v of the modified Korteweg–de Vries equation

$$v_t = v_3 - 6v^2 v_1 \tag{1.25}$$

into a solution of the Korteweg–de Vries equation (1.2). In the case of equation (1.1), a Miura-type transformation has the form

$$u = s(v, v_1, \ldots v_k), \qquad k > 0, \tag{1.26}$$

and transforms an equation of the form

$$v_t = \hat{f}(v, v_1, v_2, v_3) \tag{1.27}$$

into equation (1.1). If a conservation law (1.6) of equation (1.1) defined by the functions

$$p = p\big(u, u_1, \ldots, u_{k_1}\big), \qquad q = q\big(u, u_1, \ldots, u_{k_2}\big)$$

is known, one obtains a conservation law $D_t\hat{p} = D\hat{q}$ for equation (1.27), defining

$$\hat{p} = p(s, Ds, \ldots, D^{k_1}s), \qquad \hat{q} = q(s, Ds, \ldots, D^{k_2}s).$$

The notion of master symmetry has been introduced in [21] and later discussed in [20, 22, 23, 46]. A master symmetry of the Korteweg–de Vries equation (1.2) has the form

$$u_\tau = xu_t + 4(u_2 + 2u^2) + 2u_1 D^{-1}u, \tag{1.28}$$

where D^{-1} is the inverse of the operator D, or an x-integral, as shown in [20, 45]. If $u_\tau = h$ is a master symmetry of equation (1.1) and p is its conserved density, then new conserved densities p_i are constructed by total τ-differentiation: $p_i = D_\tau^i p, i \geqslant 1$. Generalized symmetries $u_{t_i} = g_i$ of equation (1.1) are constructed as follows (cf equation (1.5)):

$$g_1 = D_t h - D_\tau f, \qquad g_2 = D_{t_1} h - D_\tau g_1, \ldots.$$

Let us now return to the problem at hand, i.e. symmetries as integrability criteria for differential difference equations. In section 2 the general theory of formal symmetries for difference-differential equations will be given in the simple case of scalar equations depending just on nearest neighbouring lattice points. In doing so, we introduce such necessary notions as generalized symmetry, conservation law, formal symmetry, formal conserved density, Miura transformation and master symmetry for differential difference equations. We then discuss the so-called integrability conditions which do not depend on the form and order of generalized symmetries and conservation laws and are expressed only in terms of the equation in the study. It will be explained how to derive the integrability conditions and how to use them for testing and classifying the equations.

In section 3 we present the classification results in the case of difference-differential equations of Volterra, Toda and relativistic Toda type. We give the integrability conditions together with the complete lists of integrable equations. Those lists of equations are obtained, using the integrability conditions, even though those conditions are only necessary conditions for the integrability. This is the reason why we also show how to construct for these equations hierarchies of generalized symmetries and conservation laws. The equations are given in lists complete up to point transformations. We also discuss non-point transformations of the Miura type, or non-point invertible transformations which relate different equations of the same list or equations belonging to two different lists. Some of the results presented here have never been published before. In section 3.1.2 we present for the first time in a complete form the Miura transformations for Volterra-type equations. Master symmetries for relativistic Toda-type equations are presented in section 3.3.5 in a form simpler than in the present literature (cf [10]).

Section 4 is devoted to some concluding remarks.

2. The generalized symmetry method for differential difference equations

This section is devoted to the general theory of the generalized symmetry method in the discrete case. It will be discussed in the simple case of Volterra-type equations defined by one arbitrary function of three variables. The main notions are illustrated in this case. At the end we extend the discussion to systems of lattice equations. Further details of the theory can be found in the papers [10, 32, 33, 56, 74, 76, 79].

In section 2.1 we discuss generalized symmetries and conservation laws and in section 2.2 derive an integrability condition, using the existence of one generalized symmetry. The notion of formal symmetry is introduced in section 2.3, and two more integrability conditions are derived using this notion. In section 2.4 we introduce formal conserved density and then obtain two additional integrability conditions. Properties of all five integrability conditions are discussed in section 2.5. The use of the Hamiltonian structure, Miura transformations and master symmetries for the construction of generalized symmetries and conservation laws is discussed in sections 2.6 and 2.7. The case of systems of equations is considered in section 2.8.

2.1. Generalized symmetries and conservation laws

In this section we will consider mainly the following class of lattice equations:

$$\dot{u}_n = f(u_{n+1}, u_n, u_{n-1}) \equiv f_n, \qquad \frac{\partial f_n}{\partial u_{n+1}} \neq 0, \qquad \frac{\partial f_n}{\partial u_{n-1}} \neq 0, \tag{2.1}$$

where $u_n = u_n(t)$, overdot denotes the derivative with respect to the continuous time variable t and the index n is an arbitrary integer. We can think of $u_n(t)$ as an infinite set of functions of one continuous variable: $\{u_n(t) : n \in \mathbb{Z}\}$ and equation (2.1) as an infinite system of ordinary differential equations defined by one arbitrary function of three variables: $f(z_1, z_2, z_3)$. The well-known Volterra equation [68]

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1}) \tag{2.2}$$

belongs to this class.

The time t of equation (2.1) may be complex and one may, when necessary, use the transformation $\tilde{t} = it$; the functions $u_n(t)$ and $f(z_1, z_2, z_3)$ are complex-valued functions of complex variables. By considering complex functions we avoid looking at many particular cases and simplify the calculation when we derive the integrability conditions or solve the classification problem. If, as a result of classification, we obtain an integrable equation, it will be also integrable in the real case, i.e. we can pass to the real variables t, u_n and real f. For instance, the Volterra equation (2.2) possesses the same infinite hierarchies of generalized symmetries and conservation laws in both cases: when t and u_n are complex or real.

Locally analytic functions. Let us consider complex-valued functions of many complex variables which are analytic on an open and connected subset of $\mathbb{C}^N$. We consider only single-valued functions and, in the case of multi-valued functions (like $\sqrt{z}$ and $\log z$), we choose a single-valued branch. We will call such functions locally analytic functions.

By reducing, if necessary, the domain of definition of the function one can apply any arithmetical operation to a locally analytic function, compose them and compute their inverses $\varphi^{-1}(z)$, or find implicitly defined functions:

$$w = \varphi(z_1, z_2) \quad \Rightarrow \quad z_1 = \psi(w, z_2).$$

Any problem under consideration (such as the classification, derivation of the integrability conditions or testing an equation for integrability) deals with a finite number of such functions and is solved in a finite number of steps.

By the classification problem we mean looking for an unknown functions of many variables, such as the function $f(z_1, z_2, z_3)$ which appears on the right-hand side of equation (2.1). From the defining equations for the existence of generalized symmetries, we get some differential-functional relations for unknown functions which must be satisfied identically. Then we derive new relations for unknown functions which also must be satisfied identically and use for that the following two properties.

Property 1. For any locally analytic function φ by reducing, if necessary, the domain of definition we have only two possible cases: either $\varphi \neq 0$ everywhere in the domain or $\varphi \equiv 0$.

Property 2. There are no divisors of zero, i.e.

$$\varphi_1\varphi_2 \equiv 0 \quad \Rightarrow \quad \varphi_1 \equiv 0 \quad \text{or} \quad \varphi_2 \equiv 0.$$

We can differentiate functions as many times as necessary and solve differential equations. As a result of the classification, we will obtain on the right-hand side of an equation such functions as $\sqrt{z}$, $\log z$, hyperbolic and elliptic functions, but not $\bar{z}$ and $|z|$. The obtained integrable equations will be expressed in terms of analytic functions defined in a domain which may be very small. But those equations remain integrable if one passes to globally defined analytic complex functions or to real functions and real time.

The equations integrable by the inverse scattering method, which, following Calogero [14], are usually called S-integrable, are known to have infinitely many generalized symmetries. Also the equations which can be transformed into a linear equation, i.e. linearizable or C-integrable equations, possess this property. Generalized symmetries of equations (2.1) will be equations of the form

$$u_{n,\tau} = g(u_{n+m}, u_{n+m-1}, \ldots, u_{n+m'+1}, u_{n+m'}) \equiv g_n, \qquad \frac{\partial g_n}{\partial u_{n+m}} \frac{\partial g_n}{\partial u_{n+m'}} \neq 0, \tag{2.3}$$

where $u_n = u_n(t, \tau)$. By the index τ we denote its τ-derivative and $m \geqslant m'$ are two finite fixed integers. The precise definition of generalized symmetry will be given below, in definition 1.

The symmetry is defined by one locally analytic function of many variables,

$$g(z_1, z_2, z_3, \ldots, z_{1+m-m'}), \tag{2.4}$$

and depends on equation (2.1). We will call equation (2.3) a local generalized symmetry of equation (2.1) as on the right-hand side it does not contain integrals or summations. Moreover, we choose this symmetry to have no explicit dependence on the discrete spatial variable n and on the time of equation (2.1) t. It is known that S-integrable local and n- and t-independent equations like equation (2.1) may possess n- and t-dependent generalized symmetries. However, these symmetries are in most cases non-local. On the other hand, S-integrable equations in (1+1)-dimensional case have infinitely many n- and t-independent local symmetries. This property is also true for many C-integrable equations. So, the existence of an infinite hierarchy of local, n- and t-independent generalized symmetries of the form (2.3) is a natural requirement for equations (2.1).

Lie point symmetries of equation (2.1) are of the form

$$u_{n,\tau} = a(t)\dot{u}_n + b_n(t, u_n) \tag{2.5}$$

and are a subcase of the generalized symmetries. We will be interested in symmetries (2.3) with $m > 1$ and $m' < -1$ which are not Lie point symmetries, more precisely, with $m = -m' > 1$. For an explanation of the last requirement, see section 2.4.1.

A generalized symmetry of equation (2.1) is an equation of the form (2.3) compatible with equation (2.1), i.e. such that they have a common set of solutions. Before giving a precise definition of generalized symmetries, we derive and discuss the conditions necessary for their existence. If $u_n(t, \tau)$ is a common solution of equations (2.1), (2.3), we have

$$\frac{\partial^2 u_n}{\partial t \partial \tau} - \frac{\partial^2 u_n}{\partial \tau \partial t} = D_t g_n - D_\tau f_n = 0, \tag{2.6}$$

where D_t, D_τ are differentiation operators corresponding to equations (2.1), (2.3):

$$D_t g_n = \sum_{j=m'}^{m} \frac{\partial g_n}{\partial u_{n+j}} f_{n+j}, \qquad D_\tau f_n = \sum_{j=-1}^{1} \frac{\partial f_n}{\partial u_{n+j}} g_{n+j}. \tag{2.7}$$

By f_{n+j}, g_{n+j} we mean

$$f_{n+j} = f(u_{n+j+1}, u_{n+j}, u_{n+j-1}), \qquad g_{n+j} = g(u_{n+j+m}, u_{n+j+m-1}, \ldots, u_{n+j+m'}).$$

Consequently from equations (2.6), (2.7) we obtain the following compatibility condition:

$$D_t g_n = \frac{\partial f_n}{\partial u_{n+1}} g_{n+1} + \frac{\partial f_n}{\partial u_n} g_n + \frac{\partial f_n}{\partial u_{n-1}} g_{n-1}, \tag{2.8}$$

which is satisfied for any common solution of equations (2.1), (2.3) and, given f_n, is an equation for the function g_n.

In the generalized symmetry method, we assume that equation (2.8) must be identically satisfied for all values of the variables

$$u_0, u_1, u_{-1}, u_2, u_{-2}, \ldots, \tag{2.9}$$

which are considered as independent, and for all $n \in \mathbb{Z}$.

To clarify this point, let us consider a simple but important difference equation

$$\phi_{n+1} - \phi_n = 0. \tag{2.10}$$

We are looking for solutions of equation (2.10), such that ϕ_n is a function defined on a finite interval of the lattice

$$\phi_n = \phi(u_{n+k}, u_{n+k-1}, \ldots, u_{n+k'}), \qquad k \geqslant k', \tag{2.11}$$

where

$$\frac{\partial \phi_n}{\partial u_{n+k}} \neq 0, \qquad \frac{\partial \phi_n}{\partial u_{n+k'}} \neq 0 \tag{2.12}$$

if ϕ_n is a nonconstant function. From equation (2.8) we often will get relations of the form (2.10) satisfied identically for all values of the variables (2.9).

Let us analyse the consequences of equation (2.10). Assuming that there exists a nonconstant solution ϕ_n of equation (2.10) and differentiating equation (2.10) with respect to $u_{n+k'}$, one can see that $\partial\phi_n/\partial u_{n+k'} = 0$ identically. Consequently, we have a contradiction with (2.12), and relation (2.10) implies that ϕ_n must be a constant function. Introducing the standard shift operator T, such that for any integer power j we have

$$T^j \phi_n = \phi_{n+j} = \phi(u_{n+j+k}, u_{n+j+k-1}, \ldots, u_{n+j+k'}), \tag{2.13}$$

we can rewrite equation (2.10) as $(T-1)\phi_n = 0$ and thus we get

$$\ker(T-1) = \mathbb{C}. \tag{2.14}$$

Definition 1. *Equation (2.3) is called a generalized symmetry of equation (2.1) if the compatibility condition (2.8) is identically satisfied for all values of the independent variables (2.9). The numbers m and m' are called, respectively, the left order (or the order) and the right order of the generalized symmetry (2.3).*

For any generalized symmetry (2.3), the integers m and m' are fixed and define essentially different cases. In order to derive integrability conditions, we will only use the left order m and, for this reason, sometimes we will call it, more shortly, the order of the generalized symmetry.

Definition 1 is constructive. For any given equation (2.1) and any given orders m and m', with $m \geqslant m'$, one is able either to find a generalized symmetry (2.3) or to prove that it does not exist. In section 3.1.1, we will show how we can construct generalized symmetries of the Volterra equation (2.2) with $m = 2$ and $m' = -2$. The resulting generalized symmetry is

$$u_{n,\tau} = u_n(u_{n+1}(u_{n+2} + u_{n+1} + u_n) - u_{n-1}(u_n + u_{n-1} + u_{n-2})). \tag{2.15}$$

It is a general property of equations integrable by the inverse scattering method in (1+1)-dimensional case that evolution local equations like (2.1), which have no explicit dependence on n and t, possess infinitely many n- and t-independent local conservation laws. We will assume that this is true for equations (2.1) and that their conservation laws have no explicit n and t dependence.

Definition 2. *A relation of the form*

$$D_t p_n = (T - 1)q_n, \tag{2.16}$$

where p_n and q_n are functions of the form (2.11), and D_t is a differentiation operator corresponding to equation (2.1) (see (2.7)), is called a local conservation law or, more shortly, a **conservation law** *of equation (2.1). Relation (2.16) must be satisfied identically for all the values of the independent variables (2.9). The function p_n is called a* **conserved density** *of equation (2.1).*

It can be easily proved, as we did in the case of equation (2.14), that if the conserved density p_n has the form

$$p_n = p\big(u_{n+m_1}, u_{n+m_1-1}, \ldots u_{n+m_2}\big), \qquad m_1 \geqslant m_2, \qquad \frac{\partial p_n}{\partial u_{n+m_1}}\frac{\partial p_n}{\partial u_{n+m_2}} \neq 0, \tag{2.17}$$

then q_n must be a function of the form

$$q_n = q\big(u_{n+m_1}, u_{n+m_1-1}, \ldots u_{n+m_2-1}\big), \qquad \frac{\partial q_n}{\partial u_{n+m_1}}\frac{\partial q_n}{\partial u_{n+m_2-1}} \neq 0. \tag{2.18}$$

It is obvious that if p_n cannot be expressed in the form (2.17), then it is a constant function, as well as q_n, and this is a trivial case. The two simplest conservation laws of the Volterra equation (2.2), with $m_1 = m_2 = 0$, are

$$D_t u_n = (T - 1)(u_n u_{n-1}), \qquad D_t \log u_n = (T - 1)(u_n + u_{n-1}). \tag{2.19}$$

Local conservation laws, as well as generalized symmetries, can be used to solve equation (2.1). A conservation law (2.16) can be used to construct constants of motion (or conserved quantities, or first integrals). Let us consider, for example, the periodic closure of equation (2.1) of period $N \geqslant 1$, i.e. such that $u_n = u_{n+N}$ for any n. Then we can write equation (2.1) as a system of N ordinary differential equations for N functions $u_1(t), u_2(t), \ldots, u_N(t)$. A constant of motion of this system is a function $I = I(u_1, u_2, \ldots, u_N)$ such that $\frac{dI}{dt} = 0$. Any conservation law (2.16) of equation (2.1) generates a constant of motion $I = \sum_{n=1}^{N} p_n$ for this finite system. In fact,

$$\frac{dI}{dt} = \sum_{n=1}^{N} D_t p_n = \sum_{n=1}^{N}(q_{n+1} - q_n) = q_{N+1} - q_1 = 0.$$

In the case of the periodically closed Volterra equation (2.2), we can obtain from (2.19) two constants of motion I_1, I_2. We have

$$I_1 = \sum_{n=1}^{N} u_n, \qquad \tilde{I}_2 = e^{I_2} = \prod_{n=1}^{N} u_n,$$

and $I_1, \tilde{I}_2$ are arbitrary t-independent constants.

Any generalized symmetry (2.3) is a nonlinear difference-differential equation which has common solutions $u_n(t,\tau)$ with equation (2.1). As in the case of Lie point symmetries, for generalized symmetries we can perform a symmetry reduction [31] by considering solutions such that $\frac{\partial u_n}{\partial \tau} = 0$, i.e. stationary solutions of equation (2.3) which satisfy the following discrete equation:

$$g(u_{n+m}, u_{n+m-1}, \ldots u_{n+m'}) = 0.$$

This is the analogue of the reduced ordinary differential equation we obtain in the case of partial differential equations in two variables. If we solve this equation, we obtain a function $u_n(t)$ which depends on arbitrary functions of t. These arbitrary functions can be obtained by introducing $u_n(t)$ into equation (2.1). In such a way we can, by symmetry reduction, construct particular solutions of (2.1) such as, for example, its soliton solutions.

Conservation laws, conserved densities and generalized symmetries generate linear spaces. In fact, the operators D_t and $T-1$ are linear. For any pair of conservation laws of equation (2.1), with p_n, q_n and $\hat{p}_n, \hat{q}_n$ such that (2.16) and $D_t\hat{p}_n = (T-1)\hat{q}_n$ are satisfied, we have the following conservation law:

$$D_t(\alpha p_n + \beta \hat{p}_n) = (T-1)(\alpha q_n + \beta \hat{q}_n),$$

where α, β are arbitrary constants. In the case of two generalized symmetries $u_{n,\tau} = g_n$ and $u_{n,\hat{\tau}} = \hat{g}_n$ of equation (2.1), the functions g_n and $\hat{g}_n$ satisfy the linear equation (2.8), as well as any of their linear combinations $\alpha g_n + \beta \hat{g}_n$. Hence the equation

$$u_{n,\tau'} = \alpha g_n + \beta \hat{g}_n$$

will be a generalized symmetry of (2.1). The function f_n, given in (2.1), satisfies the compatibility condition (2.8) for any equation (2.1), i.e. the equation $u_{n,\tau'} = f_n$ is a trivial generalized symmetry of (2.1) and can be used in linear combinations with any other generalized symmetries to simplify them.

Conserved densities possess an important additional property: total differences can be added to them. Given any conservation law (2.16) and any function (2.11), we can construct for equation (2.1) the following conservation law:

$$D_t(p_n + (T-1)\phi_n) = (T-1)(q_n + D_t\phi_n). \tag{2.20}$$

The conservation laws (2.16), (2.20) do not essentially differ one from the other and will be considered as equivalent.

Definition 3. *Two functions a_n and b_n of the form (2.11) are said to be* equivalent, *and we will write $a_n \sim b_n$, if the difference $a_n - b_n$ is given by the following equation:*

$$a_n - b_n = (T-1)c_n,$$

where c_n also is a function of the form (2.11).

This equivalence relation allows us to split conserved densities and conservation laws into equivalence classes. Using it, we are able to transform any conserved density into a simplified reduced form and to define the order of conservation law.

In particular, $a_n \sim 0$ iff $a_n = (T-1)c_n$, where a_n, c_n are of the form (2.11). For example, in the case of conservation law (2.16), one can write $D_t p_n \sim 0$. It follows from formulae (2.16), (2.20) that if $a_n \sim b_n$ and a_n is a conserved density of equation (2.1), then b_n also is a conserved density. So conservation laws with equivalent conserved densities are equivalent.

The equivalence relation introduced by definition 3 is standard. It obviously has the following three properties:

$$a_n \sim a_n,$$
$$a_n \sim b_n \quad \Rightarrow \quad b_n \sim a_n,$$
$$a_n \sim b_n, b_n \sim c_n \quad \Rightarrow \quad a_n \sim c_n.$$

Moreover, it is easy to see that

$$a_n \sim b_n, c_n \sim d_n \quad \Rightarrow \quad \alpha a_n + \beta c_n \sim \alpha b_n + \beta d_n \tag{2.21}$$

for any complex constants α and β. One also has

$$a_n = a_{n+1} - (T-1)a_n \sim a_{n+1}, \qquad a_n = a_{n-1} + (T-1)a_{n-1} \sim a_{n-1},$$

hence

$$a_n \sim a_{n+i} \qquad \text{for all} \qquad i \in \mathbb{Z}. \tag{2.22}$$

It follows from equations (2.21), (2.22) that

$$a_n + b_n \sim a_{n+i} + b_{n+j} \qquad \text{for all} \qquad i, j \in \mathbb{Z}. \tag{2.23}$$

We can prove, in the same way as we did for property (2.14), that if (2.11) is a total difference, i.e. $\phi_n = \phi(u_{n+k}, u_{n+k-1}, \ldots u_{n+k'}) \sim 0$, then

$$\phi_n = \text{const} \quad \text{or} \quad k = k' \quad \Rightarrow \quad \phi_n = 0, \tag{2.24}$$

$$k > k' \quad \Rightarrow \quad \frac{\partial^2 \phi_n}{\partial u_{n+k} \partial u_{n+k'}} = 0. \tag{2.25}$$

We present now a theorem which helps us to simplify functions of the form (2.11), remaining inside a class of equivalence.

Theorem 1. *Any function of the form*

$$a_n = a\big(u_{n+k_1}, u_{n+k_1-1}, \ldots, u_{n+k_2}\big), \qquad k_1 \geqslant k_2, \tag{2.26}$$

where $\frac{\partial a_n}{\partial u_{n+k_1}} \frac{\partial a_n}{\partial u_{n+k_2}} \neq 0$ *if* a_n *is a nonconstant function, can be expressed in the form*

$$a_n = b_n + (T-1)c_n, \tag{2.27}$$

where c_n *is a function of the form (2.11), and for the function* b_n *we have*

$$b_n = b\big(u_{n+k_3}, u_{n+k_3-1}, \ldots, u_{n+k_4}\big), \qquad k_1 \geqslant k_3 \geqslant k_4 \geqslant k_2, \tag{2.28}$$

where only one of the following two possibilities takes place:

$$b_n = \text{const} \quad \textit{or} \quad k_3 = k_4, \tag{2.29}$$

$$k_3 > k_4, \qquad \frac{\partial^2 b_n}{\partial u_{n+k_3} \partial u_{n+k_4}} \neq 0. \tag{2.30}$$

The relation $a_n \sim 0$ *is possible only in the case (2.29) if* $b_n = 0$.

Proof. In this proof we will show how to construct the functions b_n and c_n. We will do so, using a trick which is applied as many times as necessary.

In the case of (2.26) with $a_n = \text{const}$, or $k_1 = k_2$, or $k_1 > k_2$ and $\frac{\partial^2 a_n}{\partial u_{n+k_1} \partial u_{n+k_2}} \neq 0$, we choose $b_n = a_n$, $c_n = 0$ and have the required result. The only remaining possibility is if

$$k_1 > k_2, \qquad \frac{\partial^2 a_n}{\partial u_{n+k_1} \partial u_{n+k_2}} = 0. \tag{2.31}$$

In this case one splits the function a_n in two components:

$$\begin{aligned} a_n &= a_n^1 + a_n^2, \\ a_n^1 &= a^1\big(u_{n+k_1}, \ldots, u_{n+k_2+1}\big), \\ a_n^2 &= a^2\big(u_{n+k_1-1}, \ldots, u_{n+k_2}\big). \end{aligned} \tag{2.32}$$

Then one can rewrite the result as

$$\begin{aligned} a_n &= a_n^3 + (T-1)a_{n-1}^1, \\ a_n^3 &= a_{n-1}^1 + a_n^2 = a^3\big(u_{n+k_1-1}, \ldots, u_{n+k_2}\big). \end{aligned} \tag{2.33}$$

If a_n^3 is nonconstant, then there exist two numbers $\hat{k}_1$, $\hat{k}_2$, such that $k_1 > \hat{k}_1 \geqslant \hat{k}_2 \geqslant k_2$, $a_n^3 = \hat{a}^3\big(u_{n+\hat{k}_1}, \ldots, u_{n+\hat{k}_2}\big)$, $\frac{\partial a_n^3}{\partial u_{n+\hat{k}_1}} \frac{\partial a_n^3}{\partial u_{n+\hat{k}_2}} \neq 0$. If $a_n^3 = \text{const}$, or $\hat{k}_1 = \hat{k}_2$, or $\hat{k}_1 > \hat{k}_2$ and $\frac{\partial^2 a_n^3}{\partial u_{n+\hat{k}_1} \partial u_{n+\hat{k}_2}} \neq 0$, we have the required result. If $\frac{\partial^2 a_n^3}{\partial u_{n+\hat{k}_1} \partial u_{n+\hat{k}_2}} = 0$, we can simplify a_n^3 by applying the procedure again. As $\hat{k}_1 < k_1$, this procedure will be applied only a finite number of times. It is clear that one is led at the end to formulae (2.27), (2.28) corresponding to the case (2.29) or (2.30).

If $a_n \sim 0$, then one has $b_n \sim 0$. In the case (2.30), we arrive at a contradiction with property (2.25). In the case (2.29), it follows from (2.24) that $b_n = 0$. □

Theorem 1 enables us to verify if a function ϕ_n of the form (2.11) is total difference, i.e. $\phi_n \sim 0$. Thus, we can check whether a function p_n is a conserved density of equation (2.1) and, in the case of positive answer, find the corresponding function q_n appearing in equation (2.16). In order to do so, one applies theorem 1 to the function $a_n = D_t p_n$ and verifies if, in the formula (2.27), $b_n = 0$. If it is so, then the function q_n is given by $q_n = c_n + \alpha$, where α is an arbitrary constant, as it follows from property (2.14).

As an example of the verification of a conserved density, let us consider the case of the Volterra equation (2.2) and $p_n = \log(u_{n+1}u_n)$. In this case

$$a_n = D_t p_n = \frac{\dot{u}_{n+1}}{u_{n+1}} + \frac{\dot{u}_n}{u_n} = u_{n+2} - u_n + u_{n+1} - u_{n-1},$$

with $k_1 = 2$, $k_2 = -1$. We are in the case (2.31) and for (2.32) one can take e.g. $a_n^1 = u_{n+2}$. Going over to (2.33), one obtains

$$a_n = 2u_{n+1} - u_n - u_{n-1} + (T-1)u_{n+1}.$$

Applying again the scheme used in the proof of theorem 1, one gets

$$a_n = u_n - u_{n-1} + (T-1)(u_{n+1} + 2u_n),$$

and we are led on the next step to the following conservation law:

$$D_t \log(u_{n+1}u_n) = (T-1)(u_{n+1} + 2u_n + u_{n-1}). \tag{2.34}$$

It is equivalent to the second part of conservation laws (2.19), as from property (2.23) we have $\log(u_{n+1}u_n) \sim 2\log u_n$.

The conserved density p_n plays a leading role in the conservation law (2.16) because if p_n is known, the function q_n can be easily found using theorem 1. For this reason, we will mainly work with conserved densities without writing down the whole conservation law.

Now we can define the order of a conserved density and of a conservation law. The notion of order will allow us to distinguish essentially different cases. In accordance with theorem 1, any conserved density p_n is equivalent to a function $\hat{p}_n$ of the following form:

$$p_n \sim \hat{p}_n = \wp\big(u_{n+\hat{m}_1}, u_{n+\hat{m}_1-1}, \ldots, u_{n+\hat{m}_2}\big).$$

If $\hat{p}_n$ is a nonconstant function, one has

$$\hat{m}_1 = \hat{m}_2, \qquad \text{or} \qquad \hat{m}_1 > \hat{m}_2, \qquad \frac{\partial^2 \hat{p}_n}{\partial u_{n+\hat{m}_1}\partial u_{n+\hat{m}_2}} \neq 0.$$

Introducing the function $\wp_n = \hat{p}_{n-\hat{m}_2}$ together with number $m = \hat{m}_1 - \hat{m}_2$ and using property (2.22), we obtain that any conserved density p_n is equivalent to a conserved density $\wp_n$,

$$p_n \sim \wp_n, \tag{2.35}$$

three possible forms of which are given in the following definition.

Definition 4. *A conserved density p_n and the corresponding conservation law (2.16) are called* trivial *if*

$$\wp_n = c \in \mathbb{C}, \tag{2.36}$$

and nontrivial *if*

$$\wp_n = \wp(u_n), \qquad \wp'(u_n) \neq 0, \tag{2.37}$$

$$\wp_n = \wp(u_{n+m}, u_{n+m-1}, \ldots, u_n), \qquad m > 0, \qquad \frac{\partial^2 \wp_n}{\partial u_{n+m}\partial u_n} \neq 0. \tag{2.38}$$

The order *of a nontrivial conserved density p_n and of the corresponding conservation law (2.16) is given by the number 0 or m, depending on (2.37) or (2.38) takes place.*

Properties (2.24), (2.25) imply that the conserved densities corresponding to cases (2.36)–(2.38) cannot be equivalent to each other. Formulae (2.35)–(2.38) define the special form of a conserved density necessary to define its order.

The following four functions

$$\begin{aligned} &p_n^1 = \log u_n, \qquad p_n^2 = u_n, \qquad p_n^3 = u_{n+1}u_n + \tfrac{1}{2}u_n^2, \\ &p_n^4 = u_{n+2}u_{n+1}u_n + u_{n+1}^2 u_n + u_{n+1}u_n^2 + \tfrac{1}{3}u_n^3 \end{aligned} \tag{2.39}$$

exemplify the simplest nontrivial conserved densities of the Volterra equation (2.2). The densities p_n^1 and p_n^2 are taken from equations (2.19). The reader easily can check that $D_t p_n^3 \sim D_t p_n^4 \sim 0$. The orders of the conserved densities (2.39) are 0, 0, 1 and 2, respectively.

Let us define the formal variational derivative of a function ϕ_n (2.11) as

$$\frac{\delta \phi_n}{\delta u_n} = \sum_{i=k'}^{k} T^{-i} \frac{\partial \phi_n}{\partial u_{n+i}} = \sum_{j=-k}^{-k'} \frac{\partial \phi_{n+j}}{\partial u_n}, \tag{2.40}$$

(see e.g. [18, 19, 53, 56, 69, 72]). The operator $\frac{\delta}{\delta u_n}$ is the discrete analogue of (1.7) and possesses similar properties (see theorem 3 in section 2.2 below). Sometimes it is called Euler operator [27, 36], but we use the name formal variational derivative, following the continuous case. We will use it to calculate the order of a conserved density.

Let us show that, for the discrete operator (2.40), an analogue of the property that equations (1.8) and (1.9) are equivalent will be true, namely

$$\phi_n = (T-1)\psi_n \quad \Rightarrow \quad \frac{\delta \phi_n}{\delta u_n} = 0, \tag{2.41}$$

where ϕ_n, ψ_n are functions of the form (2.11). From property (2.24) we see that we need to consider only the nontrivial case when $k > k'$, $\frac{\partial \phi_n}{\partial u_{n+k}} \frac{\partial \phi_n}{\partial u_{n+k'}} \neq 0$. In this case

$\psi_n = \psi(u_{n+k-1}, \dots, u_{n+k'})$ and

$$\frac{\delta\phi_n}{\delta u_n} = \sum_{j=-k}^{-k'} \frac{\partial}{\partial u_n}(\psi_{n+j+1} - \psi_{n+j}) = \frac{\partial}{\partial u_n}(\psi_{n-k'+1} - \psi_{n-k})$$
$$= \frac{\partial}{\partial u_n}(\psi(u_{n+k-k'}, \dots, u_{n+1}) - \psi(u_{n-1}, \dots, u_{n+k'-k})) = 0,$$

as in the last expression $k - k' \geqslant 1, k' - k \leqslant -1$ and the function does not depend on the variable u_n.

To find the order of a conserved density p_n, we can calculate its formal variational derivative $\delta p_n/\delta u_n$. It follows from (2.41) and the fact that $\delta/\delta u_n$ is a linear operator that the result of the application of this operator to equivalent conserved densities is the same. For this reason $\frac{\delta p_n}{\delta u_n} = \frac{\delta \wp_n}{\delta u_n}$, where $\wp_n$ is the special form of the conserved density defined by equations (2.35)–(2.38). The form of $\delta\wp_n/\delta u_n$ is obvious in the cases (2.36) and (2.37), while in the case (2.38) we have

$$\frac{\delta\wp_n}{\delta u_n} = \varrho_n = \varrho(u_{n+m}, u_{n+m-1}, \dots, u_{n-m}),$$
$$\frac{\partial\varrho_n}{\partial u_{n+m}} = \frac{\partial^2\wp_n}{\partial u_{n+m}\partial u_n} \neq 0, \qquad \frac{\partial\varrho_n}{\partial u_{n-m}} = \frac{\partial^2\wp_{n-m}}{\partial u_n\partial u_{n-m}} = T^{-m}\frac{\partial\varrho_n}{\partial u_{n+m}} \neq 0.$$

We thus introduce the function

$$\varrho_n = \frac{\delta p_n}{\delta u_n}, \tag{2.42}$$

which can have one of the following three essentially different forms:

$$\varrho_n = 0, \tag{2.43}$$
$$\varrho_n = \varrho(u_n) \neq 0, \tag{2.44}$$
$$\varrho_n = \varrho(u_{n+m}, u_{n+m-1}, \dots, u_{n-m}), \quad m > 0, \qquad \frac{\partial\varrho_n}{\partial u_{n+m}}\frac{\partial\varrho_n}{\partial u_{n-m}} \neq 0. \tag{2.45}$$

In the case (2.43), the conserved density p_n and the corresponding conservation law (2.16) are trivial. In the cases (2.44) and (2.45), they are nontrivial, and the corresponding orders equal 0 or $m > 0$, respectively.

As an example, let us consider the conserved densities (2.39). We have

$$\frac{\delta p_n^1}{\delta u_n} = \frac{1}{u_n}, \qquad \frac{\delta p_n^2}{\delta u_n} = 1, \qquad \frac{\delta p_n^3}{\delta u_n} = u_{n+1} + u_n + u_{n-1},$$
$$\frac{\delta p_n^4}{\delta u_n} = u_{n+2}u_{n+1} + u_{n+1}^2 + 2u_{n+1}u_n + u_n^2 + u_{n+1}u_{n-1} + 2u_nu_{n-1} + u_{n-1}^2 + u_{n-1}u_{n-2},$$

and hence the orders of p_n^1, p_n^2, p_n^3, p_n^4 are, respectively, equal to 0, 0, 1, 2.

Definition 2 of local conservation law and of conserved density is as constructive as definition 1 of generalized symmetry. For any given equation of the form (2.1) and any order, it is possible to find all conservation laws of that order or to prove that no conservation law exists. In section 3.1.1, in the example of the Volterra equation, we will construct all conservation laws of the first order, using this definition.

2.2. *First integrability condition*

We discuss in this subsection how to obtain a generalized symmetry (2.3) of an equation of the form (2.1), using the compatibility condition (2.8). On this way, the first integrability condition for equation (2.1) will arise. At the end, we briefly describe the general scheme of the generalized symmetry method.

So, given the function f_n, one looks for g_n with $m > 0$, $m' < 0$. For convenience we introduce the following notation:

$$f_n^{(i)} = \frac{\partial f_n}{\partial u_{n+i}}, \qquad g_n^{(i)} = \frac{\partial g_n}{\partial u_{n+i}}, \tag{2.46}$$

and consequently the compatibility condition is given by

$$D_t g_n = \sum_{i=m'}^{m} g_n^{(i)} f_{n+i} = f_n^{(1)} g_{n+1} + f_n^{(0)} g_n + f_n^{(-1)} g_{n-1}. \tag{2.47}$$

If $m \geqslant 1$, applying the operator $\frac{\partial}{\partial u_{n+m+1}}$ to equation (2.47), one obtains the following relation:

$$g_n^{(m)} f_{n+m}^{(1)} = g_{n+1}^{(m)} f_n^{(1)}, \qquad m \geqslant 1. \tag{2.48}$$

Applying $\frac{\partial}{\partial u_{n+m}}$ and $\frac{\partial}{\partial u_{n+m-1}}$ to equation (2.47), other two analogous relations can be derived:

$$D_t g_n^{(m)} + g_n^{(m)} f_{n+m}^{(0)} + g_n^{(m-1)} f_{n+m-1}^{(1)} = g_n^{(m)} f_n^{(0)} + g_{n+1}^{(m-1)} f_n^{(1)}, \qquad m \geqslant 2, \tag{2.49}$$

$$\begin{aligned} D_t g_n^{(m-1)} &+ g_n^{(m)} f_{n+m}^{(-1)} + g_n^{(m-1)} f_{n+m-1}^{(0)} + g_n^{(m-2)} f_{n+m-2}^{(1)} \\ &= g_{n-1}^{(m)} f_n^{(-1)} + g_n^{(m-1)} f_n^{(0)} + g_{n+1}^{(m-2)} f_n^{(1)}, \qquad m \geqslant 3. \end{aligned} \tag{2.50}$$

Introducing for any $N \geqslant 0$ the function

$$\Phi_n^{(N)} = f_n^{(1)} f_{n+1}^{(1)} \cdots f_{n+N}^{(1)}, \tag{2.51}$$

taking into account that $f_n^{(1)} \neq 0$ (see (2.1)) and dividing equations (2.48)–(2.50) by $\Phi_n^{(m)}$, $\Phi_n^{(m-1)}$, $\Phi_n^{(m-2)}$, respectively, we obtain

$$(T-1)\frac{g_n^{(m)}}{\Phi_n^{(m-1)}} = 0, \tag{2.52}$$

$$(T-1)\frac{g_n^{(m-1)}}{\Phi_n^{(m-2)}} = \Theta_n^{(1)}\big(g_n^{(m)}\big), \tag{2.53}$$

$$(T-1)\frac{g_n^{(m-2)}}{\Phi_n^{(m-3)}} = \Theta_n^{(2)}\big(g_n^{(m-1)}, g_n^{(m)}\big). \tag{2.54}$$

Here the left-hand sides are total differences, and the functions $\Theta_n^{(i)}$ $(i = 1, 2)$ depend on the partial derivatives $g_n^{(j)}$ which are defined by the previous relations.

Due to property (2.14), equation (2.52) can be easily solved and $g_n^{(m)}$ can be found. Hence the right-hand side of (2.53) is known and the following condition appears: the function $\Theta_n^{(1)}(g_n^{(m)})$ must be a total difference. If this condition is satisfied, one finds $g_n^{(m-1)}$. Then the function $g_n^{(m-2)}$ can be found from equation (2.54) if an analogous condition is satisfied. In a quite similar way, we can write down equations for the other partial derivatives $g_n^{(i)}$, such that $0 < i \leqslant m$ or $m' \leqslant i < 0$. Those equations have the same structure and lead to analogous conditions for the functions we already have found. In the integrable cases, i.e. if all such conditions are satisfied, we can define the function g_n up to arbitrary constants and an arbitrary function of u_n which easily can be specified using equation (2.47). This will be done in section 3.1.1 in the example of the Volterra equation for a generalized symmetry of the orders $m = 2$, $m' = -2$.

Let us pass now to the case when not only the function g_n but also f_n are unknown, i.e. when we consider the problem of classifying equations of the form (2.1) which have generalized symmetries. It turns out that in this case some conditions can be derived from equations of the type (2.53), (2.54), and those conditions do not depend on g_n and are expressed only in terms of equation (2.1) itself. One obtains the same conditions using any generalized symmetry of any high enough order m. Those integrability conditions will be necessary for the existence of high enough order generalized symmetries. The first of them is given by the following theorem.

Theorem 2. *If an equation of the form (2.1) possesses a generalized symmetry of the form (2.3) of order $m \geqslant 2$, then there must exist a function $q_n^{(1)}$ of the form (2.11), such that*

$$\dot{p}_n^{(1)} = (T-1)q_n^{(1)} \qquad \textit{with} \qquad p_n^{(1)} = \log \frac{\partial f_n}{\partial u_{n+1}}, \tag{2.55}$$

where $\dot{p}_n^{(1)} = D_t p_n^{(1)}$ and f_n is defined in equation (2.1).

Proof. In the case of symmetries of the order $m \geqslant 2$, we can use equations (2.48), (2.49) and then equations (2.52), (2.53). From equation (2.52) it follows that

$$g_n^{(m)} = \alpha \Phi_n^{(m-1)}, \tag{2.56}$$

where the constant α does not vanish due to (2.3). It is easy to see that the right-hand side of equation (2.53) has the form

$$\Theta_n^{(1)}\big(g_n^{(m)}\big) = \alpha D_t \log \Phi_n^{(m-1)} + \alpha\big(f_{n+m}^{(0)} - f_n^{(0)}\big). \tag{2.57}$$

From equation (2.53) it follows that $\Theta_n^{(1)}(g_n^{(m)})$ must be equivalent to zero, and property (2.23) implies that $f_{n+m}^{(0)} - f_n^{(0)} \sim 0$. The same property (2.23) together with the formulae $p_n^{(1)} = \log f_n^{(1)}$ and (2.51) provide the following result:

$$D_t \log \Phi_n^{(m-1)} = \dot{p}_n^{(1)} + \dot{p}_{n+1}^{(1)} + \cdots + \dot{p}_{n+m-1}^{(1)} \sim m\dot{p}_n^{(1)}. \tag{2.58}$$

Dividing equation (2.57) by αm and using the equivalence relations discussed above, we can see that $\dot{p}_n^{(1)} \sim 0$. This shows, in accordance with definition 3, that the function $\dot{p}_n^{(1)}$ can be expressed in the form (2.55). □

Condition (2.55) has the form of a local conservation law, and theorem 2 tells us that if there is a generalized symmetry of an order $m \geqslant 2$, then equation (2.1) must have a conservation law with conserved density $p_n^{(1)}$ defined by equation (2.1) itself. If condition (2.55) is satisfied for an equation, then one automatically obtains for that equation a conserved density. *A priori* this conserved density may be trivial or its order may be equal to 0, 1 or 2. Any of these possibilities is realized in the examples we present in section 3.1.2. In the case of the Volterra equation, for instance, $p_n^{(1)} = \log u_n$, and this is nothing but the first conserved density (2.39) of order 0.

In order to check the first integrability condition (2.55) for a given equation, one can use theorem 1. Such checking can be simplified, using the discrete analogue of the result that relations (1.8), (1.9) are equivalent.

Theorem 3. *A variational derivative (2.40) satisfies the following property:*

$$\frac{\delta \phi_n}{\delta u_n} = 0 \quad \Leftrightarrow \quad \phi_n = \sigma + (T-1)\psi_n, \tag{2.59}$$

where σ is a constant, ϕ_n and ψ_n are two functions of the form (2.11).

Proof. One part of the proof follows from property (2.41), as $\frac{\delta\sigma}{\delta u_n} = 0$. Let us discuss the other part and suppose that $\frac{\delta\phi_n}{\delta u_n} = 0$. According to theorem 1, for $a_n = \phi_n$, we have representations (2.27), (2.28) with equations (2.29) or (2.30). In the second case (2.30),

$$\frac{\delta\phi_n}{\delta u_n} = \frac{\delta a_n}{\delta u_n} = \frac{\delta b_n}{\delta u_n} = B_n = B(u_{n+K}, u_{n+K-1}, \ldots, u_{n-K}),$$

where $K = k_3 - k_4 > 0$ and

$$\frac{\partial B_n}{\partial u_{n+K}} = \frac{\partial^2 b_{n-k_4}}{\partial u_{n+K}\partial u_n} \neq 0.$$

This is in contradiction with the request that $\frac{\delta\phi_n}{\delta u_n} = 0$. In the case (2.29), the function b_n can be written as $b_n = b\big(u_{n+k_4}\big)$. Hence

$$\frac{\delta\phi_n}{\delta u_n} = \frac{\delta b_n}{\delta u_n} = \frac{\partial b_{n-k_4}}{\partial u_n} = b'(u_n) = 0,$$

i.e. b_n is a constant. □

To check the first integrability condition (2.55), we can use property (2.59) with $\phi_n = \dot{p}_n^{(1)}$. At first we check if

$$\frac{\delta}{\delta u_n}\dot{p}_n^{(1)} = 0. \tag{2.60}$$

Then, if this is true, using theorem 1 we represent $\dot{p}_n^{(1)}$ as

$$\dot{p}_n^{(1)} = \sigma + (T-1)q_n^{(1)}. \tag{2.61}$$

Here σ is a constant and $p_n^{(1)}$ will be a conserved density only if $\sigma = 0$.

After having introduced all the necessary notions, we can briefly describe the standard scheme of the generalized symmetry method. At first we choose a class of equations, as e.g. equations (2.1), with an unknown right-hand side. Then, assuming the existence of generalized symmetries and/or conservation laws of a high enough order, we derive a few integrability conditions like (2.55). These conditions have no dependence on the generalized symmetries and conservation laws and are expressed in terms of the given class of equations. Then we try to describe the class of equations satisfying the integrability conditions. The aim is to obtain a list of equations, in which there are no arbitrary functions, but only arbitrary constants. If this is impossible, we obtain more integrability conditions. We will show in the following that one can derive as many conditions as necessary to characterize any class of equations. We usually choose as starting point of the classification a class of equations which is invariant under point transformations. In the case of equations (2.1) point transformations have the form $\tilde{u}_n = s(u_n), \tilde{t} = \mu t$, where μ is a constant. In this way a complete, up to point transformations, list of equations is obtained. Equations of this list will satisfy a finite number of integrability conditions and will contain no arbitrary functions.

As will be shown, the integrability conditions are necessary conditions for the existence of generalized symmetries and conservation laws. For this reason we will prove, using the Miura transformations, master symmetries and the Hamiltonian structures, the existence of infinite hierarchies of generalized symmetries and conservation laws (see sections 2.6 and 2.7). By doing so we obtain an exhaustive list of integrable equations of the given form. An example of such exhaustive classification will be given in section 3.1.1.

2.3. Formal symmetries and further integrability conditions

From the compatibility condition (2.8), in addition to equation (2.55), we can derive more integrability conditions. However, the calculation will become more and more complicated on each step. These calculations can be drastically simplified using the notion of formal symmetry. In this section, we introduce and discuss formal symmetries and then derive a second and a third integrability conditions.

Let us introduce, in analogy to the continuous case, the discrete Frechet derivative of a function ϕ_n of the form (2.11) as the following operator:

$$\phi_n^* = \sum_{i=k'}^{k} \frac{\partial \phi_n}{\partial u_{n+i}} T^i. \tag{2.62}$$

This is the discrete analogue of the operator ϕ_* given by equations (1.12). For a function f_n given by equation (2.1), f_n^* is the operator,

$$f_n^* = f_n^{(1)} T + f_n^{(0)} + f_n^{(-1)} T^{-1}, \tag{2.63}$$

with coefficients defined according to (2.46).

Formal symmetries are closely related to the following Lax equation:

$$\dot{L}_n = [f_n^*, L_n], \tag{2.64}$$

where $[f_n^*, L_n] = f_n^* L_n - L_n f_n^*$ is the standard commutator. The solutions of equation (2.64) will be formal series in powers of the shift operator T and will have the following form:

$$L_n = \sum_{i=-\infty}^{N} l_n^{(i)} T^i, \qquad l_n^{(N)} \neq 0, \tag{2.65}$$

where the coefficients $l_n^{(i)}$ are functions of the form (2.11). The number N will be called the order of L_n, and we write $\operatorname{ord} L_n = N$. The series $\dot{L}_n$ is obtained by applying the operator D_t given by equation (2.7) to coefficients of L_n:

$$\dot{L}_n = \dot{l}_n^{(N)} T^N + \dot{l}_n^{(N-1)} T^{N-1} + \cdots.$$

The Frechet derivative operator f_n^* is a particular case of the series (2.65). In the case of f_n^*, $N = 1$ and $l_n^{(i)} = 0$ for all $i \leqslant -2$. The set of series (2.65) forms a linear space. Such series can be multiplied according to the rule: $l_n T^i \circ \hat{l}_n T^j = l_n \hat{l}_{n+i} T^{i+j}$, where $T^0 = 1$. The inverse of (2.65),

$$L_n^{-1} = \sum_{i=-\infty}^{\hat{N}} \hat{l}_n^{(i)} T^i, \qquad \hat{l}_n^{(\hat{N})} \neq 0, \tag{2.66}$$

is uniquely defined by the equations $L_n^{-1} L_n = L_n L_n^{-1} = 1$. In fact,

$$L_n^{-1} L_n = \hat{l}_n^{(\hat{N})} l_{n+\hat{N}}^{(N)} T^{\hat{N}+N} + \left(\hat{l}_n^{(\hat{N})} l_{n+\hat{N}}^{(N-1)} + \hat{l}_n^{(\hat{N}-1)} l_{n+\hat{N}-1}^{(N)}\right) T^{\hat{N}+N-1} + \cdots,$$

where the first coefficient cannot vanish. Hence $\hat{N} = -N$, and the first coefficients of equation (2.66) are defined by

$$\hat{l}_n^{(-N)} = \left(l_{n-N}^{(N)}\right)^{-1}, \qquad \hat{l}_n^{(-N-1)} = -\left(l_{n-N}^{(N)}\right)^{-1} l_{n-N}^{(N-1)} \left(l_{n-N-1}^{(N)}\right)^{-1}, \ldots. \tag{2.67}$$

Let us introduce the operator A,

$$A(L_n) = \dot{L}_n - [f_n^*, L_n]. \tag{2.68}$$

We can easily check the following two general formulae:

$$A(L_n^{-1}) = -L_n^{-1} A(L_n) L_n^{-1}, \tag{2.69}$$

$$A(L_n \tilde{L}_n) = A(L_n)\tilde{L}_n + L_n A(\tilde{L}_n). \tag{2.70}$$

These formulae show that given any two solutions L_n and $\tilde{L}_n$ of equation (2.64), their product $L_n\tilde{L}_n$ and the inverse L_n^{-1} satisfy the same equation. So any integer power L_n^i, where $L_n^0 = 1$, will also satisfy equation (2.64).

The solution L_n of equation (2.64) is nothing but the recursion operator of the integrable hierarchy of equations because it transforms the right-hand side g_n of a generalized symmetry (2.3) into the right-hand side $L_n g_n$ of a new generalized symmetry. The compatibility condition (2.8) can be written in terms of the Frechét derivative f_n^* as

$$(D_t - f_n^*)g_n = 0. \tag{2.71}$$

Then, using equations (2.64), (2.71), one can easily check that

$$D_t L_n g_n = \dot{L}_n g_n + L_n \dot{g}_n = (f_n^* L_n - L_n f_n^*)g_n + L_n f_n^* g_n = f_n^* L_n g_n,$$

i.e. $u_{n,\tau'} = L_n g_n$ is a new generalized symmetry, maybe non-local. Taking into account that any integer power L_n^i satisfies equation (2.64), as well as the fact that f_n is a trivial solution of equation (2.71), we obtain infinitely many generalized symmetries of equation (2.1):

$$u_{n,t_i} = L_n^i f_n, \tag{2.72}$$

where $i \in \mathbb{Z}$, $t_0 = t$.

As we will prove in section 2.6,

$$\Lambda_n = u_n + u_n(u_{n+1}T - u_{n-1}T^{-2})(1 - T^{-1})^{-1}u_n^{-1} \tag{2.73}$$

is the recursion operator of the Volterra equation (2.2), i.e. it satisfies equation (2.64). Using the formula

$$(1 - T^{-1})^{-1} = 1 + T^{-1} + T^{-2} + \cdots = \sum_{i=-\infty}^{0} T^i,$$

one can rewrite Λ_n as

$$\Lambda_n = u_n T + u_{n+1} + u_n + \frac{u_{n+1}u_n}{u_{n-1}} T^{-1} + \dot{u}_n \left(\sum_{i=-\infty}^{-2} T^i \right) u_n^{-1}. \tag{2.74}$$

In this way we can write down the coefficients $l_n^{(i)}$ of the representation (2.65). It can be proved that formula (2.72) provides a local generalized symmetry for any $i \geqslant 1$. The orders m and m' of this symmetry, defined in equation (2.3), are such that $m = -m' = i + 1$. As equation (2.2) can be written as

$$\dot{u}_n = u_n(1 - T^{-1})(u_{n+1} + u_n),$$

from (2.73) we obtain that

$$u_{n,\tau} = \Lambda_n \dot{u}_n = u_n \dot{u}_n + u_n(u_{n+1}T - u_{n-1}T^{-2})(u_{n+1} + u_n)$$

is the generalized symmetry (2.15). So, in the case of $i = 1$, formula (2.72) provides the generalized symmetry (2.15).

The recursion operator allows one to construct not only generalized symmetries but also conserved densities. This will be demonstrated by theorem 4 which will allow us to derive some new integrability conditions. Let us define the residue of a series (2.65) as the coefficient at T^0, i.e.

$$\operatorname{res} L_n = l_n^{(0)}. \tag{2.75}$$

From this definition it follows that if $N < 0$, then $\operatorname{res} L_n = 0$.

Theorem 4. *Let a series L_n (2.65) with $N > 0$ satisfy equation (2.64). Then*

$$\log l_n^{(N)}, \tag{2.76}$$

$$\operatorname{res} L_n^i, \quad i \geqslant 1, \tag{2.77}$$

are conserved densities of equation (2.1).

Proof. First of all we prove that

$$\operatorname{res}[\tilde{L}_n, \hat{L}_n] \sim 0 \tag{2.78}$$

for any formal series $\tilde{L}_n$ and $\hat{L}_n$ of the form (2.65). Let $\operatorname{ord} \tilde{L}_n = N_1$ and $\operatorname{ord} \hat{L}_n = N_2$. If $\operatorname{ord}[\tilde{L}_n, \hat{L}_n] = N_1 + N_2 \geqslant 0$, then

$$\operatorname{res}[\tilde{L}_n, \hat{L}_n] = \operatorname{res}\left[\sum_{i=-\infty}^{N_1} \tilde{l}_n^{(i)} T^i, \sum_{j=-\infty}^{N_2} \hat{l}_n^{(j)} T^j\right] = \sum_{i=-N_2}^{N_1} \operatorname{res}\big[\tilde{l}_n^{(i)} T^i, \hat{l}_n^{(-i)} T^{-i}\big]. \tag{2.79}$$

The last sum in equation (2.79) is a total difference, as

$$\operatorname{res}\big[\tilde{l}_n^{(i)} T^i, \hat{l}_n^{(-i)} T^{-i}\big] = \tilde{l}_n^{(i)} \hat{l}_{n+i}^{(-i)} - \hat{l}_n^{(-i)} \tilde{l}_{n-i}^{(i)} \sim 0$$

due to property (2.23). If $N_1 + N_2 < 0$, then $\operatorname{res}[\tilde{L}_n, \hat{L}_n] = 0$.

As any integer power L_n^i satisfies equation (2.64), one has

$$D_t \operatorname{res} L_n^i = \operatorname{res} D_t\big(L_n^i\big) = \operatorname{res}[f_n^*, L_n^i] \sim 0,$$

i.e. the functions $\operatorname{res} L_n^i$ are conserved densities. The series L_n given by equation (2.65) is such that $N > 0$, hence the functions $\operatorname{res} L_n^i$ are equal to 0 or 1 if $i \leqslant 0$, and these densities are trivial. These conserved densities can be nontrivial only if $i \geqslant 1$. In formula (2.77) we have exactly this case.

Equation (2.64) implies that

$$\dot{L}_n L_n^{-1} = \big[f_n^* L_n^{-1}, L_n\big]. \tag{2.80}$$

It follows from equations (2.78), (2.80) that

$$\operatorname{res}\big(\dot{L}_n L_n^{-1}\big) = \dot{l}_n^{(N)} \big(l_n^{(N)}\big)^{-1} = D_t \log l_n^{(N)} \sim 0.$$

This is the reason why the function (2.76) is another conserved density. □

A priori, we do not know whether the conserved densities (2.76), (2.77) are nontrivial, and which are their orders. All such conserved densities can be trivial in the case of a linearizable equation. In the case of known equations integrable by the inverse scattering method, the recursion operator provides an infinite hierarchy of conserved densities of arbitrarily high order. This is the case of the Volterra equation (2.2). Using formula (2.74) for its recursion operator, one easily checks that

$$\log u_n = p_n^1, \qquad \operatorname{res} \Lambda_n = u_{n+1} + u_n \sim 2p_n^2,$$

$$\operatorname{res} \Lambda_n^2 = u_{n+2}u_{n+1} + 3u_{n+1}u_n + u_{n+1}^2 + u_n^2 \sim 4p_n^3,$$

where p_n^i are conserved densities (2.39) of equation (2.2). Moreover, it is possible to prove that the conserved densities $\operatorname{res} \Lambda_n^i$ have the order $i - 1$ for any $i \geqslant 1$.

Remark. We can also construct conserved densities of the Volterra equation, using the well-known $L - A$ pair,

$$\dot{L}_n = [A_n, L_n], \tag{2.81}$$

$$L_n = u_{n+1}^{1/2} T + u_n^{1/2} T^{-1}, \quad A_n = \tfrac{1}{2} u_{n+2}^{1/2} u_{n+1}^{1/2} T^2 - \tfrac{1}{2} u_n^{1/2} u_{n-1}^{1/2} T^{-2}. \tag{2.82}$$

We can prove here an analogue of theorem 4, as in its proof we do use neither the precise formula (2.63) for f_n^* nor the fact that L_n is an infinite series. One can check, for example, that

$$\log u_{n+1}^{1/2} \sim \tfrac{1}{2} p_n^1, \qquad \operatorname{res} L_n = 0, \qquad \operatorname{res} L_n^2 \sim 2 p_n^2,$$
$$\operatorname{res} L_n^3 = 0, \qquad \operatorname{res} L_n^4 \sim 4 p_n^3,$$

i.e. densities (2.39) arise again. Moreover, we have $\operatorname{res} L_n^i = 0$ for all odd positive i, and $\operatorname{res} L_n^{2j}$ is a conserved density of order $j - 1$ for all $j \geqslant 1$.

In practice, it is difficult to construct a recursion operator and difficult to prove that the generalized symmetry (2.72) is local, i.e. its right-hand side is of the form (2.11). We shall be interested below in approximate solutions of equation (2.64). They are easy to construct and can be used for deriving the integrability conditions. These solutions can be called approximate recursion operators, but we prefer to use the name formal symmetry because of its close connection with generalized symmetry (see theorem 5 below).

Let us note that for any series L_n of order N given by (2.65), the series $A(L_n)$ given by (2.68) can be expressed as

$$A(L_n) = a_n^{(N+1)} T^{N+1} + a_n^{(N)} T^N + a_n^{(N-1)} T^{N-1} + \cdots. \tag{2.83}$$

Definition 5. *The series (2.65) is called a formal symmetry of equation (2.1) of the length l (we write $\operatorname{lgt} L_n = l$) if the first l coefficients of the series $A(L_n)$ (2.83) vanish:*

$$a_n^{(i)} = 0, \qquad N + 1 \geqslant i \geqslant N + 2 - l. \tag{2.84}$$

We assume, moreover, that $l \geqslant 1$ and $a_n^{(N+1-l)} \neq 0$.

The recursion operator L_n of order N is such that all coefficients $a_n^{(i)}$ of $A(L_n)$ vanish. The equations $a_n^{(N+1-j)} = 0, 0 \leqslant j \leqslant l - 1$, define the coefficients $l_n^{(N-j)}$ of the recursion operator L_n and of a formal symmetry L_n, such that $\operatorname{lgt} L_n = l$, $\operatorname{ord} L_n = N$. So, the first l coefficients of such formal symmetry and of the Nth order recursion operator are defined by the same equations.

In order to find the length l of a given formal symmetry L_n of order N, we have to specify formula (2.83). If

$$A(L_n) = \sum_{i=-\infty}^{k} a_n^{(i)} T^i, \qquad a_n^{(k)} \neq 0, \tag{2.85}$$

then we have $l = N + 1 - k$. So, for any formal symmetry, we have

$$\operatorname{lgt} L_n = \operatorname{ord} L_n + 1 - \operatorname{ord} A(L_n). \tag{2.86}$$

Recalling that the Frechét derivative of the right-hand side g_n of a generalized symmetry (2.3) is the following operator:

$$g_n^* = \sum_{i=m'}^{m} \frac{\partial g_n}{\partial u_{n+i}} T^i = \sum_{i=m'}^{m} g_n^{(i)} T^i \tag{2.87}$$

(see equations (2.46), (2.62)), we state the following theorem.

Theorem 5. *If equation (2.1) has a generalized symmetry (2.3) of order $m \geqslant 1$, then it has a formal symmetry L_n with* $\operatorname{ord} L_n = m$, $\operatorname{lgt} L_n \geqslant m$ *defined by*

$$L_n = g_n^* + \sum_{i=-\infty}^{m'-1} 0T^i. \tag{2.88}$$

Proof. Let us apply the Frechét derivative to both sides of the defining equation for the generalized symmetry (2.71). We see that

$$\begin{aligned}(D_t g_n)^* &= \left(\sum_{i=m'}^{m} g_n^{(i)} f_{n+i}\right)^* = \sum_{i,j} \frac{\partial^2 g_n}{\partial u_{n+i}\partial u_{n+j}} f_{n+i} T^j + \sum_{i,j} \frac{\partial g_n}{\partial u_{n+i}} \frac{\partial f_{n+i}}{\partial u_{n+j}} T^j \\ &= \sum_{j=m'}^{m} \dot{g}_n^{(j)} T^j + \left(\sum_{i=m'}^{m} g_n^{(i)} T^i\right)\left(\sum_{\sigma=-1}^{1} f_n^{(\sigma)} T^\sigma\right),\end{aligned}$$

where $\sigma = j - i$ and the functions $f_n^{(\sigma)}$ are defined by equation (2.46). We can express the result in terms of the Frechét derivatives g_n^* (2.87) and f_n^* (2.63):

$$(D_t g_n)^* = \dot{g}_n^* + g_n^* f_n^*. \tag{2.89}$$

As $f_n^* g_n = D_\tau f_n$ (see equation (2.7)), we can write the following analogue of (2.89):

$$(f_n^* g_n)^* = (D_\tau f_n)^* = f_{n,\tau}^* + f_n^* g_n^*. \tag{2.90}$$

Using equations (2.68), (2.89), (2.90), from equation (2.71) we obtain the relation

$$A(g_n^*) = f_{n,\tau}^* = f_{n,\tau}^{(1)} T + f_{n,\tau}^{(0)} + f_{n,\tau}^{(-1)} T^{-1}, \tag{2.91}$$

where $f_{n,\tau}^{(i)}$ are the τ-derivatives of $f_n^{(i)}$.

Introducing the series (2.88), we see that $\operatorname{ord} L_n = m$, as $g_n^{(m)} \neq 0$ (see (2.3)), and from equation (2.91) we get

$$\operatorname{ord} A(L_n) = \operatorname{ord} f_{n,\tau}^* \leqslant 1,$$

as $f_{n,\tau}^{(i)}$ may vanish. Formula (2.86) implies that $\operatorname{lgt} L_n \geqslant m \geqslant 1$, i.e. this series L_n is a formal symmetry. □

Theorem 5 shows how to obtain a formal symmetry from the generalized symmetry. To derive the integrability conditions, we need to use these formal symmetries. The coefficients of these formal symmetries have, due to equation (2.88), the same structure as the right-hand side of a generalized symmetry. This is the reason why the coefficients $l_n^{(i)}$ of a formal symmetry, which is a series of the form (2.65), have no explicit dependence on n and t and are functions of the form (2.11).

As we have shown before, formal series can be multiplied and inverted. The same is also true for formal symmetries. Using relations (2.69), (2.70) together with formula (2.86), we can check that, if L_n and $\tilde{L}_n$ are formal symmetries, then the series L_n^{-1} and $L_n\tilde{L}_n$ also are formal symmetries, and we can find their orders and lengths. In fact, we always have $\operatorname{ord}(L_n\tilde{L}_n) = \operatorname{ord} L_n + \operatorname{ord} \tilde{L}_n$ and

$$\begin{aligned}\operatorname{ord} A(L_n\tilde{L}_n) &\leqslant \max(\operatorname{ord}(A(L_n)\tilde{L}_n), \operatorname{ord}(L_n A(\tilde{L}_n))) \\ &= \max(\operatorname{ord} A(L_n) + \operatorname{ord} \tilde{L}_n, \operatorname{ord} L_n + \operatorname{ord} A(\tilde{L}_n)) \\ &= \max(\operatorname{ord} L_n + 1 - \operatorname{lgt} L_n + \operatorname{ord} \tilde{L}_n, \operatorname{ord} L_n + \operatorname{ord} \tilde{L}_n + 1 - \operatorname{lgt} \tilde{L}_n) \\ &= \operatorname{ord} L_n + \operatorname{ord} \tilde{L}_n + 1 - \min(\operatorname{lgt} L_n, \operatorname{lgt} \tilde{L}_n).\end{aligned}$$

Formula (2.86) implies

$$\operatorname{lgt}(L_n\tilde{L}_n) \geqslant \operatorname{ord}(L_n\tilde{L}_n) + 1 - (\operatorname{ord} L_n + \operatorname{ord}\tilde{L}_n + 1) + \min(\operatorname{lgt} L_n, \operatorname{lgt}\tilde{L}_n).$$

Thus for any formal symmetries L_n and $\tilde{L}_n$ we get

$$\begin{aligned} &\operatorname{ord}(L_n\tilde{L}_n) = \operatorname{ord} L_n + \operatorname{ord}\tilde{L}_n, \\ &\operatorname{lgt}(L_n\tilde{L}_n) \geqslant \min(\operatorname{lgt} L_n, \operatorname{lgt}\tilde{L}_n). \end{aligned} \tag{2.92}$$

In a quite similar way, one can derive from equations (2.69), (2.86) that, for any formal symmetry L_n,

$$\operatorname{ord} L_n^{-1} = -\operatorname{ord} L_n, \qquad \operatorname{lgt} L_n^{-1} = \operatorname{lgt} L_n. \tag{2.93}$$

Taking into account equations (2.92), (2.93) and that $L_n^0 = 1$ is the solution of equation (2.64) of order 0, one gets, for any integer power i of the formal symmetry L_n, the following result:

$$\operatorname{ord} L_n^i = \mathrm{i}\operatorname{ord} L_n, \qquad \operatorname{lgt} L_n^i \geqslant \operatorname{lgt} L_n. \tag{2.94}$$

In the following we will need formal symmetries of order 1. We will make the ansatz, valid in the case of the Volterra equation, that equation (2.1) has two generalized symmetries of the left orders m and $m+1$, with a high enough number m. So, let g_n and $\hat{g}_n$ be the right-hand sides of two generalized symmetries with the left orders $m \geqslant 1$ and $m+1$, respectively. Theorem 5 shows that the Frechét derivative g_n^* is a formal symmetry of order m and $\operatorname{lgt} g_n^* \geqslant m$, and $\hat{g}_n^*$ is such that $\operatorname{ord}\hat{g}_n^* = m+1$, $\operatorname{lgt}\hat{g}_n^* \geqslant m+1$.

We can construct the following series:

$$L_n = \hat{g}_n^*(g_n^*)^{-1}. \tag{2.95}$$

As it follows from (2.92), (2.93), this series will be a formal symmetry of equation (2.1) of order 1 and $\operatorname{lgt} L_n \geqslant m$. This result is formulated in the following theorem.

Theorem 6. *If equation (2.1) possesses two generalized symmetries $u_{n,\tau} = g_n$ and $u_{n,\hat{\tau}} = \hat{g}_n$ of left orders $m \geqslant 1$ and $m+1$, then it possesses a formal symmetry L_n given by formula (2.95), such that* $\operatorname{ord} L_n = 1$ *and* $\operatorname{lgt} L_n \geqslant m$.

A formal symmetry (2.95) of the first order can be written as

$$L_n = l_n^{(1)}T + l_n^{(0)} + l_n^{(-1)}T^{-1} + l_n^{(-2)}T^{-2} + \cdots, \qquad l_n^{(1)} \neq 0. \tag{2.96}$$

Its length $l = \operatorname{lgt} L_n$ can be as high as necessary. This formal symmetry generates a number of conserved densities for equation (2.1) as well as a recursion operator in theorem 4. In fact, if $l \geqslant 3$, then the highest three coefficients of the series (2.83) with $N = 1$ vanish: $a_n^{(2)} = a_n^{(1)} = a_n^{(0)} = 0$. One can show, following theorem 4, that $l_n^{(0)} = \operatorname{res} L_n$ is a conserved density. In the case $l \geqslant 4$, it follows from equation (2.94) that $\operatorname{ord} L_n^2 = 2$ and $\operatorname{lgt} L_n^2 \geqslant 4$. This means that the coefficients at T^i ($\geqslant i \geqslant 0$) of the series $A\left(L_n^2\right)$ are equal to zero, and thus also the function $\operatorname{res} L_n^2$ is a conserved density. In this way we prove the following general statement: if $\operatorname{lgt} L_n \geqslant 3$, then the functions

$$\operatorname{res} L_n^i, \qquad 1 \leqslant i \leqslant \operatorname{lgt} L_n - 2, \tag{2.97}$$

are conserved densities of equation (2.1).

Relation (2.80), which is used to obtain another conserved density, can be written, taking into account equation (2.68), as $A(L_n)L_n^{-1} = 0$. In the case $l \geqslant 2$, one can see from equation (2.86) that $\operatorname{ord} A(L_n) \leqslant 0$, hence $\operatorname{ord}(A(L_n)L_n^{-1}) < 0$. For this reason one can show, following the proof of theorem 4, that the function

$$\log l_n^{(1)} \tag{2.98}$$

is a conserved density. Consequently, we can state the following theorem, which is an analogue of theorem 4.

Theorem 7. *If equation (2.1) has a formal symmetry (2.96) of the first order and if* $\lg L_n \geqslant 2$, *then function (2.98) is one of its conserved densities. If* $\lg L_n \geqslant 3$, *the functions (2.97) are also conserved densities of equation (2.1).*

In particular, starting from two generalized symmetries of orders $m \geqslant 2$ and $m+1$ and using theorems 6 and 7, we can construct $m-1$ conserved densities which, however, may be trivial.

In the following one considers two new theorems, based on theorems 6 and 7, where further integrability conditions are written down. Here, instead of considering the existence of generalized symmetries and the compatibility condition (2.8), we use the Lax equation (2.64) and the existence of a formal symmetry of the first order and of a high enough length (2.96). Such formal symmetry not only makes the calculation much simpler but also provides us with integrability conditions which *a priori* have no dependence on the order m of the generalized symmetry (cf theorem 2 and its proof).

Theorem 8. *If equation (2.1) has a formal symmetry (2.96) of first order and if* $\lg L_n \geqslant 3$, *then it satisfies condition (2.55). Let* $q_n^{(1)}$ *be a function obtained from relation (2.55). Then there exists a function* $q_n^{(2)}$ *of the form (2.11), such that*

$$\dot{p}_n^{(2)} = (T-1)q_n^{(2)} \quad \textit{with} \quad p_n^{(2)} = q_n^{(1)} + \frac{\partial f_n}{\partial u_n}. \tag{2.99}$$

Proof. In the case when $\lg L_n \geqslant 3$, the first three coefficients of the series $A(L_n)$ (2.68), (2.83) with $N = 1$ must be equal to zero: $a_n^{(2)} = a_n^{(1)} = a_n^{(0)} = 0$. This request will give us some equations for the first three coefficients of the formal symmetry L_n: $l_n^{(1)}, l_n^{(0)}, l_n^{(-1)}$. From theorem 7, we also have two conserved densities given by equations (2.98), (2.97) with $i = 1$. These three equations for the coefficients of L_n are the direct analogues of equations (2.48)–(2.50). The first of them, $a_n^{(2)} = 0$, can be written in the form

$$l_n^{(1)} f_{n+1}^{(1)} = f_n^{(1)} l_{n+1}^{(1)}, \tag{2.100}$$

(see equations (2.46), (2.63) for the used notation). Dividing equation (2.100) by $f_n^{(1)} f_{n+1}^{(1)}$ and using (2.14), we are led to the following formula: $l_n^{(1)} = c f_n^{(1)}, c \neq 0 \in \mathbb{C}$.

As the operator A defined by (2.68) is linear, a formal symmetry can be multiplied by any nonzero constant, and the length is not changed. Dividing L_n by c, we obtain, without loss of generality, the following formula for $l_n^{(1)}$:

$$l_n^{(1)} = f_n^{(1)}. \tag{2.101}$$

Theorem 7 guarantees that the function

$$\log l_n^{(1)} = \log f_n^{(1)} = \log \frac{\partial f_n}{\partial u_{n+1}} = p_n^{(1)}$$

is a conserved density of equation (2.1), i.e. condition (2.55) is satisfied.

The second equation, $a_n^{(1)} = 0$, reads

$$\dot{l}_n^{(1)} + l_n^{(1)} f_{n+1}^{(0)} + l_n^{(0)} f_n^{(1)} = f_n^{(0)} l_n^{(1)} + f_n^{(1)} l_{n+1}^{(0)}. \tag{2.102}$$

Dividing equation (2.102) by $f_n^{(1)}$ and using equation (2.101), one obtains

$$\dot{p}_n^{(1)} = (T-1)\big(l_n^{(0)} - f_n^{(0)}\big). \tag{2.103}$$

Comparing equation (2.103) with equation (2.55) and using property (2.14), one gets $q_n^{(1)} = l_n^{(0)} - f_n^{(0)} + \alpha$, where α is a constant. Then one has

$$l_n^{(0)} + \alpha = q_n^{(1)} + f_n^{(0)} = p_n^{(2)}, \tag{2.104}$$

where $p_n^{(2)}$ is the function given by (2.99). Theorem 7 guarantees that the function $\operatorname{res} L_n = l_n^{(0)}$ is a conserved density of equation (2.1), i.e. the function $p_n^{(2)}$ is also a conserved density. □

Theorem 9. *Let equation (2.1) have a formal symmetry (2.96) with length* $\operatorname{lgt} L_n \geqslant 4$. *Let* $q_n^{(1)}$ *be a function defined by (2.55), while* $p_n^{(2)}, q_n^{(2)}$ *be functions given by (2.99). Then there exists a function* $q_n^{(3)}$ *of the form (2.11), such that*

$$\dot{p}_n^{(3)} = (T-1)q_n^{(3)} \quad \mathit{with} \quad p_n^{(3)} = q_n^{(2)} + \frac{1}{2}\left(p_n^{(2)}\right)^2 + \frac{\partial f_n}{\partial u_{n+1}}\frac{\partial f_{n+1}}{\partial u_n}. \tag{2.105}$$

Proof. This proof is a direct continuation of the calculations we did to prove theorem 8. We will need to compute the coefficient $l_n^{(-2)}$ of the formal symmetry L_n. We could consider the equation $a_n^{(-1)} = 0$; however, we prefer to use the conserved density $\operatorname{res} L_n^2$ provided by theorem 7.

Let us write down the equation $a_n^{(0)} = 0$ explicitly:

$$\dot{l}_n^{(0)} + l_n^{(1)} f_{n+1}^{(-1)} + l_n^{(0)} f_n^{(0)} + l_n^{(-1)} f_{n-1}^{(1)} = f_n^{(-1)} l_{n-1}^{(1)} + f_n^{(0)} l_n^{(0)} + f_n^{(1)} l_{n+1}^{(-1)}. \tag{2.106}$$

Using equations (2.101) and (2.104), equation (2.106) can be rewritten as

$$\dot{l}_n^{(0)} = \dot{p}_n^{(2)} = (T-1)\left(f_{n-1}^{(1)}\left(l_n^{(-1)} - f_n^{(-1)}\right)\right).$$

As in the case of (2.103), we can now express $l_n^{(-1)}$ in terms of $q_n^{(2)}$ defined by (2.99):

$$l_n^{(-1)} = \left(q_n^{(2)} + \beta\right) / f_{n-1}^{(1)} + f_n^{(-1)}, \tag{2.107}$$

where β is a constant.

Let us write down the formula for the conserved density $\operatorname{res} L_n^2$. Using the equivalence relation (2.23), we get

$$\operatorname{res} L_n^2 = l_n^{(1)} l_{n+1}^{(-1)} + \left(l_n^{(0)}\right)^2 + l_n^{(-1)} l_{n-1}^{(1)} \sim 2 l_n^{(1)} l_{n+1}^{(-1)} + \left(l_n^{(0)}\right)^2.$$

The densities can be multiplied by a nonzero constant, and taking into account (2.101), (2.104), (2.107) and formula for $p_n^{(3)}$ given by (2.105), one has

$$\begin{aligned}\tfrac{1}{2}\operatorname{res} L_n^2 &\sim q_{n+1}^{(2)} + \beta + f_n^{(1)} f_{n+1}^{(-1)} + \tfrac{1}{2}\left(p_n^{(2)} - \alpha\right)^2 \\ &\sim q_n^{(2)} + \tfrac{1}{2}\left(p_n^{(2)}\right)^2 + f_n^{(1)} f_{n+1}^{(-1)} + \beta - \alpha p_n^{(2)} + \tfrac{1}{2}\alpha^2 = p_n^{(3)} - \alpha p_n^{(2)} + \tfrac{1}{2}\alpha^2 + \beta.\end{aligned}$$

A constant is a trivial conserved density, and conserved densities of equation (2.1) generate a linear space. This is the reason why the function $p_n^{(3)}$ must be a conserved density of equation (2.1), and hence condition (2.105) is satisfied. □

As one can see from equation (2.14), the functions $q_n^{(1)}$ and $q_n^{(2)}$ of conditions (2.55), (2.99) are defined up to arbitrary constants. Therefore, the functions $p_n^{(2)}$ and $p_n^{(3)}$, given by equations (2.99) and (2.105), may depend in general on those constants. As we have shown in the proofs of theorems 8 and 9, $p_n^{(2)}$ and $p_n^{(3)}$ are conserved densities for all values of those constants, and one has no need to take into account those arbitrary constants, when checking the integrability conditions. In another words, checking the integrability conditions (2.55), (2.99), (2.105), any choice of the functions $q_n^{(1)}, q_n^{(2)}$ will give the same result.

While proving theorems 8 and 9, we have given a scheme for deriving the integrability conditions. This scheme corresponds to finding the coefficients of the first-order formal symmetry and to the application of theorem 7. Starting from a formal symmetry of a sufficiently big length, we can obtain as many integrability conditions as necessary, and all those conditions will have the form of local conservation laws.

As in the case of condition (2.55), theorem 3 is essential for checking the integrability conditions (2.99), (2.105) as one does in equations (2.60), (2.61). If, given an equation (2.1), all these integrability conditions are satisfied, we obtain three conserved densities of low orders. In the case of the Volterra equation (2.2), for example, one gets

$$p_n^{(1)} = p_n^1, \qquad p_n^{(2)} \sim 2p_n^2 + c_1, \qquad p_n^{(3)} \sim 2p_n^3 + 2c_1 p_n^2 + \tfrac{1}{2}c_1^2 + c_2,$$

where p_n^1, p_n^2, p_n^3 are the conserved densities given in the list (2.39), with c_1, c_2 arbitrary constants. The conserved densities $p_n^{(1)}, p_n^{(2)}$ and $p_n^{(3)}$ have the orders 0, 0 and 1, respectively.

For any integrable equation of the Volterra type we get from theorem 6 an arbitrarily long formal symmetry of the first order. The coefficients of such formal symmetry and theorem 7 provide us with as many conserved densities as we need. Formulae (2.101), (2.104), (2.107) give the first three coefficients of such formal symmetry in terms of $q_n^{(1)}$ and $q_n^{(2)}$. In the case of the Volterra equation, we have

$$f_n^{(1)} = u_n, \qquad f_n^{(0)} = u_{n+1} - u_{n-1}, \qquad f_n^{(-1)} = -u_n,$$
$$q_n^{(1)} = u_n + u_{n-1} + c_1, \qquad q_n^{(2)} = u_{n+1}u_n + u_n u_{n-1} + c_2,$$

where c_1, c_2 are constants, and denoting $c_3 = c_1 - \alpha$, $c_4 = c_2 + \beta$, we can write down explicitly the first three terms of the formal symmetry:

$$L_n = u_n T + u_{n+1} + u_n + c_3 + \frac{u_{n+1}u_n + c_4}{u_{n-1}} T^{-1} + \cdots$$

(cf this result with the form of the recursion operator (2.74)).

2.4. Formal conserved density

We have obtained in sections 2.2 and 2.3 the integrability conditions (2.55), (2.99), (2.105) which follow from the existence of generalized symmetries. However, in order to carry out the exhaustive classification of integrable equations of the form (2.1), we need some additional integrability conditions which come from the conservation laws[1]. So, starting from the conservation laws, we introduce and discuss in this section the formal conserved densities, in analogy with the formal symmetries, and then derive two new integrability conditions. At the end we will prove a general statement which explains why an equation, possessing a higher order local conservation law, must be in a sense symmetrical.

First of all we introduce and discuss an equation for the variational derivative ϱ_n, defined by equations (2.40), (2.42), of a conserved density p_n of equation (2.1),

$$\left(D_t + f_n^{*\dagger}\right)\varrho_n = 0. \qquad (2.108)$$

This is the analogue of equation (2.71). If f_n^*, given by equation (2.63), is the Frechét derivative of f_n, the operator $f_n^{*\dagger}$ is its adjoint operator defined by

$$(a_n T^i)^\dagger = T^{-i} \circ a_n = a_{n-i} T^{-i}. \qquad (2.109)$$

Then

$$f_n^{*\dagger} = T^{-1} \circ f_n^{(1)} + f_n^{(0)} + T \circ f_n^{(-1)} = f_{n+1}^{(-1)} T + f_n^{(0)} + f_{n-1}^{(1)} T^{-1}. \qquad (2.110)$$

[1] In the case of equations (1.1), the classification problem can be solved without using this kind of integrability conditions (see e.g. the review [39]). Such conditions help to make the problem easier and lead to a shorter list of equations. In the case of the lattice equations (2.1), these additional integrability conditions seem to be necessary to solve the problem.

Taking into account equations (2.46), equation (2.110) can be rewritten as

$$f_n^{*\dagger} = \sum_{i=-1}^{1} f_{n+i}^{(-i)} T^i = \sum_{i=-1}^{1} \frac{\partial f_{n+i}}{\partial u_n} T^i. \tag{2.111}$$

Then we can prove the following theorem.

Theorem 10. *For any conserved density p_n of equation (2.1), its variational derivative ϱ_n satisfies equation (2.108).*

Proof. Let p_n be a conserved density of equation (2.1), then

$$\dot{p}_n = \sum_i \frac{\partial p_n}{\partial u_{n+i}} f_{n+i} \sim \sum_i \frac{\partial p_{n-i}}{\partial u_n} f_n = \varrho_n f_n \sim 0, \tag{2.112}$$

where we have used definition (2.40). On the other hand,

$$\frac{\partial \varrho_n}{\partial u_{n+i}} = \frac{\partial}{\partial u_{n+i}} \sum_j \frac{\partial p_{n+j}}{\partial u_n} = \frac{\partial}{\partial u_n} \sum_j \frac{\partial p_{n+j}}{\partial u_{n+i}} = \frac{\partial}{\partial u_n} T^i \sum_\sigma \frac{\partial p_{n+\sigma}}{\partial u_n},$$

i.e. we have

$$\frac{\partial \varrho_n}{\partial u_{n+i}} = \frac{\partial \varrho_{n+i}}{\partial u_n} \qquad \text{for any} \qquad i \in \mathbb{Z}. \tag{2.113}$$

Using theorem 3 together with relation (2.112), we get

$$\frac{\delta \dot{p}_n}{\delta u_n} = \frac{\delta(\varrho_n f_n)}{\delta u_n} = 0. \tag{2.114}$$

Moreover, using equations (2.111), (2.113), we obtain

$$\begin{aligned}\frac{\delta(\varrho_n f_n)}{\delta u_n} &= \sum_i \frac{\partial \varrho_{n+i}}{\partial u_n} f_{n+i} + \sum_i \varrho_{n+i} \frac{\partial f_{n+i}}{\partial u_n} \\ &= \sum_i \frac{\partial \varrho_n}{\partial u_{n+i}} f_{n+i} + \sum_i \frac{\partial f_{n+i}}{\partial u_n} \varrho_{n+i} = \left(D_t + f_n^{*\dagger}\right)\varrho_n.\end{aligned}$$

This formula together with equation (2.114) imply equation (2.108). □

The following equation, analogous to equation (2.64), plays the main role in this section:

$$\dot{S}_n + S_n f_n^* + f_n^{*\dagger} S_n = 0. \tag{2.115}$$

Here S_n is a formal series of the same type as L_n (2.65):

$$S_n = \sum_{i=-\infty}^{M} s_n^{(i)} T^i, \qquad s_n^{(M)} \neq 0, \tag{2.116}$$

and $s_n^{(i)}$ are functions of the form (2.11). Examples of exact solutions of equation (2.115) will be given in section 2.6. We know that the exact solution L_n of equation (2.64) is the recursion operator for equation (2.1). The solution S_n of equation (2.115) is the inverse of a Noether or Hamiltonian operator. The details of this statement will be discussed in section 2.6.

Equations (2.64), (2.115) are closely related. Let us introduce an operator B, such that

$$B(S_n) = \dot{S}_n + S_n f_n^* + f_n^{*\dagger} S_n. \tag{2.117}$$

For any formal series L_n, S_n, $\tilde{S}_n$ of the form (2.65), (2.116), the following identities take place:

$$B(S_n L_n) = B(S_n) L_n + S_n A(L_n), \tag{2.118}$$

$$A\left(S_n^{-1}\tilde{S}_n\right) = S_n^{-1}B(\tilde{S}_n) - S_n^{-1}B(S_n)S_n^{-1}\tilde{S}_n. \tag{2.119}$$

Identity (2.118) shows that, for any solutions S_n of equation (2.115) and L_n of equation (2.64), the series S_nL_n will be a new solution of equation (2.115). On the other hand, as it follows from equation (2.119), if S_n and $\tilde{S}_n$ are any two solutions of equation (2.115), then the series $S_n^{-1}\tilde{S}_n$ satisfies equation (2.64).

As in the case of equation (2.64), we are interested here in approximate solutions of equation (2.115). Such solutions will be called formal conserved densities (see definition 6 below) because of their close connection with conserved densities, which is shown in theorem 11 below.

For the formal series $B(S_n)$, defined by equation (2.117), we obtain in general

$$B(S_n) = b_n^{(M+1)}T^{M+1} + b_n^{(M)}T^M + b_n^{(M-1)}T^{M-1} + \cdots, \tag{2.120}$$

where S_n is a series of the form (2.116) and thus has the order M.

Definition 6. *If a series S_n (2.116) is such that the first $l \geqslant 1$ coefficients of the series $B(S_n)$ (2.120) vanish, i.e.*

$$B(S_n) = b_n^{(M+1-l)}T^{M+1-l} + b_n^{(M-l)}T^{M-l} + \cdots, \qquad b_n^{(M+1-l)} \neq 0, \tag{2.121}$$

then S_n is called a formal conserved density *of equation (2.1) of* the order *M and* the length *l, and we will write* $\operatorname{ord} S_n = M$, $\operatorname{lgt} S_n = l$.

Comparing equations (2.120), (2.121), we easily obtain the following formula:

$$\operatorname{lgt} S_n = \operatorname{ord} S_n + 1 - \operatorname{ord} B(S_n) \tag{2.122}$$

relating the length and the order of formal conserved density S_n. This is the analogue of equation (2.86) which we obtained in the case of formal symmetries.

Theorem 11. *If equation (2.1) possesses a conserved density p_n of order $m \geqslant 2$, then it has a formal conserved density S_n, such that* $\operatorname{ord} S_n = m$ *and* $\operatorname{lgt} S_n \geqslant m-1$. *This formal conserved density S_n is given by the formula*

$$S_n = \varrho_n^* + \sum_{i=-\infty}^{-m-1} 0T^i, \qquad \varrho_n = \frac{\delta p_n}{\delta u_n}, \tag{2.123}$$

where ϱ_n has the form (2.45), and thus ϱ_n^* is given by

$$\varrho_n^* = \sum_{i=-m}^{m} \frac{\partial \varrho_n}{\partial u_{n+i}} T^i. \tag{2.124}$$

Proof. Theorem 10 allows us to pass from a conserved density p_n to equation (2.108). Then we apply the Frechét derivative to both sides of this equation. Using the formula (2.111) and definition (2.62), we check that

$$\begin{aligned}\left(f_n^{*\dagger}\varrho_n\right)^* &= \left(\sum_{i=-1}^{1} \frac{\partial f_{n+i}}{\partial u_n}\varrho_{n+i}\right)^* = \sum_{i,j} \frac{\partial^2 f_{n+i}}{\partial u_n \partial u_{n+j}}\varrho_{n+i}T^j + \sum_{i,j} \frac{\partial f_{n+i}}{\partial u_n}\frac{\partial \varrho_{n+i}}{\partial u_{n+j}}T^j \\ &= \sum_j \left(\sum_i \frac{\partial^2 f_{n+i}}{\partial u_n \partial u_{n+j}}\varrho_{n+i}\right)T^j + \left(\sum_{i=-1}^{1} \frac{\partial f_{n+i}}{\partial u_n}T^i\right)\left(\sum_{\sigma=-m}^{m} \frac{\partial \varrho_n}{\partial u_{n+\sigma}}T^\sigma\right),\end{aligned}$$

where $\sigma = j - i$. The first term has the form $\sum_{j=-2}^{2} c_n^{(j)} T^j$, as $\frac{\partial^2 f_{n+i}}{\partial u_n \partial u_{n+j}} = 0$ for $j > 2$ and $j < -2$ (see equation (2.1)). The second term in the last expression is equal to $f_n^{*\dagger}\varrho_n^*$ (see equation (2.124)). Hence one is led to the following result:

$$\left(f_n^{*\dagger}\varrho_n\right)^* = f_n^{*\dagger}\varrho_n^* + \sum_{j=-2}^{2} c_n^{(j)} T^j. \tag{2.125}$$

On the other hand,

$$(D_t\varrho_n)^* = \dot{\varrho}_n^* + \varrho_n^* f_n^* \tag{2.126}$$

(cf equation (2.89)). Now relations (2.125), (2.126) together with equations (2.108), (2.117) imply

$$B(\varrho_n^*) = -\sum_{j=-2}^{2} c_n^{(j)} T^j. \tag{2.127}$$

Introducing the series (2.123), i.e. $S_n = \varrho_n^*$, we see that ord $S_n = m$ due to equations (2.45), (2.124). Formula (2.127) provides the inequality ord $B(S_n) \leqslant 2$, and equation (2.122) implies that lgt $S_n \geqslant m - 1$. This means that the formal series S_n is a formal conserved density. □

From relations (2.118), (2.119) it follows that there is the same connection between formal symmetries and formal conserved densities of equation (2.1) as in the case of exact solutions of the Lax equation (2.64) and equation (2.115). Two formal conserved densities S_n and $\tilde{S}_n$ give a formal symmetry $L_n = S_n^{-1}\tilde{S}_n$. Formal conserved density S_n together with formal symmetry L_n generate another formal conserved density $\tilde{S}_n = S_n L_n$. The orders and lengths of the resulting formal symmetries and conserved densities can easily be found, using formulae (2.118), (2.119), (2.86) and (2.122). For example, the following analogues of relations (2.92) take place:

$$\begin{aligned} &\operatorname{ord}(S_n L_n) = \operatorname{ord} S_n + \operatorname{ord} L_n, \\ &\operatorname{lgt}(S_n L_n) \geqslant \min(\operatorname{lgt} S_n, \operatorname{lgt} L_n). \end{aligned} \tag{2.128}$$

Let us consider a formal conserved density S_n (2.123) and a first-order formal symmetry L_n (2.96) such that lgt $L_n \geqslant$ lgt S_n. We can consider a new formal conserved density $\hat{S}_n = S_n L_n^i$. Its length will satisfy the inequality lgt $\hat{S}_n \geqslant$ lgt S_n, as it follows from equations (2.94), (2.128). In this way, we can obtain a formal conserved density S_n which has the order 1 or 0 and an arbitrarily big length. This provides a simple calculation of the coefficients of S_n and an easy derivation of additional integrability conditions (cf theorems 8, 9).

However, for the classification of equations (2.1), we need only two additional integrability conditions. These can be obtained, using just one conservation law of the order $m \geqslant 3$. More precisely, we can and shall derive those integrability conditions using equation (2.115) and one formal conserved density S_n of the order m and lgt $S_n \geqslant m - 1$ obtained from theorem 11.

Theorem 12. *Let equation (2.1) have a conservation law of the order $m \geqslant 3$ and a generalized symmetry of the order $m \geqslant 2$. Then there will exist functions $\sigma_n^{(1)}$ and $\sigma_n^{(2)}$ of the form (2.11) which satisfy the following relations:*

$$r_n^{(1)} = (T-1)\sigma_n^{(1)}, \qquad r_n^{(1)} = \log\left(-\frac{\partial f_n}{\partial u_{n+1}} \bigg/ \frac{\partial f_n}{\partial u_{n-1}}\right), \tag{2.129}$$

$$r_n^{(2)} = (T-1)\sigma_n^{(2)}, \qquad r_n^{(2)} = \dot{\sigma}_n^{(1)} + 2\frac{\partial f_n}{\partial u_n}. \tag{2.130}$$

Proof. Theorem 2 has shown that the existence of a generalized symmetry of order $m \geqslant 2$ guarantees that condition (2.55) is satisfied, i.e. $\dot{p}_n^{(1)} \sim 0$. According to theorem 11, a conservation law of order $m \geqslant 3$ implies the existence of a formal conserved density S_n of the form (2.116) of order $M = m$ and length l, such that $l \geqslant m-1 \geqslant 2$. From equation (2.120) and definition 6 it follows that we have $b_n^{(m+1)} = b_n^{(m)} = 0$.

Using formulae (2.63), (2.110), (2.117), the equation $b_n^{(m+1)} = 0$ is written as

$$s_n^{(m)} f_{n+m}^{(1)} + f_{n+1}^{(-1)} s_{n+1}^{(m)} = 0, \tag{2.131}$$

where $f_n^{(1)} f_n^{(-1)} s_n^{(m)} \neq 0$ due to equations (2.1), (2.116). Applying the operator T^{-1} and then dividing by $s_{n-1}^{(m)} f_n^{(-1)}$, one gets

$$\frac{s_n^{(m)}}{s_{n-1}^{(m)}} = -\frac{f_{n+m-1}^{(1)}}{f_n^{(-1)}} = -\frac{f_n^{(1)}}{f_n^{(-1)}} \frac{\Phi_{n+1}^{(m-2)}}{\Phi_n^{(m-2)}},$$

where $\Phi_n^{(N)}$ is defined by equation (2.51). Applying the logarithm to both sides of this relation, one obtains the condition

$$r_n^{(1)} = \log\left(-f_n^{(1)}/f_n^{(-1)}\right) = (T-1)\left(\log s_{n-1}^{(m)} - \log \Phi_n^{(m-2)}\right).$$

It is easy to see now that there will exist a function $\sigma_n^{(1)}$ satisfying condition (2.129), with general form $\sigma_n^{(1)} = c + \log\left(s_{n-1}^{(m)}/\Phi_n^{(m-2)}\right)$, where c is a constant. The function $s_n^{(m)}$ can be expressed in terms of $\sigma_n^{(1)}$:

$$s_n^{(m)} = \Phi_{n+1}^{(m-2)} \mathrm{e}^{\sigma_{n+1}^{(1)} - c}. \tag{2.132}$$

The equation $b_n^{(m)} = 0$ reads

$$\dot{s}_n^{(m)} + s_n^{(m)} f_{n+m}^{(0)} + s_n^{(m-1)} f_{n+m-1}^{(1)} + f_{n+1}^{(-1)} s_{n+1}^{(m-1)} + f_n^{(0)} s_n^{(m)} = 0.$$

We exclude $f_{n+1}^{(-1)}$, using equation (2.131) and then divide the result by $s_n^{(m)}$:

$$D_t \log s_n^{(m)} + f_{n+m}^{(0)} + f_n^{(0)} + (1-T) \frac{s_n^{(m-1)} f_{n+m-1}^{(1)}}{s_n^{(m)}} = 0. \tag{2.133}$$

Formulae (2.51), (2.132) and property (2.23) allow one to check that

$$\begin{aligned} D_t \log s_n^{(m)} &= D_t \log \Phi_{n+1}^{(m-2)} + \dot{\sigma}_{n+1}^{(1)} = \sum_{i=1}^{m-1} D_t \log f_{n+i}^{(1)} + \dot{\sigma}_{n+1}^{(1)} \\ &\sim (m-1) D_t \log f_n^{(1)} + \dot{\sigma}_n^{(1)} = (m-1)\dot{p}_n^{(1)} + \dot{\sigma}_n^{(1)}, \end{aligned}$$

where $p_n^{(1)}$ is the function defined by equation (2.55). As $\dot{p}_n^{(1)} \sim 0$, then $D_t \log s_n^{(m)} \sim \dot{\sigma}_n^{(1)}$, and hence equation (2.133) can be rewritten as $\dot{\sigma}_n^{(1)} + 2 f_n^{(0)} \sim 0$. Consequently, the second point of theorem 12 has been proved. □

Using equation (2.115), we can derive arbitrarily many integrability conditions analogous to equations (2.129), (2.130). In the general case, it is easier to check these integrability conditions applying theorem 3. However, we do not need this theorem in the simple case of the Volterra equation (2.2). In fact, $f_n = u_n(u_{n+1} - u_{n-1})$, hence $r_n^{(1)} = 0$, i.e. condition (2.129) is trivially satisfied. Moreover, $\sigma_n^{(1)}$ is a constant function and thus $r_n^{(2)} = 2(u_{n+1} - u_{n-1}) \sim 0$.

2.4.1. Why integrable equations on the lattice are symmetrical. From the results presented up to now, we can obtain a theorem which explains why only symmetrical equations may possess higher order conservation laws. Then we will illustrate the result by the example of the discrete Burgers equation.

Here we will consider n- and t-independent equations of the following very general form:

$$\dot{u}_n = f_n = f(u_{n+N}, u_{n+N-1}, \ldots, u_{n+M}), \tag{2.134}$$

$$N \geqslant M, \qquad \frac{\partial f_n}{\partial u_{n+N}} \frac{\partial f_n}{\partial u_{n+M}} \neq 0, \tag{2.135}$$

where for a given equation N and M are fixed integers. The definitions of conservation laws, conserved densities and their orders are given by definitions 2 and 4. Formulae (2.42)–(2.45) will provide us with the way of finding orders also in this case.

In a quite similar way, we can prove the following analogue of theorem 10: if p_n is a conserved density of equation (2.134), then its variational derivative ϱ_n satisfies equation (2.108), where

$$D_t = \sum_i f_{n+i} \frac{\partial}{\partial u_{n+i}}, \qquad f_n^{*\dagger} = \sum_i \frac{\partial f_{n+i}}{\partial u_n} T^i.$$

As we will consider a conserved density p_n of the order $m > 0$, formula (2.45) for ϱ_n is valid, and we can rewrite equation (2.108) as

$$\sum_{i=-m}^{m} \frac{\partial \varrho_n}{\partial u_{n+i}} f_{n+i} + \sum_{i=-N}^{-M} \frac{\partial f_{n+i}}{\partial u_n} \varrho_{n+i} = 0. \tag{2.136}$$

We can now formulate and prove the following theorem.

Theorem 13. *If an equation of the form (2.134), (2.135) possesses a conservation law of order m, such that*

$$m > \min(|N|, |M|), \tag{2.137}$$

then $N = -M$ *and* $N \geqslant 0$.

Proof. As we have said before, if $m > 0$, we can define the variational derivative ϱ_n of the conserved density p_n which will satisfy equations (2.45), (2.136).

The following table will be helpful:

$$\begin{array}{llll}
\dfrac{\partial}{\partial u_{n+j}} \dfrac{\partial \varrho_n}{\partial u_{n+i}} = 0 & \text{for} \quad j < -m & \text{or} & j > m \\[2ex]
\dfrac{\partial}{\partial u_{n+j}} \dfrac{\partial f_{n+i}}{\partial u_n} = 0 & \text{for} \quad j < M - N & \text{or} & j > N - M \\[2ex]
\dfrac{\partial \varrho_n}{\partial u_{n+i}} \dfrac{\partial f_{n+i}}{\partial u_{n+j}} = 0 & \text{for} \quad j < M - m & \text{or} & j > N + m \\[2ex]
\dfrac{\partial f_{n+i}}{\partial u_n} \dfrac{\partial \varrho_{n+i}}{\partial u_{n+j}} = 0 & \text{for} \quad j < -m - N & \text{or} & j > m - M.
\end{array} \tag{2.138}$$

This result obtained, using only the formulae (2.45), (2.134)), does not depend on the number i. We will need to take into account the table when we will differentiate equation (2.136) with respect to u_{n+j}.

The proof of this theorem is based on the formulation of two conditions which will lead to a contradiction. The first condition means that m, N, M satisfy the following inequalities:

$$N > 0, \qquad m > -M, \qquad N > -M. \tag{2.139}$$

In this case, differentiating (2.136) with respect to u_{n+N+m} and using (2.138), we obtain

$$\frac{\partial}{\partial u_{n+N+m}}\left(\frac{\partial \varrho_n}{\partial u_{n+m}} f_{n+m}\right) = \frac{\partial \varrho_n}{\partial u_{n+m}} \frac{\partial f_{n+m}}{\partial u_{n+N+m}} = \frac{\partial \varrho_n}{\partial u_{n+m}} T^m \frac{\partial f_n}{\partial u_{n+N}} = 0.$$

This result is in contradiction with conditions (2.45), (2.135). The situation is quite similar in the second case, when

$$-M > 0, \qquad -M > N, \qquad m > N. \tag{2.140}$$

Here we can differentiate (2.136) with respect to u_{n+m-M} and are led to

$$\frac{\partial}{\partial u_{n+m-M}}\left(\frac{\partial f_{n-M}}{\partial u_n} \varrho_{n-M}\right) = \frac{\partial f_{n-M}}{\partial u_n} \frac{\partial \varrho_{n-M}}{\partial u_{n+m-M}} = T^{-M}\left(\frac{\partial f_n}{\partial u_{n+M}} \frac{\partial \varrho_n}{\partial u_{n+m}}\right) = 0.$$

Also in this case the result is again in contradiction with equations (2.45), (2.135).

Now we are going to prove that we must have $N = -M$, considering the following two possible cases:

$$\begin{aligned} &\textit{Case 1}: \quad N > -M \\ &\textit{Case 2}: \quad -M > N \end{aligned}$$

and using the results just obtained. Case 1 is compatible with (2.139). As $N \geqslant M$, we have

$$N > -M \geqslant -N, \qquad N \geqslant M > -N, \tag{2.141}$$

and hence $N > 0$. This result together with equation (2.141) imply $|N| = N \geqslant |M| \geqslant -M$. Now, using condition (2.137), we obtain the following result: $m > -M$. So, condition (2.139) must take place, i.e. case 1 is impossible. Case 2 is considered in a very similar way. As $N \geqslant M$, then

$$-M > N \geqslant M, \qquad -M \geqslant -N > M,$$

hence $-M > 0$. Now $|M| = -M \geqslant |N| \geqslant N$, and therefore $m > N$ due to equation (2.137). So, condition (2.140) has been obtained, and thus case 2 is also impossible. Consequently, $N = -M$. As $N \geqslant M$, we also have $N \geqslant 0$. □

In all known integrable cases, the generalized symmetry and corresponding equation of the form (2.1) possess the same infinite hierarchy of conservation laws. That is why generalized symmetries of equations (2.1) are symmetrical in the sense of theorem 13.

Let us discuss the example of the discrete Burgers equation which is not symmetrical and is linearizable:

$$\dot{u}_n = u_n(u_{n+1} - u_n). \tag{2.142}$$

This equation has an infinite hierarchy of generalized symmetries, but no local conservation laws of a positive order. It can be obtained from the linear equation

$$\dot{v}_n = v_{n+1} \tag{2.143}$$

by the transformation

$$u_n = v_{n+1}/v_n. \tag{2.144}$$

For any integer k, equations of the form

$$v_{n,t_k} = v_{n+k} \tag{2.145}$$

are compatible with equation (2.143), i.e. are its symmetries. The transformation (2.144) allows one to obtain from equations (2.145) symmetries for (2.142). In the case $k = 0$, we have a trivial symmetry: $u_{n,t_0} = 0$. If $k > 1$ or $k < 0$, we are led to the nontrivial ones. For $k = 2$ and $k = -1$ we get, for example,

$$u_{n,t_2} = u_n u_{n+1}(u_{n+2} - u_n), \qquad u_{n,t_{-1}} = 1 - u_n/u_{n-1}. \tag{2.146}$$

The reader can check that these equations are generalized symmetries of the discrete Burgers equation. According to theorem 13, equation (2.142), as well as the generalized symmetries (2.146), cannot have conservation laws of the order $m > 0$. The function $\log u_n$ is the conserved density of all equations (2.142), (2.146), e.g.

$$(\log u_n)_{t_{-1}} = 1/u_n - 1/u_{n-1} \sim 0,$$

but it is of order $m = 0$. In the case of the linear equations (2.145) with $k \neq 0$, theorem 13 only guarantees that there is no conservation law of the order $m > |k|$.

2.5. *Discussion of the integrability conditions*

Here in the following, we will discuss some properties of the integrability conditions (2.55), (2.99), (2.105), (2.129), (2.130). More precisely, we will see

- how to derive the integrability conditions, starting from the existence of two conservation laws without using the generalized symmetries,
- how to obtain an explicit form of the integrability conditions convenient for testing the integrability of a given equation,
- when the integrability conditions (2.55), (2.99), (2.105) allow one to construct nontrivial conservation laws and
- the problem of the left and right orders of the generalized symmetry, and one more set of integrability conditions.

2.5.1. Derivation of integrability conditions from the existence of conservation laws. As it will be explained in section 3.1, conditions (2.55), (2.129), (2.130) are sufficient to provide an exhaustive classification of integrable equations of the form (2.1). The other conditions (2.99), (2.105) are automatically satisfied by all the equations of the resulting list and will be used for the construction of conservation laws for equations of the list.

Let us consider the three integrability conditions (2.55), (2.129), (2.130) derived in theorems 2 and 12, starting from the existence of one generalized symmetry of the order $m \geqslant 2$ and one conservation law of the order $m \geqslant 3$. Now, instead of that, we require the existence of two conservation laws of orders m_1 and m_2: $m_1 > m_2 \geqslant 3$.

In accordance with theorem 11, from these conservation laws we can obtain two formal conserved densities S_n and $\tilde{S}_n$, such that

$$\operatorname{ord} S_n < \operatorname{ord} \tilde{S}_n, \qquad \operatorname{lgt} S_n \geqslant 2, \qquad \operatorname{lgt} \tilde{S}_n \geqslant 2.$$

Using relations (2.70) and (2.119), we can pass from these formal conserved densities S_n and $\tilde{S}_n$ to the following formal symmetry:

$$L_n = \left(S_n^{-1} \tilde{S}_n\right)^2. \tag{2.147}$$

This formal symmetry L_n will be such that $\operatorname{ord} L_n \geqslant 2$ and $\operatorname{lgt} L_n \geqslant 2$, as we have (cf (2.92), (2.94), (2.128)):

$$\operatorname{ord} L_n = 2(\operatorname{ord} \tilde{S}_n - \operatorname{ord} S_n), \qquad \operatorname{lgt} L_n \geqslant \min(\operatorname{lgt} S_n, \operatorname{lgt} \tilde{S}_n).$$

The series L_n has the form (2.65) with $N \geqslant 2$. As $\mathrm{lgt}\, L_n \geqslant 2$, we obtain from equation (2.64) the following system of equations for the coefficients $l_n^{(N)}, l_n^{(N-1)}$:

$$\begin{aligned} &l_n^{(N)} f_{n+N}^{(1)} = l_{n+1}^{(N)} f_n^{(1)}, \\ &\dot{l}_n^{(N)} + l_n^{(N)} f_{n+N}^{(0)} + l_n^{(N-1)} f_{n+N-1}^{(1)} = l_n^{(N)} f_n^{(0)} + l_{n+1}^{(N-1)} f_n^{(1)}. \end{aligned} \tag{2.148}$$

The integrability condition (2.55) has been derived in section 2.2 from equation (2.47) for the generalized symmetry (2.3) of equation (2.1). However, the proof of theorem 2 uses only the system of equations (2.48), (2.49) for the functions $g_n^{(m)}$ and $g_n^{(m-1)}$, which follows from the compatibility condition (2.47). As the structure of two systems (2.148) and (2.48), (2.49) is the same, one can derive the integrability condition (2.55) from the system (2.148). The proof of theorem 12 uses condition (2.55) instead of the existence of a generalized symmetry of the order $m \geqslant 2$. This means that we can write down an obvious modification of theorem 12 in order to obtain conditions (2.129), (2.130). So, we are led to the following result.

Theorem 14. *If equation (2.1) has two conservation laws with orders $m_1 > m_2 \geqslant 3$, then it satisfies the integrability conditions (2.55), (2.129), (2.130).*

One further result of this kind can be obtained with a reasoning similar to that used to prove theorem 14. Let us make the ansatz, valid in the case of the Volterra equation, that there are two conservation laws of orders m and $m + 1$, with $m \geqslant 5$. In this case, in accordance with theorem 11, we can pass to a pair of formal conserved densities S_n and $\tilde{S}_n$:

$$\mathrm{ord}\, S_n = m, \qquad \mathrm{ord}\, \tilde{S}_n = m + 1, \qquad \mathrm{lgt}\, S_n \geqslant 4, \qquad \mathrm{lgt}\, \tilde{S}_n \geqslant 5.$$

Therefore, a formal symmetry L_n, given by $L_n = S_n^{-1} \tilde{S}_n$, is such that $\mathrm{ord}\, L_n = 1$ and $\mathrm{lgt}\, L_n \geqslant 4$. Theorems 8 and 9 provide us with three integrability conditions. Using theorem 12, we can derive two other conditions. More precisely, the following result takes place: if equation (2.1) possesses two conservation laws of the orders $m \geqslant 5$ and $m + 1$, then this equation satisfies the five integrability conditions (2.55), (2.99), (2.105), (2.129) and (2.130).

2.5.2. Explicit form of the integrability conditions. It will be proved in section 3.1 that the three integrability conditions (2.55), (2.129), (2.130), which can be written in the form

$$\dot{p}_n^{(1)} \sim 0, \qquad r_n^{(1)} \sim 0, \qquad r_n^{(2)} \sim 0, \tag{2.149}$$

are not only necessary but also sufficient for the integrability of an equation of the form (2.1). For this reason, conditions (2.149) can be used for testing the integrability of a given equation. To be able to do so, we rewrite these conditions in an explicit form.

The first two conditions (2.149) are explicit, as the functions $\dot{p}_n^{(1)}$ and $r_n^{(1)}$, given by equations (2.55), (2.129), are explicitly defined in terms of the right-hand side of equation (2.1). One can easily rewrite $r_n^{(2)}$, given by equation (2.130), in an explicit form. In fact, as it follows from equation (2.129), the function $\sigma_n^{(1)}$ may only depend on the variables u_n and u_{n-1}, as it is defined by the relation $\sigma_{n+1}^{(1)} - \sigma_n^{(1)} = r_n^{(1)}$. Differentiating it with respect to u_{n+1} and u_{n-1}, we can find the partial derivatives $\frac{\partial \sigma_n^{(1)}}{\partial u_n}, \frac{\partial \sigma_n^{(1)}}{\partial u_{n-1}}$ and then rewrite the time derivative

$$\dot{\sigma}_n^{(1)} = \frac{\partial \sigma_n^{(1)}}{\partial u_n} f_n + \frac{\partial \sigma_n^{(1)}}{\partial u_{n-1}} f_{n-1}$$

in the definition of $r_n^{(2)}$. Thus we get the following explicit expression for $r_n^{(2)}$:

$$r_n^{(2)} = \frac{\partial r_{n-1}^{(1)}}{\partial u_n} f_n - \frac{\partial r_n^{(1)}}{\partial u_{n-1}} f_{n-1} + 2\frac{\partial f_n}{\partial u_n}. \tag{2.150}$$

One can also use the following form of conditions (2.149): $\frac{\delta p_n^{(1)}}{\delta u_n} = 0$, $\frac{\delta r_n^{(1)}}{\delta u_n} = 0$, $\frac{\delta r_n^{(2)}}{\delta u_n} = 0$, as it has been proved in theorem 3 that these two forms of conditions are equivalent up to some integration constants.

2.5.3. Construction of conservation laws from the integrability conditions. The integrability conditions (2.55), (2.99), (2.105) provide an easy way for constructing conservation laws for integrable equations (2.1), the complete list of which will be presented in section 3.1.2. We explain in this section why, for most of those equations, these three conservation laws are nontrivial.

In the case of the Volterra equation (2.2), such conservation laws have low orders 0, 0, 1, as has been shown in section 2.3. In the case of the equation

$$\dot{u}_n = (u_{n+1} - u_n)^{1/2}(u_n - u_{n-1})^{1/2}, \tag{2.151}$$

all these three conservation laws are trivial. In fact, introducing the function $w_n = u_n - u_{n-1}$, we obtain for instance that

$$p_n^{(1)} = \log\tfrac{1}{2} - \tfrac{1}{2}(T-1)\log w_n, \qquad p_n^{(2)} = c - \tfrac{1}{2}(T-1)\tfrac{w_{n-1}^{1/2}}{w_n^{1/2}},$$

where c is a constant, $p_n^{(1)}$ and $p_n^{(2)}$ are given by (2.55), (2.99).

However, if an equation satisfies the condition

$$\Theta_n = \frac{\partial^2 p_n^{(1)}}{\partial u_{n+1}\partial u_{n-1}} \neq 0, \tag{2.152}$$

then we prove in theorem 15 that the conserved densities $p_n^{(1)}, p_n^{(2)}, p_n^{(3)}$ defined by (2.55), (2.99), (2.105) are nontrivial and have rather high orders 2, 3, 4. One can easily check that for most integrable equations, presented in section 3.1.2, condition (2.152) is satisfied. This condition means that the conserved density $p_n^{(1)}$ is of order 2, as $p_n^{(1)} \sim p_{n+1}^{(1)} = \wp(u_{n+2}, u_{n+1}, u_n)$.

Theorem 15. *Let us assume that an equation of the form (2.1) satisfies the integrability conditions (2.55), (2.99), (2.105), and that the conserved density $p_n^{(1)}$ is of order 2, i.e. condition (2.152) is satisfied. Then the conserved densities $p_n^{(2)}$ and $p_n^{(3)}$ are of orders 3 and 4, respectively.*

Proof. We shall use here, in addition to (2.152), the following condition: $f_n^{(-1)} \neq 0$, which is obtained from equations (2.1), (2.46). First we obtain some information on $q_n^{(1)}$ and $q_n^{(2)}$, using relations (2.55), (2.99). The function $q_n^{(1)}$ may depend on the variables $u_{n+1}, u_n, u_{n-1}, u_{n-2}$ only, and one has

$$\frac{\partial q_n^{(1)}}{\partial u_{n-2}} = -\frac{\partial p_n^{(1)}}{\partial u_{n-1}} f_{n-1}^{(-1)}. \tag{2.153}$$

It is easy to see that $p_n^{(2)}$ must depend on the same variables. Then $q_n^{(2)}$ may only depend on $u_{n+1}, u_n, \ldots, u_{n-3}$, and due to equations (2.99), (2.153) we have:

$$\frac{\partial q_n^{(2)}}{\partial u_{n-3}} = -\frac{\partial p_n^{(2)}}{\partial u_{n-2}} f_{n-2}^{(-1)} = -\frac{\partial q_n^{(1)}}{\partial u_{n-2}} f_{n-2}^{(-1)} = \frac{\partial p_n^{(1)}}{\partial u_{n-1}} f_{n-1}^{(-1)} f_{n-2}^{(-1)}. \tag{2.154}$$

Using relations (2.99), (2.152), (2.153), one can show that

$$\frac{\partial^2 p_n^{(2)}}{\partial u_{n+1}\partial u_{n-2}} = \frac{\partial^2 q_n^{(1)}}{\partial u_{n+1}\partial u_{n-2}} = -\Theta_n f_{n-1}^{(-1)} \neq 0,$$

i.e. the density $p_n^{(2)}$ has order 3. The conserved density $p_n^{(3)}$, given by equation (2.105), has the following structure:

$$p_n^{(3)} \sim q_n^{(2)} + \tfrac{1}{2}\big(p_n^{(2)}\big)^2 + f_{n-1}^{(1)} f_n^{(-1)} = \wp_n = \wp(u_{n+1}, u_n, \ldots, u_{n-3}).$$

Moreover, we derive from equations (2.152), (2.154) that

$$\frac{\partial^2 \wp_n}{\partial u_{n+1}\partial u_{n-3}} = \frac{\partial^2 q_n^{(2)}}{\partial u_{n+1}\partial u_{n-3}} = \Theta_n f_{n-1}^{(-1)} f_{n-2}^{(-1)} \neq 0,$$

and therefore $p_n^{(3)}$ has order 4. □

We illustrate theorem 15, considering the equation

$$\dot{u}_n = (u_{n+1} - u_{n-1})^{-1}, \tag{2.155}$$

which satisfies all five integrability conditions. Introducing the function $w_n = u_{n+1} - u_{n-1}$, one can check that $p_n^{(1)} = \log\big(-w_n^{-2}\big)$, $\Theta_n = -2w_n^{-2}$, and condition (2.152) is satisfied. We then easily find that

$$p_n^{(2)} \sim c_1 - 2w_{n+1}^{-1} w_n^{-1},$$
$$p_n^{(3)} \sim c_2 - 2c_1 w_{n+1}^{-1} w_n^{-1} + w_{n+1}^{-2} w_n^{-2} + 2w_{n+2}^{-1} w_{n+1}^{-2} w_n^{-1},$$

where c_1, c_2 are arbitrary constants. So, these conserved densities have orders 3 and 4, respectively.

2.5.4. Left and right order of generalized symmetries. In deriving the integrability conditions (2.55), (2.99), (2.105), we have considered the left order m of a generalized symmetry (2.3) only. According to theorems 2, 6, 8 and 9, m was required to be sufficiently high. On the other hand, we used only the first of two conditions given by (2.1), namely $f_n^{(1)} \neq 0$.

One can assume that the right order m' of a generalized symmetry (2.3) is sufficiently low and use the condition $f_n^{(-1)} \neq 0$. Following the proof of theorem 2, one differentiates compatibility condition (2.47) with respect to

$$u_{n+m'-1}, \quad u_{n+m'}, \quad u_{n+m'+1}, \quad u_{n+m'+2}, \quad \ldots.$$

Then, following the proofs of theorems 8 and 9, one considers instead of (2.96) a formal symmetry of the form

$$L_n = l_n^{(-1)} T^{-1} + l_n^{(0)} + l_n^{(1)} T + l_n^{(2)} T^2 + \cdots, \qquad l_n^{(-1)} \neq 0,$$

which is a formal series in positive powers of the shift operator T.

One more set of integrability conditions can be obtained in this way which have the form

$$D_t \hat{p}_n^{(i)} = (T-1)\hat{q}_n^{(i)}, \qquad i = 1, 2, 3, \ldots, \tag{2.156}$$

where, for example,

$$\hat{p}_n^{(1)} = \log f_n^{(-1)}, \qquad \hat{p}_n^{(2)} = \hat{q}_n^{(1)} - f_n^{(0)}. \tag{2.157}$$

Conditions (2.156) are the analogous of equations (2.55), (2.99), (2.105), and the functions (2.157) are similar to the conserved densities $p_n^{(1)}$, $p_n^{(2)}$ which can be written as

$$p_n^{(1)} = \log f_n^{(1)}, \qquad p_n^{(2)} = q_n^{(1)} + f_n^{(0)}. \tag{2.158}$$

Let us explain why the integrability conditions (2.156) are not important. In fact, such conditions can be obtained as corollaries of the integrability conditions like (2.55), (2.99), (2.105), (2.129), (2.130). This will be demonstrated, considering as an example conditions (2.156) with $i = 1, 2$ and using in addition to equations (2.157), (2.158) the following formulae for the functions $r_n^{(1)}$ and $r_n^{(2)}$, given by (2.129), (2.130):

$$r_n^{(1)} = \log\left(-f_n^{(1)}/f_n^{(-1)}\right), \qquad r_n^{(2)} = \dot{\sigma}_n^{(1)} + 2f_n^{(0)}.$$

If the integrability conditions (2.55), (2.129) are satisfied, then

$$D_t\hat{p}_n^{(1)} = D_t\left(p_n^{(1)} - r_n^{(1)}\right) = (T-1)\left(q_n^{(1)} - \dot{\sigma}_n^{(1)}\right).$$

This means that condition (2.156) with $i = 1$ is satisfied too, a function $\hat{q}_n^{(1)}$ exists, and its general form is: $\hat{q}_n^{(1)} = q_n^{(1)} - \dot{\sigma}_n^{(1)} + c$, where c is an arbitrary integration constant. From conditions (2.99), (2.130) we obtain

$$D_t\hat{p}_n^{(2)} = D_t\left(q_n^{(1)} - \dot{\sigma}_n^{(1)} - f_n^{(0)}\right) = D_t\left(p_n^{(2)} - r_n^{(2)}\right) \sim 0,$$

i.e. the integrability condition (2.156) with $i = 2$ is obtained as a corollary of equations (2.99), (2.130).

2.6. Hamiltonian equations and their properties

We discuss here Hamiltonian lattice equations of the form (2.1) and explain why such equations are useful in the generalized symmetry method.

Let us consider anti-symmetric operators K_n, such that $K_n^\dagger = -K_n$, where the definition of an adjoint operator is given by equation (2.109). The operators K_n have the form

$$K_n = \sum_{j=1}^{\nu}\left(k_n^{(j)}T^j - k_{n-j}^{(j)}T^{-j}\right). \tag{2.159}$$

These operators are supposed to satisfy an equation similar to that for the formal conservation densities (2.115):

$$\dot{K}_n = f_n^* K_n + K_n f_n^{*\dagger}. \tag{2.160}$$

Hamiltonian equations are given by

$$\dot{u}_n = f_n, \qquad f_n = K_n\frac{\delta h_n}{\delta u_n}, \tag{2.161}$$

where h_n is any function of the form (2.11). It is more convenient in the case of the generalized symmetry method to introduce the following definition for Hamiltonian equations and operators.

Definition 7. *An equation (2.1) is called* Hamiltonian *if it can be written as equation (2.161), where K_n is anti-symmetric operator of the form (2.159) satisfying equation (2.160). The operator K_n and function h_n are called* Hamiltonian operator *and* Hamiltonian density, *respectively.*

The name Hamiltonian density is due to the fact that h_n is a conserved density of equation (2.161), as follows from property (2.159). In fact,

$$\begin{aligned} D_t h_n &= \sum_i \frac{\partial h_n}{\partial u_{n+i}} f_{n+i} \sim \frac{\delta h_n}{\delta u_n} f_n = \frac{\delta h_n}{\delta u_n} K_n \frac{\delta h_n}{\delta u_n} \\ &= \sum_{j=1}^{\nu}(1 - T^{-j})\left(\frac{\delta h_n}{\delta u_n} k_n^{(j)} \frac{\delta h_{n+j}}{\delta u_{n+j}}\right) \sim 0. \end{aligned}$$

As the operator K_n satisfies equation (2.160), we can construct, starting from any conserved density, generalized symmetries of equation (2.161).

Theorem 16. *If p_n is a conserved density of equation (2.161), then the equation*

$$u_{n,\tau} = g_n, \qquad g_n = K_n \frac{\delta p_n}{\delta u_n}, \tag{2.162}$$

is its generalized symmetry.

Proof. If p_n is a conserved density of equation (2.161), then its variational derivative $\varrho_n = \frac{\delta p_n}{\delta u_n}$ solves, according to theorem 10, equation (2.108). From equations (2.108), (2.160) it follows that the function $g_n = K_n \varrho_n$ satisfies equation (2.71):

$$D_t g_n = \dot{K}_n \varrho_n + K_n \dot{\varrho}_n = \left(f_n^* K_n + K_n f_n^{*\dagger}\right)\varrho_n - K_n f_n^{*\dagger} \varrho_n = f_n^* K_n \varrho_n = f_n^* g_n.$$

This means that equation (2.162) is a generalized symmetry of equation (2.161). □

It is obvious that theorem 16 is also true if K_n is an infinite formal series of the form (2.65) and satisfies equation (2.160). However in this case, one has to check that equation (2.161) and its symmetry (2.162) are local, i.e. have the form (2.1) and (2.3), respectively. So, an infinite formal series satisfying equation (2.160) also may map conserved densities into generalized symmetries. Sometimes, such formal series is called Noether operator [24].

Let us consider, for example, equations of the form

$$\dot{u}_n = P(u_n)(u_{n+1} - u_{n-1}), \tag{2.163}$$

where P is any non-zeroth function. Any equation (2.163) is Hamiltonian. The Hamiltonian density h_n is given by $h_n = \int \frac{u_n}{P(u_n)}\, \mathrm{d}u_n$. The operator

$$K_n = P(u_n)(T - T^{-1})P(u_n) = P(u_n)P(u_{n+1})T - P(u_n)P(u_{n-1})T^{-1} \tag{2.164}$$

is Hamiltonian as one can prove, checking equation (2.160) by direct calculation.

There are two nonlinear integrable equations in this class. These are the Volterra equation (2.2) and the modified Volterra equation:

$$\dot{u}_n = \left(c^2 - u_n^2\right)(u_{n+1} - u_{n-1}), \tag{2.165}$$

where c is an arbitrary constant. Omitting constants of integration, which play no role in this case, we can write down the Hamiltonian densities as

$$h_n = u_n, \qquad h_n = -\tfrac{1}{2} \log\left(c^2 - u_n^2\right), \tag{2.166}$$

respectively. In the case of the Volterra equation, if we use formula (2.162) and the conserved densities (2.39), we obtain the trivial symmetry $u_{n,\tau'} = 0$ from p_n^1 and the generalized symmetry (2.15) from p_n^3. Let us note that the case of $p_n^2 = h_n$ is not interesting here. The conserved density p_n^4 is transformed into a generalized symmetry (2.3) with $m = 3$, $m' = -3$.

Theorem 17. *If K_n is a Hamiltonian operator of equation (2.161), then its inverse $S_n = K_n^{-1}$ is a solution of equation (2.115).*

Proof. This follows immediately from equation (2.160). In fact,

$$\begin{aligned}\dot{S}_n + S_n f_n^* + f_n^{*\dagger} S_n &= -K_n^{-1}\dot{K}_n K_n^{-1} + K_n^{-1} f_n^* + f_n^{*\dagger} K_n^{-1} \\ &= K_n^{-1}\left(-\dot{K}_n + f_n^* K_n + K_n f_n^{*\dagger}\right)K_n^{-1} = 0.\end{aligned}$$

□

The same is true for any formal series of the form (2.65) satisfying equation (2.160), i.e. for Noether operators. As has been said in section 2.3, the exact solution of equation (2.64) is the recursion operator. Theorem 17 states that exact solutions of equation (2.115) are the inverses of Hamiltonian and Noether operators.

Let us pass to an important application of theorem 17. If a Hamiltonian equation (2.161) possesses generalized symmetries of high enough orders, then it has a formal symmetry L_n of the first order and of length l as big as necessary (see theorem 6 of section 2.3). In this case, we have not only the exact solution $S_n = K_n^{-1}$ of equation (2.115) but also formal conserved densities $K_n^{-1}L_n^i$ of any order and of length l (section 2.4). Additional integrability conditions similar to (2.129), (2.130), which come from the existence of conservation laws, are automatically satisfied in this case.

So, when studying Hamiltonian equations like equation (2.163), we can use the two properties as follows.

- Generalized symmetries can be constructed, starting from conservation laws. Moreover, it will be shown in section 2.7 that conservation laws can be obtained, using Miura-type transformations. The Hamiltonian structure provides the equations with generalized symmetries.
- Additional integrability conditions similar to equations (2.129), (2.130) are automatically satisfied. If we classify integrable Hamiltonian equations or test a given Hamiltonian equation for integrability, we can use only integrability conditions similar to equations (2.55), (2.99), (2.105) which come from the existence of generalized symmetries.

For Toda and relativistic Toda-type lattice equations, there are wide classes of Hamiltonian equations, and these two properties will be very useful.

Bi-Hamiltonian equations, i.e. equations possessing two Hamiltonian structures, are known to be integrable. Let us briefly provide an explanation of this fact. If an equation has two representations (2.161) with Hamiltonian operators K_n and $\hat{K}_n$ of different orders ($\operatorname{ord} K_n > \operatorname{ord} \hat{K}_n$), we can introduce the formal series $S_n = K_n^{-1}$, $\hat{S}_n = \hat{K}_n^{-1}$ and then, due to equation (2.119), the series $L_n = S_n^{-1}\hat{S}_n = K_n\hat{K}_n^{-1}$. Equations (2.69), (2.70), (2.118) imply that L_n^i and $S_nL_n^i$, where i is an arbitrary integer, are the exact solutions of equations (2.64) and (2.115), respectively. This means that all integrability conditions are satisfied, as those conditions are derived from equations (2.64), (2.115). Moreover, L_n is the recursion operator, and one can construct, using it, an infinite number of conserved densities and generalized symmetries.

The Volterra equation (2.2) exemplifies a bi-Hamiltonian equation [48]. It is easy to check that both operators,

$$K_n = u_n(T - T^{-1})u_n,$$
$$\hat{K}_n = u_n(u_{n+1}T^2 + (u_{n+1} + u_n)T - (u_n + u_{n-1})T^{-1} - u_{n-1}T^{-2})u_n,$$

together with the functions $h_n = u_n$ and $\hat{h}_n = \frac{1}{2}\log u_n$, define Hamiltonian representations (2.161) for the Volterra equation. As we shall show, $\Lambda_n = \hat{K}_nK_n^{-1}$ is the formal series (2.73). This will prove that (2.73) is an exact solution of equation (2.64), i.e. is the recursion operator. Moreover, the formula $K_n^{-1}\Lambda_n^i$ ($i \in \mathbb{Z}$) will give for the Volterra equation exact solutions of (2.115).

In fact, the Hamiltonian operators K_n and $\hat{K}_n$ can be rewritten as

$$K_n = u_n(1 - T^{-1})(T + 1)u_n,$$
$$\hat{K}_n = u_n(u_n(1 - T^{-1}) + u_{n+1}T - u_{n-1}T^{-2})(T + 1)u_n,$$

and hence the inverse of K_n is given by

$$K_n^{-1} = u_n^{-1}(T + 1)^{-1}(1 - T^{-1})^{-1}u_n^{-1}.$$

That is why we obtain the formal series Λ_n (2.73):

$$\hat{K}_n K_n^{-1} = u_n(u_n(1-T^{-1}) + u_{n+1}T - u_{n-1}T^{-2})(1-T^{-1})^{-1}u_n^{-1} = \Lambda_n.$$

2.7. Discrete Miura transformations and master symmetries

If, using integrability conditions, we obtain a list of equations, we pass to a next problem, taking into account that the integrability conditions are only necessary conditions for the existence of generalized symmetries and conservation laws. The problem is to construct for resulting equations higher order generalized symmetries and conservation laws.

Here are some possibilities. One can construct a few conservation laws, using the integrability conditions (2.55), (2.99), (2.105) as presented in sections 2.2 and 2.3. One can find coefficients of the formal series L_n (2.96) of the first order and then, using theorem 7 contained in section 2.3, obtain conserved densities. One more solution is obtained by using the recursion operator considered in section 2.3, which generates the infinite hierarchies of conservation laws and generalized symmetries. If an equation is Hamiltonian, one can, using the results presented in section 2.6, construct generalized symmetries, starting from conserved densities.

In the review paper, we provide equations by infinite hierarchies of conservation laws and generalized symmetries, and we mainly do that using Miura-type transformations and local master symmetries together with Hamiltonian and Lagrangian structures.

Let us consider Miura-type transformations. The discrete analogue of the Miura transformation (1.26) is given by the following definition.

Definition 8. *The equation*

$$\dot{v}_n = \hat{f}_n = \hat{f}(v_{n+1}, v_n, v_{n-1}) \tag{2.167}$$

is transformed into equation (2.1) by the transformation

$$u_n = s_n = s(v_n, v_{n+1}, \ldots, v_{n+k}), \quad k > 0, \qquad \frac{\partial s_n}{\partial v_n}\frac{\partial s_n}{\partial v_{n+k}} \neq 0, \tag{2.168}$$

if s_n satisfies

$$D_t s_n = \sum_{j=0}^{k} \frac{\partial s_n}{\partial v_{n+j}} \hat{f}_{n+j} = f(s_{n+1}, s_n, s_{n-1}). \tag{2.169}$$

Transformation (2.168) is called a Miura-type transformation.

The name Miura-type transformation is due to the fact that such transformations are similar to the original Miura transformation (1.24). As an example, let us present the transformation

$$\tilde{u}_n = (c + u_n)(c - u_{n+1}), \tag{2.170}$$

which brings solutions u_n of the modified Volterra equation (2.165) into solutions $\tilde{u}_n$ of the Volterra equation (2.2).

Miura transformations, unlike point transformations which have the form $u_n = s(v_n)$, are not invertible. Definition 8 is constructive because, for any given pair of equations (2.1), (2.167) and for any number k, we can find a Miura-type transformation (2.168) or prove that it does not exist.

If equation (2.1) possesses a conservation law, it can be rewritten easily as a conservation law of equation (2.167). To do so, one has to replace the dependent variables u_{n+i} by the functions s_{n+i}. It is important that nontrivial conservation laws of positive order remain nontrivial. This will be shown in detail for conservation laws written in the special form (2.35)–(2.38).

Let the relation

$$\begin{aligned} &D_t p_n = (T-1)q_n, \\ &p_n = p(u_n, u_{n+1}, \ldots, u_{n+m}), \\ &q_n = q(u_{n-1}, u_n, \ldots, u_{n+m}), \end{aligned} \tag{2.171}$$

be a conservation law of equation (2.1) of the special form which has positive order m. This means that

$$\frac{\partial^2 p_n}{\partial u_n \partial u_{n+m}} \neq 0. \tag{2.172}$$

Then the following theorem will provide a conservation law for equation (2.167).

Theorem 18. *Let transformation (2.168) transform equation (2.167) into equation (2.1). Let us assume that there exists a conservation law (2.171) of equation (2.1) of positive order m. Then equation (2.167) possesses a conservation law*

$$\begin{aligned} &D_t \hat{p}_n = (T-1)\hat{q}_n, \\ &\hat{p}_n = \hat{p}(v_n, v_{n+1}, \ldots, v_{n+k+m}) = p(s_n, s_{n+1}, \ldots, s_{n+m}), \\ &\hat{q}_n = \hat{q}(v_{n-1}, v_n, \ldots, v_{n+k+m}) = q(s_{n-1}, s_n, \ldots, s_{n+m}), \end{aligned} \tag{2.173}$$

and its order is equal to $k+m$.

Proof. Let us prove the first part of the statement, namely, that (2.173) is a conservation law of equation (2.167). We use the formula $\frac{\partial \hat{p}_n}{\partial v_{n+i}} = \sum_j \frac{\partial p_n}{\partial u_{n+j}} \frac{\partial s_{n+j}}{\partial v_{n+i}}$. By the following calculation

$$\begin{aligned} D_t \hat{p}_n &= \sum_i \frac{\partial \hat{p}_n}{\partial v_{n+i}} \hat{f}_{n+i} = \sum_i \left(\sum_j \frac{\partial p_n}{\partial u_{n+j}} \frac{\partial s_{n+j}}{\partial v_{n+i}} \right) \hat{f}_{n+i} \\ &= \sum_j \frac{\partial p_n}{\partial u_{n+j}} \left(\sum_i \frac{\partial s_{n+j}}{\partial v_{n+i}} \hat{f}_{n+i} \right) = \sum_j \frac{\partial p_n}{\partial u_{n+j}} D_t s_{n+j}, \end{aligned}$$

using equations (2.1), (2.168), (2.169), (2.171) together with formulae for $\hat{p}_n$ and $\hat{q}_n$, given by (2.173), we obtain

$$\begin{aligned} D_t \hat{p}_n &= \sum_j \frac{\partial p_n}{\partial u_{n+j}} f(s_{n+j+1}, s_{n+j}, s_{n+j-1}) = \sum_j \frac{\partial p_n}{\partial u_{n+j}} f(u_{n+j+1}, u_{n+j}, u_{n+j-1}) \\ &= \sum_j \frac{\partial p_n}{\partial u_{n+j}} \dot{u}_{n+j} = D_t p_n = (T-1)q_n = (T-1)\hat{q}_n. \end{aligned}$$

In this way we have proved the first part of theorem 18. As $m > 0$, we easily check, using equations (2.168), (2.172), that

$$\frac{\partial^2 \hat{p}_n}{\partial v_n \partial v_{n+k+m}} = \frac{\partial}{\partial v_n} \left(\frac{\partial p_n}{\partial u_{n+m}} \frac{\partial s_{n+m}}{\partial v_{n+k+m}} \right) = \frac{\partial^2 p_n}{\partial u_n \partial u_{n+m}} \frac{\partial s_n}{\partial v_n} T^m \frac{\partial s_n}{\partial v_{n+k}} \neq 0,$$

i.e. this conservation law is of order $k+m$. □

Theorem 18 provides a way of constructing conservation laws and it shows that if equation (2.1) is integrable, in the sense that it possesses conservation laws of arbitrarily high orders, and equation (2.167) is transformed into it by a Miura-type transformation, then equation (2.167) is integrable in the same sense.

Let us give a few examples. As has been said before, the modified Volterra equation (2.165) is transformed into the Volterra equation (2.2) by transformation (2.170). Let us now use theorem 18 and the conserved densities (2.39) of (2.2) to construct two conserved densities for equation (2.165). Starting from p_n^1, we obtain the following conserved density:

$$\log(c+u_n)+\log(c-u_{n+1}) \sim \log\left(c^2-u_n^2\right)=\hat{p}_n^1.$$

Starting from p_n^2, we are led to

$$(c+u_n)(c-u_{n+1})=-u_n u_{n+1}+c^2-c(T-1)u_n.$$

Omitting the trivial terms and multiplying the result by -1, we obtain for equation (2.165) the density $\hat{p}_n^2=u_n u_{n+1}$.

The modified Volterra equation is Hamiltonian (see (2.163), (2.164), (2.166)), and one now can use theorem 16 to construct a generalized symmetry. The case of $\hat{p}_n^1=-2h_n$ (see equation (2.166)) is trivial. In the case of $\hat{p}_n^2$, one obtains $\frac{\delta \hat{p}_n^2}{\delta u_n}=u_{n+1}+u_{n-1}$ and is led, using formula (2.162), to the following generalized symmetry of equation (2.165):

$$u_{n,\tau}=\left(c^2-u_n^2\right)\left(\left(c^2-u_{n+1}^2\right)\left(u_{n+2}+u_n\right)-\left(c^2-u_{n-1}^2\right)(u_n+u_{n-2})\right). \tag{2.174}$$

Another example is given by equation (2.155) discussed in section 2.5. It can be transformed, by the transformation $\tilde{u}_n=(u_{n+1}-u_{n-1})^{-1}$, into the modified Volterra equation (2.165) with $c=0$. This proves that equation (2.155) is integrable.

Let us consider equation (2.151). It can be shown that not only three conserved densities $p_n^{(1)}, p_n^{(2)}, p_n^{(3)}$ given by equations (2.55), (2.99), (2.105) are trivial in this case but also any conserved density which can be obtained using equation (2.64) and formulae (2.76), (2.77). Nevertheless, equation (2.151) has an infinite hierarchy of nontrivial conserved densities. In fact, if we introduce the function

$$w_n=(u_{n+1}-u_n)^{1/2}, \tag{2.175}$$

the transformed equation reads

$$2\dot{w}_n=w_{n+1}-w_{n-1}, \tag{2.176}$$

i.e. equation (2.151) is linearizable. It is easy to check that the functions $p_n^m=w_n w_{n+m}$ $(m\geqslant 0)$ are conserved densities of this linear equation, as

$$2D_t p_n^m=(T-1)(w_{n-1}w_{n+m}+w_n w_{n+m-1}).$$

The order of such density p_n^m equals m. Using transformation (2.175), we obtain from p_n^m the following conserved density for equation (2.151):

$$(u_{n+1}-u_n)^{1/2}(u_{n+m+1}-u_{n+m})^{1/2}. \tag{2.177}$$

If $m=0$, such density is trivial. If $m>0$, one can see that conserved density (2.177) is nontrivial and has order $m+1$, as theorem 18 guarantees.

Let us consider the master symmetries. The notion of master symmetry has been introduced in [21], see also [20, 22, 23, 46]. We consider in this review only local master symmetries, i.e. such master symmetries whose right-hand side, unlike equation (1.28), contains no operators like $(T-1)^{-1}$. Such master symmetry has been found for the first time in [23] for the Landau–Lifshitz equation. It is known that there are many local master symmetries in the case of discrete-differential equations [10, 11, 16, 17, 47, 48, 80].

The master symmetry is an equation which depends explicitly on the spatial variable and may depend on its time. In the case of equations (2.1), we consider local master symmetries of the form

$$u_{n,\tau}=\phi_n(\tau,u_{n+1},u_n,u_{n-1}). \tag{2.178}$$

If there is an essential dependence on τ, then the corresponding equation (2.1) and its generalized symmetries (2.3) will also depend on τ which, for these equations, is an outer parameter. More details will be given at the end of this section. This is the reason why the evolution differentiation D_τ corresponding to equation (2.178) is defined by

$$D_\tau = \frac{\partial}{\partial \tau} + \sum_j \phi_{n+j} \frac{\partial}{\partial u_{n+j}}, \tag{2.179}$$

(cf (2.7)). An example of a master symmetry is given by

$$u_{n,\tau} = u_n((n+2)u_{n+1} + u_n - (n-1)u_{n-1}). \tag{2.180}$$

This equation, introduced in [29], is a master symmetry of the Volterra equation (2.2) (see [48]), as can be checked, using the following definition 9.

Let us define a Lie algebra structure on the set of functions ϕ_n of the form (2.11) and $\phi_n(\tau, u_{n+1}, u_n, u_{n-1})$ defined in equation (2.178). For any functions ϕ_n and $\hat{\phi}_n$, we introduce the equations $u_{n,\tau} = \phi_n$ and $u_{n,\hat{\tau}} = \hat{\phi}_n$ and corresponding evolution differentiations D_τ and $D_{\hat{\tau}}$. A new function is defined as follows:

$$[\phi_n, \hat{\phi}_n] \equiv D_\tau \hat{\phi}_n - D_{\hat{\tau}} \phi_n. \tag{2.181}$$

Here [,] is a Lie bracket, as it is obviously anti-symmetric: $[\phi_n, \hat{\phi}_n] = -[\hat{\phi}_n, \phi_n]$, and one can check by a direct calculation that it satisfies the Jacobi identity:

$$[[\phi_n, \hat{\phi}_n], \tilde{\phi}_n] = [[\phi_n, \tilde{\phi}_n], \hat{\phi}_n] + [\phi_n, [\hat{\phi}_n, \tilde{\phi}_n]]. \tag{2.182}$$

The right-hand side g_n of a generalized symmetry (2.3) of equation (2.1) satisfies equation (2.6), i.e. $[g_n, f_n] = 0$. In the case of the master symmetry (2.178), the function

$$g_n = [\phi_n, f_n] \tag{2.183}$$

is the right-hand side of a generalized symmetry. This generalized symmetry must be nontrivial, i.e. in equation (2.3) $m > 1$ and $m' < -1$. The function ϕ_n satisfies the following equation:

$$[[\phi_n, f_n], f_n] = 0. \tag{2.184}$$

Any generalized symmetry (2.3) gives its trivial solution: $\phi_n = g_n$. The master symmetry corresponds to a nontrivial solution of equation (2.184).

Definition 9. *Equation (2.178) is a master symmetry of equation (2.1) if the function ϕ_n satisfies equation (2.184), and the function (2.183) is the right-hand side of a generalized symmetry (2.3) with orders $m > 1$ and $m' < -1$.*

In the case of the local master symmetry, this definition is constructive because, for any given equation (2.1), one can find a master symmetry (2.178) or prove that it does not exist.

The master symmetry is closely related to a t-dependent generalized symmetry of equation (2.1), where t is the time of (2.1). Let us show this in the case if there is, in equation (2.178), no dependence on τ, and thus $D_\tau = \sum_j \phi_{n+j} \frac{\partial}{\partial u_{n+j}}$ (cf (2.179)). This generalized symmetry is of the form $u_{n,\hat{\tau}} = \hat{g}_n = t g_n + \phi_n$, where g_n given by equation (2.183) is the right-hand side of a generalized symmetry $u_{n,\tau'} = g_n$. One easily checks that

$$\begin{aligned}[\hat{g}_n, f_n] &= D_{\hat{\tau}} f_n - D_t \hat{g}_n = \sum_j \frac{\partial f_n}{\partial u_{n+j}} (t g_{n+j} + \phi_{n+j}) - D_t(t g_n + \phi_n) \\ &= t D_{\tau'} f_n + D_\tau f_n - g_n - t D_t g_n - D_t \phi_n = t[g_n, f_n] + [\phi_n, f_n] - g_n = 0.\end{aligned}$$

Master symmetries enable one to construct infinite hierarchies of generalized symmetries. Let us introduce an operator ad_{ϕ_n} corresponding to the master symmetry (2.178),

$$\mathrm{ad}_{\phi_n}\hat{\phi}_n = [\phi_n, \hat{\phi}_n]. \tag{2.185}$$

Then, in terms of its powers $\mathrm{ad}^i_{\phi_n}$, we can construct generalized symmetries for any $i \geqslant 1$:

$$u_{n,t_i} = g_n^{(i)} = \mathrm{ad}^i_{\phi_n} f_n. \tag{2.186}$$

In spite of the fact that equation (2.178) has an explicit dependence on the variable n, the resulting generalized symmetries (2.186) do not depend on n. It is clear that the way of constructing generalized symmetries is simpler, using local master symmetry, than in the case of non-local master symmetries or recursion operators, as non-local functions can never appear when one applies the operator ad_{ϕ_n}.

In the generic case, it is not easy to prove that equations (2.186) are generalized symmetries and do not depend explicitly on n. This can be proved only for some integrable equations using specific additional properties (see e.g. [17], where the Volterra equation is discussed). Definition 9 implies that equation (2.186) with $i = 1$ is a generalized symmetry of equation (2.1). We only prove here that also equation (2.186) with $i = 2$ is a generalized symmetry (see e.g. [20]).

Theorem 19. *If equation (2.178) is the master symmetry of equation (2.1), then (2.186) with $i = 2$ is a generalized symmetry of this equation.*

Proof. Introducing the notation $g_n^{(0)} = f_n$, we obtain from equations (2.185), (2.186) the following result for all $i \geqslant 0$:

$$g_n^{(i+1)} = \mathrm{ad}_{\phi_n} g_n^{(i)} = \left[\phi_n, g_n^{(i)}\right]. \tag{2.187}$$

Then, using the Jacobi identity (2.182) and the fact that equation (2.186) with $i = 1$ is a generalized symmetry, we have

$$\begin{aligned}\left[g_n^{(2)}, f_n\right] &= \left[\left[\phi_n, g_n^{(1)}\right], f_n\right] = \left[[\phi_n, f_n], g_n^{(1)}\right] + \left[\phi_n, \left[g_n^{(1)}, f_n\right]\right] \\ &= \left[g_n^{(1)}, g_n^{(1)}\right] + [\phi_n, 0] = 0,\end{aligned}$$

i.e equation (2.186) with $i = 2$ is a generalized symmetry of equation (2.1). □

It can be checked, using definition 9, that equation (2.180) and

$$u_{n,\tau} = \left(c^2 - u_n^2\right)((n+1)u_{n+1} - (n-1)u_{n-1}) \tag{2.188}$$

are master symmetries of the Volterra equation (2.2) and the modified Volterra equation (2.165) (see [80] in the second case). Formula (2.186) with $i = 1$ gives the generalized symmetry (2.15) in the first case and (2.174) in the second one. Formula (2.186) for $i = 2$ provides in both cases a generalized symmetry (2.3) of orders $m = 3, m' = -3$.

Using local master symmetries, one can easily construct not only generalized symmetries but also conserved densities. Indeed, if p_n is a conserved density of equation (2.1), then the functions

$$D_\tau^j p_n, \qquad j \geqslant 1, \tag{2.189}$$

also are its conserved densities. Here D_τ^j are powers of the operator D_τ (2.179). Adding total differences to the functions (2.189), one can not only simplify the obtained conserved densities but also remove the explicit dependence on the variable n.

We cannot prove formula (2.189) in the general case. For a given equation, such proof requires using additional properties, see e.g. [10, 16, 17], where a proof is given for the Volterra equation, using the $L - A$ pair. It is a general property of integrable equations that the equation and its generalized symmetries possess common conserved densities. We consider below the case when a function p_n is the common conserved density for an equation (2.1) and corresponding generalized symmetries (2.186). In this case we prove formula (2.189), which is a corollary of the following theorem.

Theorem 20. *Let equation (2.178) be a master symmetry of equation (2.1) and let p_n be the common conserved density of equation (2.1) and its generalized symmetries (2.186). Then the function $D_\tau p_n$ also is a common conserved density of equations (2.1), (2.186).*

Proof. Introducing the notations $t_0 = t$ and $g_n^{(0)} = f_n$, using the derivative operator $D_{t_i} = \sum_j g_{n+j}^{(i)} \frac{\partial}{\partial u_{n+j}}$ for all $i \geqslant 0$, and taking into account equations (2.186) together with equations (2.179), (2.181), (2.187), we obtain

$$\begin{aligned} D_\tau D_{t_i} - D_{t_i} D_\tau &= \sum_j \left(D_\tau g_{n+j}^{(i)} - D_{t_i}\phi_{n+j}\right)\frac{\partial}{\partial u_{n+j}} \\ &= \sum_j \left[\phi_{n+j}, g_{n+j}^{(i)}\right]\frac{\partial}{\partial u_{n+j}} = \sum_j g_{n+j}^{(i+1)}\frac{\partial}{\partial u_{n+j}}. \end{aligned}$$

So, for any $i \geqslant 0$, we have the following general formula:

$$[D_\tau, D_{t_i}] = D_\tau D_{t_i} - D_{t_i} D_\tau = D_{t_{i+1}}. \tag{2.190}$$

If a function p_n is the common conserved density of equations (2.1), (2.186), then one has the set of conservation laws:

$$D_{t_i} p_n = (T-1)\omega_n^{(i)}, \qquad i \geqslant 0. \tag{2.191}$$

Relations (2.190), (2.191) imply

$$D_{t_i} D_\tau p_n = D_\tau D_{t_i} p_n - D_{t_{i+1}} p_n = (T-1)\left(D_\tau \omega_n^{(i)} - \omega_n^{(i+1)}\right),$$

i.e. $D_\tau p_n$ is also a conserved density of equations (2.1), (2.186). □

Let us consider, as an example, master symmetry (2.180) of the Volterra equation (2.2) and conserved densities (2.39). We see that

$$\begin{aligned} D_\tau p_n^1 = (n+2)u_{n+1} + u_n - (n-1)u_{n-1} &= (n+1)u_n \\ &+ (T-1)((n+1)u_n) + u_n - nu_n + (T-1)((n-1)u_{n-1}) \sim 2u_n = 2p_n^2. \end{aligned}$$

In particular, the explicit dependence on n disappears after passing to an equivalent density. Moreover, one can check that $D_\tau p_n^2 \sim 2p_n^3$, $D_\tau p_n^3 \sim 3p_n^4$, i.e. master symmetry (2.180) allows one to construct, starting from p_n^1, the conserved densities p_n^2, p_n^3, p_n^4.

The explicit dependence of the master symmetry on its time is more difficult to understand and will be discussed separately. We do that by considering the following example:

$$\dot{u}_n = \frac{u_{n+1} + u_{n-1} + 2u_n + c}{u_{n+1} - u_{n-1}}, \tag{2.192}$$

where c is a constant. If we try to find a master symmetry (2.178) for equation (2.192), using definition 9, we fail. However, if we consider the equation

$$\dot{u}_n = \frac{u_{n+1} + u_{n-1} + 2u_n + a(\tau)}{u_{n+1} - u_{n-1}}, \tag{2.193}$$

where $a(\tau)$ is an unknown function and τ is the time of the master symmetry, we find a master symmetry if $a'(\tau) = -4$.

Let us choose a solution of this ODE, satisfying the initial condition $a(0) = c$, and write down the master symmetry of equation (2.193) as

$$u_{n,\tau} = n\dot{u}_n, \qquad a(\tau) = -4\tau + c. \tag{2.194}$$

Generalized symmetries and conserved densities, generated by (2.194) for equation (2.193), explicitly depend on τ and remain generalized symmetries and conserved densities for any value of the parameter τ (unlike the master symmetry). Putting $\tau = 0$, we obtain generalized symmetries and conserved densities for equation (2.192) with any given number c. So, a master symmetry is constructed for the generalization (2.193) of equation (2.192) depending on τ, and that master symmetry provides generalized symmetries and conserved densities for both equations (2.192), (2.193).

2.8. Remarks on generalized symmetries for systems of lattice equations

Here we discuss the generalized symmetry method in the case of systems of lattice equations. We do that by example of Toda-type equations which will be written as systems of two equations on the lattice. We will see that general theory of the method in this case is quite similar to the scalar one. The main difference is that coefficients of formal symmetries and conserved densities are 2×2 matrices.

Let us consider the following class of equations:

$$\ddot{u}_n = f_n = f(\dot{u}_n, u_{n+1}, u_n, u_{n-1}), \qquad \frac{\partial f_n}{\partial u_{n+1}} \frac{\partial f_n}{\partial u_{n-1}} \neq 0, \tag{2.195}$$

which includes the well-known Toda model [65–67]:

$$\ddot{u}_n = \mathrm{e}^{u_{n+1}-u_n} - \mathrm{e}^{u_n - u_{n-1}}. \tag{2.196}$$

This class will be discussed in section 3.2.

Local conservation laws of equations (2.195) have the form (2.16), where the scalar functions p_n, q_n are analogous to those given by (2.11), but depend on a finite number of the variables $u_{n+j}, \dot{u}_{n+j}$. The time derivatives $\mathrm{d}^i u_{n+j}/\mathrm{d}t^i$ with $i \geqslant 2$ are expressed in terms of these variables in virtue of equation (2.195). The evolution differentiation D_t is defined in this case as follows:

$$D_t = \sum_j \dot{u}_{n+j} \frac{\partial}{\partial u_{n+j}} + \sum_j f_{n+j} \frac{\partial}{\partial \dot{u}_{n+j}}. \tag{2.197}$$

Generalized symmetries of equation (2.195) are equations of the form:

$$u_{n,\tau} = \varphi_n = \varphi(u_{n+m}, \dot{u}_{n+m}, u_{n+m-1}, \dot{u}_{n+m-1}, \ldots, u_{n+m'}, \dot{u}_{n+m'}), \tag{2.198}$$

where $m \geqslant m'$. Using the fact that equations (2.195), (2.198) have common solutions $u_n(t,\tau)$ and applying the operator D_τ to (2.195), we obtain the compatibility condition for equations (2.195), (2.198), i.e. an equation for the function φ_n:

$$D_t^2 \varphi_n = D_\tau f_n. \tag{2.199}$$

From the viewpoint of the generalized symmetry method, it is more convenient to rewrite equation (2.195) in the form of a system of two equations. Let us introduce the function $v_n = \dot{u}_n$ and rewrite equation (2.195) as the system

$$\dot{u}_n = v_n, \qquad \dot{v}_n = f_n = f(v_n, u_{n+1}, u_n, u_{n-1}), \tag{2.200}$$

which is equivalent to equation (2.195) from the viewpoint of definitions of generalized symmetries and conservation laws. The following formula

$$\phi_n = \phi\big(u_{n+k_1}, v_{n+k_2}, u_{n+k_1-1}, v_{n+k_2-1}, \ldots, u_{n+k_1'}, v_{n+k_2'}\big), \tag{2.201}$$

with finite $k_1 \geqslant k_1', k_2 \geqslant k_2'$, expresses the most general form of functions. Conservation laws for (2.200) have the same form (2.16), p_n and q_n are functions of the form (2.201) and a formula for the operator D_t is obviously rewritten from equation (2.197).

The system (2.200) can be written in vector form

$$\dot{U}_n = F_n = F(U_{n+1}, U_n, U_{n-1}), \qquad U_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \qquad F_n = \begin{pmatrix} v_n \\ f_n \end{pmatrix}. \tag{2.202}$$

Then, its generalized symmetry reads

$$U_{n,\tau} = G_n = G(U_{n+m}, U_{n+m-1}, \ldots, U_{n+m'}), \qquad G_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}, \tag{2.203}$$

where $m \geqslant m'$ and φ_n, ψ_n are functions of the form (2.201). The standard compatibility condition $D_t G_n = D_\tau F_n$ implies the relations $D_t \varphi_n = v_{n,\tau}$ and $D_t \psi_n = D_\tau f_n$. Thus, we see that ψ_n is expressed via φ_n: $\psi_n = D_t \varphi_n$ and the function φ_n satisfies condition (2.199).

Main formulae, notions, definitions and theorems presented in this section are very similar to ones presented in sections 2.1, 2.3, 2.4, 2.6. Let us introduce the following notation for vector-function G_n defined by (2.198):

$$\frac{\partial G_n}{\partial U_{n+j}} = \begin{pmatrix} \partial \varphi_n / \partial u_{n+j} & \partial \varphi_n / \partial v_{n+j} \\ \partial \psi_n / \partial u_{n+j} & \partial \psi_n / \partial v_{n+j} \end{pmatrix}. \tag{2.204}$$

We suppose for generalized symmetry (2.198) that $\frac{\partial G_n}{\partial U_{n+m}} \frac{\partial G_n}{\partial U_{n+m'}} \neq 0$ (i.e. G_n really depends on $U_{n+m}, U_{n+m'}$) and call the numbers m and m' the left and right orders of this symmetry. The order of conserved densities and conservation laws is also defined as in the scalar case. As in definition 4, we say that a conserved density p_n (which is the scalar function) is called trivial if it is equivalent to a constant. If $p_n \sim \wp(U_n)$, where $\wp$ is not a constant function, then p_n is of order 0. A conserved density p_n has order $m > 0$ if

$$p_n \sim \wp_n = \wp(U_n, U_{n+1}, \ldots, U_{n+m}), \qquad \begin{pmatrix} \frac{\partial^2 \wp_n}{\partial u_n \partial u_{n+m}} & \frac{\partial^2 \wp_n}{\partial u_n \partial v_{n+m}} \\ \frac{\partial^2 \wp_n}{\partial v_n \partial u_{n+m}} & \frac{\partial^2 \wp_n}{\partial v_n \partial v_{n+m}} \end{pmatrix} \neq 0.$$

We can easily calculate the order of a conserved density, using the formal variational derivative. Let us introduce, as in (2.40), the variational derivatives with respect to u_n and v_n for any function ϕ_n (2.201) as

$$\frac{\delta \phi_n}{\delta u_n} = \sum_{j=-k_1}^{-k_1'} \frac{\partial \phi_{n+j}}{\partial u_n}, \qquad \frac{\delta \phi_n}{\delta v_n} = \sum_{j=-k_2}^{-k_2'} \frac{\partial \phi_{n+j}}{\partial v_n}, \tag{2.205}$$

and then define for the conserved density p_n a variational derivative with respect to U_n:

$$\varrho_n = \frac{\delta p_n}{\delta U_n} = \begin{pmatrix} \delta p_n / \delta u_n \\ \delta p_n / \delta v_n \end{pmatrix}. \tag{2.206}$$

As in equations (2.42)–(2.45), we have $\varrho_n = 0$ if p_n is a trivial conserved density, $\varrho_n = \varrho(U_n) \neq 0$ if it is of order 0, and

$$\varrho_n = \varrho(U_{n+m}, U_{n+m-1}, \ldots, U_{n-m}), \qquad \frac{\partial \varrho_n}{\partial U_{n+m}} \frac{\partial \varrho_n}{\partial U_{n-m}} \neq 0,$$

if p_n has the order $m > 0$.

Following equations (2.62), (2.109), the Frechét derivative G_n^* of vector-function G_n (2.198) and its adjoint operator $G_n^{*\dagger}$ are defined in this case as

$$G_n^* = \sum_{j=m'}^{m} \frac{\partial G_n}{\partial U_{n+j}} T^j, \qquad G_n^{*\dagger} = \sum_{j=m'}^{m} \left(\frac{\partial G_{n-j}}{\partial U_n} \right)^{\dagger} T^{-j}, \tag{2.207}$$

where the coefficients of $G_n^{*\dagger}$ are the transposed matrices. We see that coefficients of the operators are matrices now and thus do not commute. In the same way and using the compact notation

$$f_n^{(j)} = \frac{\partial f_n}{\partial u_{n+j}}, \qquad f_n^{(v)} = \frac{\partial f_n}{\partial v_n}, \tag{2.208}$$

(see (2.200)), we obtain the following formulae for the operators F_n^*, $F_n^{*\dagger}$ in the case of F_n given by (2.202)

$$F_n^* = \begin{pmatrix} 0 & 0 \\ f_n^{(1)} & 0 \end{pmatrix} T + \begin{pmatrix} 0 & 1 \\ f_n^{(0)} & f_n^{(v)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ f_n^{(-1)} & 0 \end{pmatrix} T^{-1}, \tag{2.209}$$

$$F_n^{*\dagger} = \begin{pmatrix} 0 & f_{n+1}^{(-1)} \\ 0 & 0 \end{pmatrix} T + \begin{pmatrix} 0 & f_n^{(0)} \\ 1 & f_n^{(v)} \end{pmatrix} + \begin{pmatrix} 0 & f_{n-1}^{(1)} \\ 0 & 0 \end{pmatrix} T^{-1}. \tag{2.210}$$

Now, as in sections 2.3 and 2.4, we can derive in place of equations (2.71), (2.108) the following equations:

$$(D_t - F_n^*)G_n = 0, \qquad \left(D_t + F_n^{*\dagger}\right)\varrho_n = 0 \tag{2.211}$$

for the right-hand side of generalized symmetry (2.203) and for variational derivative (2.206). Here $D_t\Phi_n = \sum_j \frac{\partial \Phi_n}{\partial U_{n+j}} F_{n+j}$ for any vector-function Φ_n.

As in the scalar case given by (2.64), (2.115), equations (2.211) are connected to the equations

$$\dot{L}_n = [F_n^*, L_n], \qquad \dot{S}_n + S_n F_n^* + F_n^{*\dagger} S_n = 0, \tag{2.212}$$

but L_n, S_n are now formal series of the form (2.65), (2.116) with 2×2 matrix coefficients. Applying the Frechét derivative to equations (2.211), one can show that

$$L_n = G_n^*, \qquad S_n = \varrho_n^* \tag{2.213}$$

provide corresponding approximate solutions of equations (2.212). In this way we can construct formal symmetries and conserved densities, and the definition of orders and lengths will be the same as before. As in theorems 5 and 11, using formulae (2.213), we obtain from the generalized symmetry of order $m \geqslant 1$ a formal symmetry L_n, such that $\operatorname{ord} L_n = m$, $\operatorname{lgt} L_n \geqslant m$, and from the conserved density of order $m \geqslant 2$ a formal conserved density S_n, such that $\operatorname{ord} S_n = m$ and $\operatorname{lgt} S_n \geqslant m - 1$.

Formal symmetries and conserved densities in the case of systems under consideration can be of two types. Namely, formal symmetries and conserved densities (2.65) and (2.116) can be such that $\det l_n^{(N)} = \det s_n^{(M)} = 0$, and then they are called degenerate, or $\det l_n^{(N)} \det s_n^{(M)} \neq 0$, and then they are nondegenerate. The nondegenerate case is equivalent to the scalar one and formal series (2.65), (2.116) can easily be inverted. The degenerate case is new and needs a detailed discussion. Let us introduce compact notations for the operator F_n^* (2.209) and the coefficients of the series L_n (2.65):

$$F_n^* = F_n^{(1)}T + F_n^{(0)} + F_n^{(-1)}T^{-1}, \qquad l_n^{(i)} = \begin{pmatrix} \alpha_n^{(i)} & \beta_n^{(i)} \\ \gamma_n^{(i)} & \delta_n^{(i)} \end{pmatrix}. \tag{2.214}$$

Then we can present the following theorem.

Theorem 21. *If the formal series L_n (2.65) is a degenerate formal symmetry of equation (2.202) of length $l \geqslant 2$, then it can be written as*

$$L_n = \begin{pmatrix} 0 & 0 \\ \gamma_n^{(N)} & 0 \end{pmatrix} T^N + \begin{pmatrix} \alpha_n^{(N-1)} & \beta_n^{(N-1)} \\ \gamma_n^{(N-1)} & \delta_n^{(N-1)} \end{pmatrix} T^{N-1} + \cdots, \tag{2.215}$$

where $\gamma_n^{(N)}\beta_n^{(N-1)} \neq 0$.

Proof. The condition $l \geqslant 2$ means that we can set to zero coefficients at T^{N+1}, T^N in the first part of equations (2.212). Taking into account (2.209), (2.214) and collecting coefficients at T^{N+1}, we obtain the condition $F_n^{(1)} l_{n+1}^{(N)} = l_n^{(N)} F_{n+N}^{(1)}$ which is equivalent to the following ones:

$$f_n^{(1)} \beta_{n+1}^{(N)} = \beta_n^{(N)} f_{n+N}^{(1)} = 0, \tag{2.216}$$

$$f_n^{(1)} \alpha_{n+1}^{(N)} = \delta_n^{(N)} f_{n+N}^{(1)}. \tag{2.217}$$

As from (2.195), (2.208) $f_n^{(1)} \neq 0$, equation (2.216) implies $\beta_n^{(N)} = 0$. Using equation (2.217) and the condition $\det l_n^{(N)} = \alpha_n^{(N)} \delta_n^{(N)} = 0$, we obtain $\alpha_n^{(N)} = \delta_n^{(N)} = 0$.

Taking from equation (2.65) the condition $l_n^{(N)} \neq 0$, one can rewrite it in the form $\gamma_n^{(N)} \neq 0$. Coefficients at T^N give the following matrix equation:

$$\dot{l}_n^{(N)} = F_n^{(1)} l_{n+1}^{(N-1)} + F_n^{(0)} l_n^{(N)} - l_n^{(N)} F_{n+N}^{(0)} - l_n^{(N-1)} F_{n+N-1}^{(1)}.$$

The left upper corner element is $\gamma_n^{(N)} = \beta_n^{(N-1)} f_{n+N-1}^{(1)}$, thus $\beta_n^{(N-1)} \neq 0$. □

One can prove in a quite similar way that the degenerate formal conserved density (2.116) of length $l \geqslant 2$ has the form

$$S_n = \begin{pmatrix} a_n^{(M)} & 0 \\ 0 & 0 \end{pmatrix} T^M + \begin{pmatrix} a_n^{(M-1)} & b_n^{(M-1)} \\ c_n^{(M-1)} & d_n^{(M-1)} \end{pmatrix} T^{M-1} + \cdots, \tag{2.218}$$

where $a_n^{(M)} d_n^{(M-1)} \neq 0$.

Theorem 21 enables us to obtain the following important result: the second power L_n^2 of a formal symmetry L_n of equation (2.202) is nondegenerate if $\operatorname{lgt} L_n \geqslant 2$. In fact, let us consider the series L_n (2.215) and, taking into account equation (2.65), write down the first two coefficients of L_n^2:

$$L_n^2 = l_n^{(N)} l_{n+N}^{(N)} T^{2N} + \left(l_n^{(N)} l_{n+N}^{(N-1)} + l_n^{(N-1)} l_{n+N-1}^{(N)}\right) T^{2N-1} + \cdots.$$

The first coefficient of this formal series is the zero matrix, and the second one reads

$$\begin{pmatrix} \beta_n^{(N-1)} \gamma_{n+N-1}^{(N)} & 0 \\ \gamma_n^{(N)} \alpha_{n+N}^{(N-1)} + \delta_n^{(N-1)} \gamma_{n+N-1}^{(N)} & \gamma_n^{(N)} \beta_{n+N}^{(N-1)} \end{pmatrix}. \tag{2.219}$$

This matrix is nondegenerate, then the formal symmetry L_n^2 is also nondegenerate and has order $2N-1$ and $\operatorname{lgt} L_n^2 \geqslant \operatorname{lgt} L_n - 1$. This is the reason why, deriving integrability conditions from the first of equations (2.212), we can consider only nondegenerate formal symmetries.

The easiest way to derive integrability conditions is to use a property of the Toda model (2.196). The Toda model has, for any order $m \geqslant 1$, two generalized symmetries and two conserved densities of order m, such that one of the corresponding formal symmetries and conserved densities is degenerate, while the other one is nondegenerate. In such a case, one can avoid considering degenerate formal conserved densities. In fact, for any degenerate formal conserved density S_n, such that $\operatorname{lgt} S_n \geqslant 2$, we can take a degenerate formal symmetry L_n, such that $\operatorname{lgt} L_n > \operatorname{lgt} S_n$. Then, using formulae (2.215), (2.218), we easily prove that the formal conserved density $S_n L_n$ is nondegenerate and

$$\operatorname{ord}(S_n L_n) = \operatorname{ord} S_n + \operatorname{ord} L_n - 1, \qquad \operatorname{lgt}(S_n L_n) \geqslant \operatorname{lgt} S_n - 1.$$

In accordance with what we have said above, one can start from the nondegenerate formal symmetries L_n, $\hat{L}_n$ and conserved density S_n:

$$\begin{array}{lll} \operatorname{ord} L_n = m, & \operatorname{ord} \hat{L}_n = m+1, & \operatorname{ord} S_n = m+1, \\ \operatorname{lgt} L_n \geqslant m, & \operatorname{lgt} \hat{L}_n \geqslant m+1, & \operatorname{lgt} S_n \geqslant m, \end{array}$$

where $m \geqslant 1$. Then the nondegenerate formal symmetry $\widetilde{L}_n = \hat{L}_n L_n^{-1}$ and conserved density $\widetilde{S}_n = S_n L_n^{-1}$, such that

$$\operatorname{ord} \widetilde{L}_n = \operatorname{ord} \widetilde{S}_n = 1, \qquad \operatorname{lgt} \widetilde{L}_n \geqslant m, \qquad \operatorname{lgt} \widetilde{S}_n \geqslant m,$$

can be used for deriving integrability conditions. For the formal symmetry

$$\widetilde{L}_n = \tilde{l}_n^{(1)} T + \tilde{l}_n^{(0)} + \tilde{l}_n^{(-1)} T^{-1} + \cdots,$$

we can write down some useful formulae for the conserved densities in terms of the matrix trace and determinant (cf theorem 7):

$$D_t \log \det \tilde{l}_n^{(1)} \sim 0,$$
$$D_t \operatorname{tr} \operatorname{res} \widetilde{L}_n^i \sim 0, \qquad 1 \leqslant i \leqslant \operatorname{lgt} \widetilde{L}_n - 2.$$

These formulae are valid if $\operatorname{lgt} \widetilde{L}_n \geqslant 2$ and $\operatorname{lgt} \widetilde{L}_n \geqslant 3$, respectively.

The integrability conditions can also be derived, starting from the existence of any two conservation laws or one generalized symmetry and one conservation law. A precise statement is given in section 3.2, but corresponding calculation would be quite long in this case. One has to consider both nondegenerate and degenerate formal conserved densities. One also has to use the fact that the degenerate formal conserved densities of equations (2.202) are invertible (as well as the degenerate formal symmetries)! Indeed, it is easy to check that the inverse S_n^{-1} of the formal series S_n (2.218) exists and is unique. First coefficients of S_n^{-1} have the following form:

$$S_n^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{d}_n^{(1-M)} \end{pmatrix} T^{1-M} + \begin{pmatrix} \tilde{a}_n^{(-M)} & \tilde{b}_n^{(-M)} \\ \tilde{c}_n^{(-M)} & \tilde{d}_n^{(-M)} \end{pmatrix} T^{-M} + \cdots,$$

$$\tilde{d}_n^{(1-M)} = \frac{1}{d_{n-M+1}^{(M-1)}}, \qquad \tilde{a}_n^{(-M)} = \frac{1}{a_{n-M}^{(M)}},$$

$$\tilde{b}_n^{(-M)} = -\frac{b_{n-M}^{(M-1)}}{a_{n-M}^{(M)} d_{n-M}^{(M-1)}}, \qquad \tilde{c}_n^{(-M)} = -\frac{c_{n-M+1}^{(M-1)}}{a_{n-M}^{(M)} d_{n-M+1}^{(M-1)}}, \ldots .$$

This allows us, starting from the existence of two conservation laws, to pass to formal conserved densities S_n, $\hat{S}_n$ and then to obtain the formal symmetry $L_n = S_n^{-1} \hat{S}_n$ even if S_n is degenerate.

As a result, we can derive for the systems (2.200) integrability conditions which have the same structure and meaning as conditions (2.55), (2.99), (2.105), (2.129), (2.130). Similarly to (2.55), the first of them is

$$D_t \log \frac{\partial f_n}{\partial u_{n+1}} = (T-1) q_n^{(1)},$$

the other ones will be presented in section 3.2. In this case, integrability conditions can be checked, using the following property formulated for functions ϕ_n of the form (2.201):

$$\frac{\delta \phi_n}{\delta u_n} = \frac{\delta \phi_n}{\delta v_n} = 0 \quad \Leftrightarrow \quad \phi_n = c + (T-1)\psi_n$$

(cf theorem 3). Here the formal variational derivatives are defined by (2.205), c is a constant and ψ_n is another function of the form (2.201).

Let us discuss in conclusion the Hamiltonian systems

$$\dot{u}_n = \varphi_n \frac{\delta h_n}{\delta v_n}, \qquad \dot{v}_n = -\varphi_n \frac{\delta h_n}{\delta u_n}, \qquad \varphi_n = \varphi(u_n, v_n), \tag{2.220}$$

where h_n is a function of the form (2.201). For example, if

$$\varphi_n = 1, \qquad h_n = \mathrm{e}^{u_{n+1}-u_n} + \tfrac{1}{2} v_n^2, \tag{2.221}$$

we obtain the Toda model (2.196). Almost all integrable equations in sections 3.2 and 3.3.2 are Hamiltonian with respect to this Hamiltonian structure.

In order to check that system (2.220) is Hamiltonian, one has to rewrite it in vector form:

$$\dot{U}_n = F_n = K_n \frac{\delta h_n}{\delta U_n}, \qquad K_n = \varphi_n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.222}$$

The vector U_n is given in equation (2.202), and the operator $\frac{\delta}{\delta U_n}$ is defined by (2.206). One can see that K_n is a Hamiltonian operator, as it is obviously anti-symmetric (i.e $K_n^\dagger = -K_n$) and satisfies the equation

$$\dot{K}_n = F_n^* K_n + K_n F_n^{*\dagger} \tag{2.223}$$

for any functions h_n, φ_n. Condition (2.223) is checked by a straightforward, but rather long calculation. It is much more easy to prove that the function h_n is the conserved density of system (2.220). In fact,

$$\begin{aligned} D_t h_n &= \sum_i \frac{\partial h_n}{\partial u_{n+i}} \varphi_{n+i} \frac{\delta h_{n+i}}{\delta v_{n+i}} - \sum_i \frac{\partial h_n}{\partial v_{n+i}} \varphi_{n+i} \frac{\delta h_{n+i}}{\delta u_{n+i}} \\ &\sim \left(\sum_j \frac{\partial h_{n+j}}{\partial u_n} \right) \varphi_n \frac{\delta h_n}{\delta v_n} - \left(\sum_j \frac{\partial h_{n+j}}{\partial v_n} \right) \varphi_n \frac{\delta h_n}{\delta u_n} = \frac{\delta h_n}{\delta u_n} \varphi_n \frac{\delta h_n}{\delta v_n} - \frac{\delta h_n}{\delta v_n} \varphi_n \frac{\delta h_n}{\delta u_n} = 0. \end{aligned}$$

In section 3.3.2, where all equations have the Hamiltonian structure (2.220), we need to write down only those integrability conditions which come from the existence of generalized symmetries, i.e. from the first part of equations (2.212), as explained in section 2.6. For all Hamiltonian equations of sections 3.2 and 3.3.2, we take into account that if p_n is a conserved density of equation (2.222), then the equation $U_{n,\tau} = K_n \frac{\delta p_n}{\delta U_n}$ is its generalized symmetry. In the case of system (2.220), such formula for the generalized symmetry takes the form

$$u_{n,\tau} = \varphi_n \frac{\delta p_n}{\delta v_n}, \qquad v_{n,\tau} = -\varphi_n \frac{\delta p_n}{\delta u_n}, \tag{2.224}$$

where p_n is its conserved density.

3. Classification results

Here we present classification results for lattice equations of the Volterra, Toda and relativistic Toda types. Classification theorems will be given together with integrability conditions and complete lists of integrable equations. The involved theorems are presented with no proof, but we refer to the original literature for them. When necessary, Miura-type transformations and master symmetries are written down in order to explain why those equations possess infinite hierarchies of generalized symmetries and conservation laws.

3.1. Volterra-type equations

As proofs of the classification theorems are not included in this review, we first show some examples of simple classification problems in order to demonstrate the standard technique of calculations.

3.1.1. Examples of classification. The following three problems are discussed here:

- how to find all generalized symmetries of given orders in the case of the Volterra equation,

- how to find for this equation all conservation laws of a given order and
- classification problem for a simple class of equations including the Volterra and modified Volterra equations.

The problem of finding generalized symmetries. Let us find for the Volterra equation (2.2) all generalized symmetries (2.3) of orders $m = 2$ and $m' = -2$. Here the equation is given by

$$f_n = u_n(u_{n+1} - u_{n-1}), \tag{3.1}$$

and the right-hand sides of symmetries are of the form

$$g_n = g(u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}), \qquad \frac{\partial g_n}{\partial u_{n+2}} \frac{\partial g_n}{\partial u_{n-2}} \neq 0. \tag{3.2}$$

In order to find the function g_n, we are going to use only compatibility condition (2.8) and take into account property (2.14).

The functions in equation (2.8) may depend only on the variables u_{n+j} with $-3 \leqslant j \leqslant 3$. Differentiating (2.8) with respect to u_{n+3}, one is led to the equation

$$\frac{\partial g_n}{\partial u_{n+2}} \frac{\partial f_{n+2}}{\partial u_{n+3}} = \frac{\partial f_n}{\partial u_{n+1}} \frac{\partial g_{n+1}}{\partial u_{n+3}}$$

which, after dividing by $u_n u_{n+1} u_{n+2}$, turns into

$$(T - 1)\left(\frac{1}{u_n u_{n+1}} \frac{\partial g_n}{\partial u_{n+2}}\right) = 0.$$

Using property (2.14), we obtain $\frac{\partial g_n}{\partial u_{n+2}} = \alpha u_n u_{n+1}$, where α is a nonzero constant due to (3.2). So, the dependence of g_n on u_{n+2} is specified by

$$g_n = \alpha u_n u_{n+1} u_{n+2} + a_n, \qquad a_n = a(u_{n+1}, u_n, u_{n-1}, u_{n-2}). \tag{3.3}$$

On the next step, we differentiate equation (2.8) with respect to u_{n+2}:

$$D_t \frac{\partial g_n}{\partial u_{n+2}} + \frac{\partial g_n}{\partial u_{n+1}} \frac{\partial f_{n+1}}{\partial u_{n+2}} + \frac{\partial g_n}{\partial u_{n+2}} \frac{\partial f_{n+2}}{\partial u_{n+2}} = \frac{\partial f_n}{\partial u_{n+1}} \frac{\partial g_{n+1}}{\partial u_{n+2}} + \frac{\partial f_n}{\partial u_n} \frac{\partial g_n}{\partial u_{n+2}}. \tag{3.4}$$

Then we divide this relation by $u_n u_{n+1}$ and, using equation (3.3), obtain

$$(T - 1)\left(\frac{1}{u_n} \frac{\partial a_n}{\partial u_{n+1}} - \alpha(2u_{n+1} + u_n)\right) = 0.$$

Equation (2.14) implies that $\frac{\partial a_n}{\partial u_{n+1}} = \alpha\left(2u_n u_{n+1} + u_n^2\right) + \beta u_n$, where β is another constant, and in such a way we specify the dependence of a_n on u_{n+1}:

$$a_n = \alpha\left(u_n u_{n+1}^2 + u_n^2 u_{n+1}\right) + \beta u_n u_{n+1} + b_n, \qquad b_n = b(u_n, u_{n-1}, u_{n-2}). \tag{3.5}$$

Compatibility condition (2.8) is symmetrical. In a quite similar way, differentiating equation (2.8) with respect to u_{n-3} and u_{n-2}, one can find the dependence on u_{n-2} and u_{n-1} of the functions appearing in equations (3.3), (3.5). Consequently, one obtains the following formula for the function g_n:

$$\begin{aligned} g_n = {} & \alpha u_n u_{n+1}(u_{n+2} + u_{n+1} + u_n) + \gamma u_n u_{n-1}(u_n + u_{n-1} + u_{n-2}) \\ & + \beta u_n u_{n+1} + \delta u_n u_{n-1} + c(u_n), \end{aligned} \tag{3.6}$$

with four arbitrary constants and one arbitrary function $c(u_n)$. Compatibility condition (2.8) takes now the form

$$\begin{aligned} & (\alpha + \gamma)u_n^2\left(u_{n+1}^2 - u_{n-1}^2 + f_n\right) + (\beta + \delta)u_n f_n \\ & \quad + u_n(c(u_{n+1}) - c(u_{n-1})) + (u_{n+1} - u_{n-1})c(u_n) = f_n c'(u_n), \end{aligned} \tag{3.7}$$

where f_n is given by (3.1).

Applying to both sides of equation (3.7) the operator $\frac{\partial^4}{\partial u_n^2 \partial u_{n+1}^2}$, we obtain the restriction $\alpha + \gamma = 0$. Then, applying $\frac{\partial^3}{\partial u_n \partial u_{n+1}^2}$, we are led to the condition $c''(u_n) = 0$, i.e. $c(u_n) = c_1 u_n + c_2$. Now, dividing (3.7) by $u_{n+1} - u_{n-1}$, we obtain

$$(\beta + \delta)u_n^2 + c_1 u_n + c_2 = 0.$$

Setting equal to zero all coefficients of this polynomial, we get g_n of the form (3.6) with

$$\gamma = -\alpha, \qquad \delta = -\beta, \qquad c(u_n) = 0.$$

In the particular case $\alpha = 1$ and $\beta = 0$, one has generalized symmetry (2.15). The general formula for a symmetry (2.3) of orders $m = 2$ and $m' = -2$ of the Volterra equation (2.2) is of the form

$$u_{n,\tau'} = \alpha u_{n,\tau} + \beta \dot{u}_n, \qquad \alpha \neq 0.$$

Here α and β are arbitrary constants, $u_{n,\tau}$ is defined by generalized symmetry (2.15) and $\dot{u}_n$ is given by the Volterra equation itself.

First order conservation laws of the Volterra equation. We look for conservation laws (2.16) of the first order with conserved densities p_n of the special form (2.38), i.e. such that

$$p_n = p(u_{n+1}, u_n), \qquad \frac{\partial^2 p_n}{\partial u_{n+1} \partial u_n} \neq 0. \tag{3.8}$$

Let us introduce the function $a_n = D_t p_n$, then $a_n \sim 0$. Using the scheme of the proof of theorem 1, we must be able to express this function in the form (2.27) with $b_n = 0$. In doing so, we will get a restriction on p_n. In fact,

$$a_n = \frac{\partial p_n}{\partial u_{n+1}}(u_{n+1}u_{n+2} - u_{n+1}u_n) + \frac{\partial p_n}{\partial u_n}(u_n u_{n+1} - u_n u_{n-1}),$$

and condition (2.31) is satisfied. According to theorem 1 we choose, for instance, $a_n^1 = \frac{\partial p_n}{\partial u_{n+1}} u_{n+1} u_{n+2}$. Then the function a_n^3 of (2.33) takes the form

$$a_n^3 = \left(\frac{\partial p_{n-1}}{\partial u_n} - \frac{\partial p_n}{\partial u_{n+1}} + \frac{\partial p_n}{\partial u_n} \right) u_n u_{n+1} - \frac{\partial p_n}{\partial u_n} u_n u_{n-1} = a^3(u_{n+1}, u_n, u_{n-1}).$$

It follows from (2.33) that $a_n \sim a_n^3 \sim 0$. Thus, due to property (2.25), the following condition must be satisfied:

$$\frac{\partial^2 a_n^3}{\partial u_{n+1} \partial u_{n-1}} = u_n(\Phi_{n-1} - \Phi_n) = 0, \qquad \Phi_n = \frac{\partial^2 p_n}{\partial u_{n+1} \partial u_n}. \tag{3.9}$$

This is the first restriction for the density p_n.

Now one can, applying the operator $-T\frac{1}{u_n}$ to equation (3.9), rewrite it in the form $(T-1)\Phi_n = 0$. This means that $\Phi_n = \alpha \in \mathbb{C}$, where $\alpha \neq 0$ due to (3.8). So, the conserved density p_n can be expressed as $p_n = \alpha u_{n+1} u_n + \varphi(u_{n+1}) + \omega(u_n)$ or equivalently as

$$p_n = \alpha u_{n+1} u_n + \psi(u_n) + (T-1)\varphi(u_n), \tag{3.10}$$

where $\psi(z) = \varphi(z) + \omega(z)$. The function ψ, which defines the nontrivial part of this conserved density, will be specified now.

The function $D_t p_n$ reads

$$D_t p_n = \Omega_n + (T-1)(\alpha u_{n+1} u_n u_{n-1} + \psi'(u_n) u_n u_{n-1} + D_t \varphi(u_n)), \tag{3.11}$$

where

$$\Omega_n = u_{n+1} u_n (\alpha u_{n+1} - \alpha u_n - \psi'(u_{n+1}) + \psi'(u_n)).$$

As $D_t p_n \sim \Omega_n \sim 0$, then property (2.25) implies that

$$\frac{\partial^2 \Omega_n}{\partial u_{n+1} \partial u_n} = 2\alpha(u_{n+1} - u_n) - \Psi(u_{n+1}) + \Psi(u_n) = (T-1)(2\alpha u_n - \Psi(u_n)) = 0,$$

where $\Psi(z) = (z\psi'(z))'$. We are led to the following ODE for the function ψ: $\Psi(z) = (z\psi'(z))' = 2\alpha z + \beta$, where β is an arbitrary constant. Solving this ODE, we obtain a formula for ψ which depends on two other arbitrary constants:

$$\psi(z) = \frac{\alpha}{2} z^2 + \beta z + \gamma \log z + \delta. \tag{3.12}$$

The function Ω_n takes now the form

$$\Omega_n = (T-1)(\gamma u_n). \tag{3.13}$$

One can see from equations (3.11), (3.13) that $D_t p_n \sim 0$, i.e. no more restriction for the density p_n can appear. Using equations (3.10)–(3.13), one obtains the following formulae for p_n and q_n defining conservation law (2.16):

$$\begin{aligned} p_n &= \alpha p_n^3 + \beta p_n^2 + \gamma p_n^1 + \delta + (T-1)\varphi(u_n), \qquad \alpha \neq 0, \\ q_n &= \alpha\big(u_{n+1}u_n u_{n-1} + u_n^2 u_{n-1}\big) + \beta u_n u_{n-1} + \gamma(u_n + u_{n-1}) + \varepsilon + D_t\varphi(u_n). \end{aligned} \tag{3.14}$$

Here p_n^j are given by (2.39) and ε is an arbitrary constant of integration.

The functions p_n and q_n depend on five arbitrary constants and one arbitrary function φ, and the conserved density p_n is nothing but the linear combination of known conserved densities p_n^j and a trivial one: $\delta + (T-1)\varphi$. Formulae (3.14) give the most general form of a conservation law with density (3.8) in the case of the Volterra equation.

An example of the classification problem. The class of equations considered here is very simple, but it includes an integrable case apart from the Volterra equation. The classification problem is solved using only integrability condition (2.55) and its corollary (2.60).

The starting point of our classification is the following class of lattice equations:

$$\dot{u}_n = P(u_n)(u_{n+1} - u_{n-1}), \tag{3.15}$$

where $P'(u_n) \neq 0$, as we are interested in the nonlinear equations. There is only one unknown function P here, and the aim is to find all equations (3.15) satisfying integrability condition (2.55).

In order to use corollary (2.60) of equation (2.55), we find

$$p_n^{(1)} = \log P(u_n), \qquad \dot{p}_n^{(1)} = P'(u_n)(u_{n+1} - u_{n-1}). \tag{3.16}$$

Now we can rewrite equation (2.60) as the relation

$$P''(u_n)(u_{n+1} - u_{n-1}) + P'(u_{n-1}) - P'(u_{n+1}) = 0, \tag{3.17}$$

which must be identically satisfied for all values of three independent variables u_{n+1}, u_n, u_{n-1}. Applying the operator $\frac{\partial^2}{\partial u_n \partial u_{n+1}}$, we see that $P'''(u_n) = 0$, i.e. P is the quadratic polynomial with arbitrary constant coefficients:

$$P(u_n) = \alpha u_n^2 + \beta u_n + \gamma. \tag{3.18}$$

With P given by equation (3.18), condition (3.17) and equation (2.60) are satisfied. Moreover, $\sigma = 0$ in representation (2.61), i.e. integrability condition (2.55) is also satisfied. This follows from the formula

$$\dot{p}_n^{(1)} = (T-1)(2\alpha u_n u_{n-1} + \beta u_n + \beta u_{n-1}),$$

see equations (3.16), (3.18). So, the polynomial (3.18) describes all equations of the form (3.15) satisfying integrability condition (2.55).

Using the following linear point transformations, $\tilde{u}_n = c_1 u_n + c_2$, where $c_1 \neq 0$ and c_2 are constants, one can transform any equation of the form (3.15), (3.18) into the Volterra equation (2.2) or into the modified Volterra equation (2.165). This means that, up to point transformations, the resulting list of integrable equations of the form (3.15) consists of the Volterra and modified Volterra equations. As is known, equation (2.165) is transformed into equation (2.2) by the discrete Miura transformation (2.170), i.e. the list of integrable equations (3.15) consists, up to Miura-type transformations, in the Volterra equation only.

The classification has been finished in this simple case because the Volterra and modified Volterra equations are known to be integrable and to have the infinite hierarchies of generalized symmetries and conservation laws.

3.1.2. Lists of equations, transformations and master symmetries. Let us discuss the classification of equations of the form (2.1). In this case only three conditions (2.55), (2.129), (2.130) are used for the classification. The other two conditions (2.99), (2.105) are used for constructing simple conservation laws. For the sake of convenience, let us formulate again some of the above results, contained in theorems 2, 12, 14, in the following theorem.

Theorem 22. *If equation (2.1) has one generalized symmetry of order $m_1 \geqslant 2$ and one conservation law of order $m_2 \geqslant 3$, or it possesses two conservation laws of orders $m_1 > m_2 \geqslant 3$, then it must satisfy the following three conditions:*

$$\begin{aligned} &D_t \log \frac{\partial f_n}{\partial u_{n+1}} \sim 0, \\ &r_n = \log\left(-\frac{\partial f_n}{\partial u_{n+1}} \Big/ \frac{\partial f_n}{\partial u_{n-1}}\right) \sim 0, \qquad D_t \sigma_n + 2\frac{\partial f_n}{\partial u_n} \sim 0, \end{aligned} \tag{3.19}$$

where σ_n is such that $(T-1)\sigma_n = r_n$.

A list of equations satisfying these conditions is given below with no index n for simplicity. That can be done because equations (2.1) have no explicit dependence on the variable n. The Volterra equation (2.2) takes in such case the form $\dot{u} = u(u_1 - u_{-1})$.

List of Volterra-type equations

$$\dot{u} = P(u)(u_1 - u_{-1}) \tag{V1}$$

$$\dot{u} = P(u^2)\left(\frac{1}{u_1 + u} - \frac{1}{u + u_{-1}}\right) \tag{V2}$$

$$\dot{u} = Q(u)\left(\frac{1}{u_1 - u} + \frac{1}{u - u_{-1}}\right) \tag{V3}$$

$$\dot{u} = \frac{R(u_1, u, u_{-1}) + \nu R(u_1, u, u_1)^{1/2} R(u_{-1}, u, u_{-1})^{1/2}}{u_1 - u_{-1}} \tag{V4}$$

$$\dot{u} = y(u_1 - u) + y(u - u_{-1}), \qquad y' = P(y) \tag{V5}$$

$$\dot{u} = y(u_1 - u)y(u - u_{-1}) + \mu, \qquad y' = P(y)/y \tag{V6}$$

$$\dot{u} = (y(u_1 - u) + y(u - u_{-1}))^{-1} + \mu, \qquad y' = P(y^2) \tag{V7}$$

$$\dot{u} = (y(u_1 + u) - y(u + u_{-1}))^{-1}, \qquad y' = Q(y) \tag{V8}$$

$$\dot{u} = \frac{y(u_1 + u) - y(u + u_{-1})}{y(u_1 + u) + y(u + u_{-1})}, \qquad y' = P(y^2)/y \tag{V9}$$

$$\dot{u} = \frac{y(u_1 + u) + y(u + u_{-1})}{y(u_1 + u) - y(u + u_{-1})}, \qquad y' = Q(y)/y \tag{V10}$$

$$\dot{u} = \frac{(1 - y(u_1 - u))(1 - y(u - u_{-1}))}{y(u_1 - u) + y(u - u_{-1})} + \mu, \qquad y' = \frac{P(y^2)}{1 - y^2} \tag{V11}$$

Here $\nu \in \{0, \pm 1\}$, the functions $P(u)$ and $Q(u)$ are polynomials of the form

$$P(u) = \alpha u^2 + \beta u + \gamma, \tag{3.20}$$

$$Q(u) = \alpha u^4 + \beta u^3 + \gamma u^2 + \delta u + \varepsilon, \tag{3.21}$$

while R is the following polynomial of three variables:

$$R(u, v, w) = (\alpha v^2 + 2\beta v + \gamma)uw + (\beta v^2 + \lambda v + \delta)(u + w) + \gamma v^2 + 2\delta v + \varepsilon. \tag{3.22}$$

Coefficients of P, Q, R and the number μ are arbitrary constants; the functions y are given by ordinary differential equations.

It should be remarked that, using n- and t-dependent transformations, one can reduce this list. For example, using the transformation $\tilde{u}_n = (-1)^n u_n$, one can rewrite equation (V2) in the form (V3), as

$$P(u^2) = \alpha u^4 + \beta u^2 + \gamma. \tag{3.23}$$

However, we do not discuss here more detailed transformations of this kind.

The form of equations (V3) and (V4) is invariant under the linear-fractional transformations

$$\tilde{u}_n = \frac{c_1 u_n + c_2}{c_3 u_n + c_4}, \tag{3.24}$$

with constant coefficients. Only coefficients of Q, R are changed, while the number ν remains unchanged. The particular case of equation (V4) with $\nu = 0$,

$$\dot{u}_n = \frac{(u_{n+1} - u_n)(u_n - u_{n-1})}{u_{n+1} - u_{n-1}},$$

is completely invariant under the action of transformations (3.24).

The classification is carried out up to point transformations of the form

$$\tilde{u}_n = s(u_n), \qquad \tilde{t} = ct, \tag{3.25}$$

where $s' \neq 0$ and $c \neq 0$ is a constant. It can be shown that integrability conditions (3.19) are invariant under (3.25). Generalized symmetries and conservation laws are transformed into generalized symmetries and conservation laws of the same order. The following theorem has been formulated in [71], and its proof can be found in [72].

Theorem 23. *Equation (2.1) satisfies conditions (3.19) if and only if it can be written, using point transformations (3.25), as one of the equations (V1)–(V11).*

The integrability of almost all equations of the list, except for equation (V4) with $\nu = 0$, can be shown, using Miura-type transformations. Equation (V6) with $P = \gamma \neq 0$ should be considered separately because it is linearizable. The function y satisfies ODE $y' = \gamma/y$, whose solution is $y(z) = (2\gamma z + c)^{1/2}$, where c is an integration constant. An obvious point transformation (3.25) allows one to set $\gamma = 1/2$. Then the transformation $w_n = y(u_{n+1} - u_n)$

transforms equation (V6) with $y' = \gamma/y$ into the linear equation (2.176). Now the conserved densities for such equation (V6) are constructed in the same way as for equation (2.151) in section 2.7 (see formula (2.177)).

Many equations of the list are transformed into the Volterra equation (2.2) by non-invertible Miura-type transformations of the form

$$\tilde{u}_n = s\big(u_{n+k_1}, u_{n+k_1-1}, \ldots, u_{n+k_2}\big), \tag{3.26}$$

where $k_1 > k_2$. Transformations (3.26) and (2.168) of section 2.7 are very similar. After the change of variables $\hat{u}_n = \tilde{u}_{n-k_2}$, equation (3.26) takes the form (2.168) with $k = k_1 - k_2$. However, one can prove that some of equations like, for example, equation (V3) cannot be transformed into the Volterra equation in this way. In such case we need to use transformations of the form

$$\begin{aligned} \tilde{u}_n &= U\big(u_{n+k_1}, u_{n+k_1-1}, \ldots, u_{n+k_2}\big), \\ \tilde{v}_n &= V\big(u_{n+k_3}, u_{n+k_3-1}, \ldots, u_{n+k_4}\big), \end{aligned} \tag{3.27}$$

with $k_1 > k_2$ and $k_3 > k_4$, in order to reduce an equation to the system

$$\dot{u}_n = u_n(v_{n+1} - v_{n-1}), \qquad \dot{v}_n = u_{n+1} - u_{n-1}. \tag{3.28}$$

Using transformations (3.26), (3.27), one can construct the conserved densities in the same way as in section 2.7 using (2.168).

Let us explain how one can obtain conserved densities for the system (3.28). This system is nothing but one of the forms of the Toda model. In fact, denoting $\tilde{u}_n = u_{2n}$, $\tilde{v}_n = v_{2n-1}$ or $\tilde{u}_n = u_{2n+1}$, $\tilde{v}_n = v_{2n}$, one obtains from equations (3.28) the following system:

$$\dot{u}_n = u_n(v_{n+1} - v_n), \qquad \dot{v}_n = u_n - u_{n-1}. \tag{3.29}$$

One can see that (3.28) consists of two copies of systems (3.29). On the other hand, the Toda model (2.196) turns into (3.29) after the transformation

$$\tilde{u}_n = \mathrm{e}^{u_{n+1}-u_n}, \qquad \tilde{v}_n = \dot{u}_n, \tag{3.30}$$

and we can call (3.29) the Toda system.

The known L–A pair of the Toda system can be rewritten for (3.28) as

$$\begin{aligned} &\dot{L}_n = [A_n, L_n], \\ &L_n = u_{n+1}^{1/2} T^2 + v_n + u_{n-1}^{1/2} T^{-2}, \qquad A_n = \tfrac{1}{2} u_{n+1}^{1/2} T^2 - \tfrac{1}{2} u_{n-1}^{1/2} T^{-2}. \end{aligned}$$

Conserved densities for (3.28) are obtained, using formulae (2.76), (2.77) of section 2.3, as in the case of the L–A pair (2.81), (2.82).

It turns out that the Volterra equation can be transformed into (3.28) too, and we have the following general theorem.

Theorem 24. *Any nonlinear equation of the form (V1)–(V11), except for equation (V4) with $\nu = 0$ and equation (V6) with $y' = \gamma/y$, can be transformed into the system (3.28) by a transformation of the form (3.27).*

Proof. The Volterra equation (2.2) is transformed into equations (3.28) by $\tilde{u}_n = u_{n+1}u_n$, $\tilde{v}_n = u_{n+1} + u_n$. As pointed out at the end of section 3.1.1, the nonlinear equations (V1) split into equation (2.2) and the modified Volterra equation (2.165), using point transformations. Equation (2.165) is transformed into (2.2) by the discrete Miura transformation (2.170).

It is easy to check that transformations of the form $\tilde{u}_n = y(u_{n+1} + u_n)$ transform equation (V9) into (V2) and equations (V8), (V10) into (V3). The transformations $\tilde{u}_n = y(u_{n+1} - u_n)$ transform equations (V5), (V6) into (V1) and equations (V7), (V11) into (V2). So, we have to discuss now only three equations: (V2), (V3) and (V4) with $\nu \neq 0$.

Equations of the form (V2) defined by (3.23) split into two cases: $\alpha = 0$ and $\alpha \neq 0$. If $\alpha \neq 0$, then the polynomial (3.23) can be written as

$$P(u^2) = \alpha(u^2 - a^2)(u^2 - b^2).$$

We can transform equations (V2) into the Volterra equation in both cases, using the following transformations:

$$\tilde{u}_n = -\frac{P\left(u_n^2\right)}{(u_{n+1} + u_n)(u_n + u_{n-1})}, \qquad \tilde{u}_n = -\alpha\frac{(u_{n+1} + a)\left(u_n^2 - b^2\right)(u_{n-1} - a)}{(u_{n+1} + u_n)(u_n + u_{n-1})}.$$

The linear-fractional transformations (3.24) can be used to simplify equations (V3), (V4). In the case of (V3), one can obtain in this way $\alpha = 0$ in the polynomial (3.21). If $\beta = 0$, then the transformation into (2.2) is given by

$$\tilde{u}_n = \Omega_n, \qquad \Omega_n = -\frac{Q(u_n)}{(u_{n+1} - u_n)(u_n - u_{n-1})}.$$

In the case $\beta \neq 0$, equation (V3) is transformed into the system (3.28) by the transformation

$$\tilde{u}_n = \Omega_{n+1}\Omega_n, \qquad \tilde{v}_n = \Omega_{n+1} + \Omega_n - \beta(u_{n+1} + u_n).$$

In the case of equation (V4) with $\nu \neq 0$, we introduce the function

$$\Delta_n = \frac{R(u_{n+1}, u_n, u_{n+1})^{1/2} + \nu R(u_{n-1}, u_n, u_{n-1})^{1/2}}{u_{n+1} - u_{n-1}}.$$

Using transformations (3.24), we obtain two cases for the polynomial (3.22): $\alpha = \beta = 0$ and $\alpha = 1, \beta = \gamma = 0$. In the first case, the transformation into the Volterra equation is given by

$$\tilde{u}_n = -\frac{\nu}{2}(\Delta_{n+1} + \gamma^{1/2})(\Delta_n - \gamma^{1/2}).$$

In the second one, we have to consider two subcases: $\nu = -1$ and $\nu = 1$. In the first of them, we transform equation (V4) into the Volterra equation using the transformation

$$\tilde{u}_n = \tfrac{1}{2}(\Delta_{n+1} + u_{n+1})(\Delta_n - u_n).$$

In the last case, the formulae

$$\begin{aligned}\tilde{u}_n &= \tfrac{1}{4}(\Delta_{n+1} + u_{n+1})\left(\Delta_n^2 - u_n^2\right)(\Delta_{n-1} - u_{n-1}),\\ \tilde{v}_n &= -\tfrac{1}{2}((\Delta_{n+1} - u_{n+1})(\Delta_n - u_n) + (\Delta_n + u_n)(\Delta_{n-1} + u_{n-1}))\end{aligned}$$

enable us to transform (V4) with $\nu = \alpha = 1, \beta = \gamma = 0$ into (3.28). □

The transformations presented above are given in their complete form only in [72], some of them can be found in [16, 17, 71, 76]. As for equation (V4) with $\nu = 0$, we can state the following two results [72].

- Equation (V4) with $\nu = 0$ and R given by (3.22) is transformed into the Volterra equation (2.2) by a transformation of the form (3.26) if and only if it can be reduced, using the linear-fractional transformations (3.24), to the case $\alpha = \beta = 0$. Then the transformation is given by

$$\tilde{u}_n = A_n, \qquad A_n = -\frac{R(u_{n+1}, u_n, u_{n+1})}{(u_{n+2} - u_n)(u_{n+1} - u_{n-1})}.$$

- The same equation is transformed into the system (3.28) by the Miura transformation (3.27) iff it can be reduced by (3.24) to the case $\alpha\gamma = \beta^2, \alpha\delta = \beta(\lambda - \gamma)$. In this case the transformation into (3.28) is given by

$$\tilde{u}_n = A_{n+1}A_n, \qquad \tilde{v}_n = B_{n+1} + B_n, \qquad B_n = A_n - \alpha u_{n+1}u_n - \beta(u_{n+1} + u_n).$$

We see that, in general, this equation cannot be transformed into (3.28). So, up to Miura-type transformations, we have in this section three nonlinear cases: the system (3.28), equation (V4) with $\nu = 0$ and equation (V6) with $y' = \gamma/y$.

The integrability of all equations (V1)–(V11) has been shown, using Miura-type transformations, except for equation (V4) with $\nu = 0$. In this case we can construct a local master symmetry [10]. The form of the master symmetry is simple:

$$u_{n,\tau} = n\dot{u}_n, \tag{3.31}$$

where $\dot{u}_n$ is given by (V4) with $\nu = 0$. However, we need to introduce an explicit dependence on the time τ of (3.31) into the master symmetry and equation itself, and we do that, following the example of equation (2.192) presented at the end of section 2.7, which is a particular case of (V4) with $\nu = 0$.

Let coefficients of the polynomial R given by (3.22) be functions of τ. Let us define a new polynomial ρ as

$$\rho(u, v) = R(u, v, u) = \alpha u^2v^2 + 2\beta uv(u + v) + \gamma(u^2 + v^2) + 2\lambda uv + 2\delta(u + v) + \varepsilon. \tag{3.32}$$

Then the dependence on τ in equations (3.31) and (V4) with $\nu = 0$ is given by requiring that ρ satisfies the following partial differential equation:

$$\frac{\partial\rho}{\partial\tau} = \rho\frac{\partial^2\rho}{\partial u\partial v} - \frac{\partial\rho}{\partial u}\frac{\partial\rho}{\partial v}. \tag{3.33}$$

On the left-hand side of this equation, we only differentiate the coefficients of ρ with respect to τ. The polynomial on the right-hand side has the same form as ρ, but with different coefficients. That is why, collecting coefficients corresponding to the same terms u^iv^j, we obtain from equation (3.33) a system of six ODEs for six coefficients of the polynomial ρ. That system has solutions $\alpha(\tau), \beta(\tau), \ldots,$ for any initial conditions $\alpha(0) = \alpha_0, \beta(0) = \beta_0, \ldots$. Therefore, as in the case of equation (2.192), we can construct conservation laws and generalized symmetries of equation (V4) with $\nu = 0$ for any given constant coefficients of the polynomial (3.22).

One can see that the existence of a pair of conservation laws or of one generalized symmetry and one conservation law, with orders given in theorem 22, implies the existence of an infinite hierarchy of conservation laws. It turns out that integrability conditions (3.19) are not only necessary but also sufficient for the integrability of equations (2.1). That is why these conditions can be used for testing a given equation for integrability. It is convenient to use for such testing an explicit form of the last of conditions (3.19) which is given by formula (2.150).

The problem of constructing the generalized symmetries for all equations (V1)–(V11) remains open. However, many equations of the form (V1)–(V3) and (V4) with $\nu \neq 0$ have local master symmetries (see [16, 17]) and therefore generalized symmetries. For example, equation (V2) defined by the polynomial

$$P(u^2) = (1 - u^2)(a^2 - b^2u^2)$$

has the following master symmetry:

$$u_{n,\tau} = P\left(u_n^2\right)\left(\frac{n}{u_{n+1} + u_n} - \frac{n-1}{u_n + u_{n-1}}\right) + b^2u_n\left(1 - u_n^2\right).$$

The dependence on the time τ is introduced into the coefficients a and b as follows:

$$a(\tau) = \lambda_1(\tau) - \lambda_2(\tau), \qquad b(\tau) = \lambda_1(\tau) + \lambda_2(\tau),$$

where both functions λ_j satisfy the same ODE, namely $\lambda_j' = \frac{1}{2}\lambda_j^3$.

The L–A pairs can also be constructed, if needed, for all integrable equations obtained by the generalized symmetry method. For instance, an L–A pair for equation (V2), (3.23) can be found in [76].

3.2. Toda-type equations

We discuss here the class (2.195) of lattice equations including the well-known Toda model (2.196). It should be remarked that a more narrow class, such that $\frac{\partial f_n}{\partial \dot{u}_n} = 0$, also contains equation (2.196), but has only one more integrable example that is a trivial modification of (2.196), given by equation (T3) below, and thus it is not interesting. The class (2.195) is the most simple nontrivial class of equations, including the Toda model, which is invariant under the point transformations (3.25).

Integrability conditions [10, 74] necessary and sufficient for the exhaustive classification are given by

$$\begin{aligned} &D_t p_n^{(i)} = (T-1)q_n^{(i)}, \qquad i = 1, 2, 3, \\ &p_n^{(1)} = \log \frac{\partial f_n}{\partial u_{n+1}}, \qquad p_n^{(2)} = 2q_n^{(1)} + \frac{\partial f_n}{\partial \dot{u}_n}, \\ &p_n^{(3)} = 2q_n^{(2)} - \frac{1}{2} D_t \frac{\partial f_n}{\partial \dot{u}_n} + \frac{1}{4}\left(\frac{\partial f_n}{\partial \dot{u}_n}\right)^2 + \frac{1}{4}\left(p_n^{(2)}\right)^2 + \frac{\partial f_n}{\partial u_n}, \end{aligned} \tag{3.34}$$

$$\begin{aligned} &r_n^{(j)} = (T-1)\sigma_n^{(j)}, \qquad j = 1, 2, \\ &r_n^{(1)} = \log\left(\frac{\partial f_n}{\partial u_{n+1}} \Big/ \frac{\partial f_n}{\partial u_{n-1}}\right), \qquad r_n^{(2)} = D_t \sigma_n^{(1)} + \frac{\partial f_n}{\partial \dot{u}_n}. \end{aligned} \tag{3.35}$$

In conditions (3.34) and (3.35), we require the existence of some functions $q_n^{(i)}, \sigma_n^{(j)}$ depending on a finite number of independent variables $u_{n+k}, \dot{u}_{n+k}$. Using conditions (3.34), one can construct for a given integrable equation low-order conservation laws.

Theorem 25. *Let us assume that equation (2.195) has one generalized symmetry of order $m_1 \geqslant 5$ and one conservation law of order $m_2 \geqslant 4$ or possesses two conservation laws of orders $m_1 > m_2 \geqslant 7$. Then this equation satisfies conditions (3.34), (3.35).*

Let us write down a complete list of lattice equations of the form (2.195) satisfying conditions (3.34), (3.35). For simplicity, we write them down using the notations $u = u_n, u_1 = u_{n+1}$ and $u_{-1} = u_{n-1}$.

List of Toda-type equations

$$\ddot{u} = P(\dot{u})(y(u_1 - u) - y(u - u_{-1})), \qquad y' = Q(y) \tag{T1}$$

$$\ddot{u} = (S(u) - \dot{u}^2)\left(\frac{1}{u_1 - u} - \frac{1}{u - u_{-1}}\right) + \frac{S'(u)}{2} \tag{T2}$$

$$\ddot{u} = \mathrm{e}^{u_1 - 2u + u_{-1}} + \mu \tag{T3}$$

$$\ddot{u} = (\dot{u}^2 - s(u))\left(\frac{1}{u_1 + u} + \frac{1}{u + u_{-1}}\right) + \frac{s'(u)}{2} \tag{T4}$$

Here P, Q, S and s are the following polynomials:

$$P(z) = \varepsilon z^2 + az + b, \qquad Q(z) = \varepsilon z^2 + cz + d, \tag{3.36}$$

$$S(z) = c_4 z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0, \qquad s(z) = c_4 z^4 + c_2 z^2 + c_0. \tag{3.37}$$

Coefficients of these polynomials and the number μ are arbitrary constants. Equation (T2) with $S = 0$ has been studied in [15]. The following theorem and its proof can be found in [74, 77].

Theorem 26. *Equation (2.195) satisfies conditions (3.34), (3.35) if and only if it can be rewritten, using the point transformations (3.25), as one of the equations (T1)–(T4).*

If, instead of transformations (3.25), we consider simple point transformations depending explicitly on n and t:

$$\tilde{u}_n = s_n(t, u_n), \qquad \tilde{t} = \theta(t), \tag{3.38}$$

we can reduce the number of arbitrary constants in the above list of equations and rewrite, for example, equations (T1) in an explicit form. Namely, one can transform any equation (T1) into one of the following lattice equations.

Detailed form of equations (T1)

$$\ddot{u} = \mathrm{e}^{u_1-u} - \mathrm{e}^{u-u_{-1}} \tag{Td1}$$

$$\ddot{u} = \dot{u}(u_1 - 2u + u_{-1}) \tag{Td2}$$

$$\ddot{u} = \dot{u}(\mathrm{e}^{u_1-u} - \mathrm{e}^{u-u_{-1}}) \tag{Td3}$$

$$\ddot{u} = (\alpha^2 - \dot{u}^2)\left(\frac{1}{u_1 - u} - \frac{1}{u - u_{-1}}\right) \tag{Td4}$$

$$\ddot{u} = (\alpha^2 - \dot{u}^2)(\tanh(u_1 - u) - \tanh(u - u_{-1})). \tag{Td5}$$

Here α is an arbitrary constant; equation (Td1) is the Toda model (2.196).

Moreover, the change of variables $\tilde{u}_n = (-1)^n u_n$ allows one to transform (T4) into an equation of the form (T2). Transformations (3.38), we use here, are invertible and enable us to rewrite solutions and, if necessary, generalized symmetries and conservation laws. Those transformations do not introduce any explicit dependence on the variables n and t into the generalized symmetries and conservation laws, but that will not be proved. For this reason, the integrability will be shown below only for equations (T2), (T3) and (Td1)–(Td5).

Equations (T1) and (T2) and, therefore, (Td1)–(Td5) can be expressed in Hamiltonian and Lagrangian forms. This is useful from the viewpoint of physical applications. The Hamiltonian form given by equations (2.220) (see also (2.205)), where $v_n = \dot{u}_n$, and formula (2.224) can be used for constructing generalized symmetries. The function φ_n and Hamiltonian density h_n are given for equation (T1) by

$$\varphi_n = P(v_n), \qquad h_n = Y(u_{n+1} - u_n) + Z(v_n), \tag{3.39}$$

$$Y'(z) = y(z), \qquad Z'(z) = z/P(z), \tag{3.40}$$

and in the case of equation (T2) by

$$\varphi_n = v_n^2 - S(u_n), \qquad h_n = \tfrac{1}{2}\log\varphi_n - \log(u_{n+1} - u_n). \tag{3.41}$$

The Lagrangian form, which is defined by formulae (3.55), (3.56), is discussed in detail in section 3.3.1. In particular, we have formulae (3.75), (3.76) for constructing two extra conservation laws. Here we only write down the Lagrangians [10] of the form

$$L = R(\dot{u}_n, u_n) - Y(u_{n+1} - u_n). \tag{3.42}$$

In the case of equation (T1), the functions R and Y are defined by

$$R = R(\dot{u}_n), \qquad R''(z) = 1/P(z), \qquad Y'(z) = y(z), \tag{3.43}$$

while in the case of equation (T2), defining

$$S_+ = \sqrt{S(u_n)} + \dot{u}_n, \qquad S_- = \sqrt{S(u_n)} - \dot{u}_n,$$

one has for $S \neq 0$:

$$R = \frac{S_+ \log S_+ + S_- \log S_-}{2\sqrt{S(u_n)}}, \qquad Y = \log(u_{n+1} - u_n).$$

Equation (T2) with $S = 0$ is nothing but (Td4) with $\alpha = 0$, and thus we can use formulae (3.43).

Theorem 27. *Any equation of the form (T3) or (Td1)–(Td5) can be transformed into the Toda system (3.29) by a Miura-type transformation.*

Proof. Equation (T3) is transformed into the Toda model (Td1) by the following transformation: $\tilde{u}_n = u_{n+1} - u_n$. All equations (Td1)–(Td5) can be rewritten as systems in the form

$$\begin{aligned} \dot{u}_n &= A(u_n)(v_{n+1} - v_n), \\ \dot{v}_n &= B(v_n)(u_n - u_{n-1}), \end{aligned} \tag{3.44}$$

with the following three possibilities:

$$\begin{array}{lll} \textit{Case 1}: & A(z) = z & B(z) = 1 \\ \textit{Case 2}: & A(z) = z & B(z) = z \\ \textit{Case 3}: & A(z) = z^2 - \alpha^2 & B(z) = z^2 - \beta^2. \end{array}$$

In fact, equation (Td1) is transformed into case 1 by (3.30). Transformations of equation (Td2) into the same case 1 and of equation (Td3) into case 2 are given by $\tilde{u}_n = \dot{u}_n, \tilde{v}_n = y(u_n - u_{n-1})$, where the function y is defined by (T1). The transformation $\tilde{u}_n = \dot{u}_n, \tilde{v}_n = -y(u_n - u_{n-1})$ brings equations (Td4), (Td5) into case 3, where $\beta = 0$ for (Td4) and $\beta = 1$ for (Td5).

Moreover, case 3 is transformed into case 2 by

$$\tilde{u}_n = (u_n + \alpha)(v_{n+1} + \beta), \qquad \tilde{v}_n = (u_n - \alpha)(v_n - \beta),$$

and case 2 is transformed into case 1 by the transformation $\tilde{u}_n = u_n v_{n+1}, \tilde{v}_n = u_n + v_n$. As the system (3.29) corresponds to case 1, the proof has been finished. □

All transformations contained in the proof of theorem 27 can be found in [54]. It turns out that we can write down local master symmetries for the systems (3.44) (see [30, 48]). Those master symmetries are of the form

$$\begin{aligned} u_{n,\tau} &= A(u_n)((2n + k)v_{n+1} - 2nv_n) + \gamma u_n^2, \\ v_{n,\tau} &= B(v_n)((2n - 1 + k)u_n - (2n - 1)u_{n-1}) + \delta v_n^2, \end{aligned} \tag{3.45}$$

where

$$\begin{array}{llll} \textit{Case 1}: & k = 4 & \gamma = 0 & \delta = 1 \\ \textit{Case 2}: & k = 3 & \gamma = 1 & \delta = 1 \\ \textit{Case 3}: & k = 2 & \gamma = 0 & \delta = 0. \end{array}$$

In order to construct new conserved densities, using formula (2.189), we need a starting conserved density. The systems (3.44) have two obvious densities:

$$p_n^+ = \int \frac{\mathrm{d}u_n}{A(u_n)}, \qquad p_n^- = \int \frac{\mathrm{d}v_n}{B(v_n)}.$$

In cases 1 and 2, both can be used as starting ones. In case 1, master symmetry (3.45) provides the system (3.44) with the following conserved density:

$$D_\tau p_n^+ \sim 2p_n^-, \qquad D_\tau p_n^- \sim 2u_n + v_n^2,$$

and in case 2, we are led to

$$D_\tau p_n^+ \sim D_\tau p_n^- \sim u_n + v_n.$$

As $D_\tau p_n^+ \sim D_\tau p_n^- \sim 0$ in case 3, one needs to start from a different density:

$$p_n^0 = \log(u_n + \alpha) + \log(v_n + \beta), \qquad D_\tau p_n^0 \sim u_n(v_{n+1} + v_n).$$

We can construct the conserved densities not only for (3.44) but also for equations (T3), (Td1)–(Td5), using the transformations given in the proof of theorem 27. Equations (Td1)–(Td5) are Hamiltonian, therefore we can obtain for them also the generalized symmetries.

Let us now discuss equation (T2). We do not know how to transform it into the Toda model. However, there is a connection with equation (V4), such that $\nu = 0$, considered in section 3.1.2. In fact, introducing the functions $\tilde{u}_n = u_{2n}$ and $\tilde{v}_n = u_{2n-1}$, we can pass from equation (V4) with $\nu = 0$ to the system

$$\dot{u}_n = \frac{2\rho}{v_{n+1} - v_n} + \rho_{v_n}, \qquad \dot{v}_n = \frac{2\rho}{u_n - u_{n-1}} - \rho_{u_n}, \qquad \rho = \rho(u_n, v_n), \tag{3.46}$$

where we denote by the indices partial derivatives of ρ, and ρ is defined by equations (3.32), (3.22). The function ρ is a quadratic polynomial of each variable, and if one considers the discriminator S with respect to v_n:

$$S = \left(\rho_{v_n}\right)^2 - 2\rho\rho_{v_n v_n}, \tag{3.47}$$

one obtains a function of u_n. Besides, $S(u_n)$ is a fourth degree polynomial. Differentiating the first part of equations (3.46) with respect to t and then using the second part, it is possible to express the result only in terms of u_{n+j}, $\dot{u}_{n+j}$ and to rewrite it as equation (T2) with S given by (3.47). That is why for any solution (u_n, v_n) of the system (3.46), the function u_n satisfies an equation of the form (T2), (3.37). However, this connection cannot be used for constructing local conservation laws and generalized symmetries of equation (T2).

We solve this problem using a master symmetry [10]. Passing from equation (T2) to an equivalent system for the functions u_n and $v_n = \dot{u}_n$, we can write down the master symmetry as

$$u_{n,\tau} = (\lambda + 2n)v_n, \qquad v_{n,\tau} = (\lambda + 2n)\dot{v}_n + u_{n,t'}. \tag{3.48}$$

Here λ is an arbitrary constant, $\dot{v}_n$ is given by equation (T2), while $u_{n,t'}$ by the equation

$$u_{n,t'} = \left(S(u_n) - v_n^2\right)\left(\frac{1}{u_{n+1} - u_n} + \frac{1}{u_n - u_{n-1}}\right).$$

This equation with $v_n = \dot{u}_n$ is nothing but the generalized symmetry of equation (T2). Conserved densities can be constructed, starting for instance from the Hamiltonian density h_n given by (3.41).

If the number λ is not an even integer, then we can exclude the function v_n from the second part of equations (3.48) using the first one. In this way, one obtains from the master symmetry an n-dependent second-order differential difference equation:

$$u_{n,\tau\tau} = \left(\eta S(u_n) - \frac{u_{n,\tau}^2}{\eta}\right)\left(\frac{\eta + 1}{u_{n+1} - u_n} - \frac{\eta - 1}{u_n - u_{n-1}}\right) + \frac{\eta^2}{2}S'(u_n), \tag{3.49}$$

where $\eta = \lambda + 2n$. Master symmetries are known to be integrable in some sense (see, in the case of lattice equations, e.g. [13, 29, 30, 66]), and equation (3.49) exemplifies a nice equation of this kind.

As we now see, conditions (3.34), (3.35) are necessary and sufficient for the integrability and can be used as a testing tool for equations of the form (2.195). As in section 2.5.2, in the case of the Volterra-type equations, all five conditions (3.34), (3.35) can be rewritten in an explicit form convenient for such testing [79].

As in the case of equations (V3), (V4) considered in section 3.1.2, the form of equation (T2) is invariant under the linear-fractional transformations (3.24). Only the coefficients of S (3.37) are changed by these transformations. Equations of this kind can appear in practice and expressed in terms of the elliptic functions. Let us consider an interesting example of the Toda type [28]:

$$\ddot{u}_n = \left(\dot{u}_n^2 - 1\right)(\zeta(u_n + u_{n+1}) + \zeta(u_n - u_{n+1}) + \zeta(u_n + u_{n-1}) + \zeta(u_n - u_{n-1}) - 2\zeta(2u_n)), \tag{3.50}$$

where by ζ we mean the ζ-function of Weierstrass. Recall that $\zeta'(z) = -\wp(z)$, where the $\wp$-function of Weierstrass is defined by the ODE $\wp'^2(z) = 4\wp^3(z) + \alpha\wp(z) + \beta$ with constants coefficients. Standard formulae for the elliptic functions allow us to rewrite equation (3.50) as follows:

$$\ddot{u}_n = \left(\dot{u}_n^2 - 1\right)\left(\frac{\wp'(u_n)}{\wp(u_n) - \wp(u_{n+1})} + \frac{\wp'(u_n)}{\wp(u_n) - \wp(u_{n-1})} - \frac{\wp''(u_n)}{\wp'(u_n)}\right).$$

Introducing $\tilde{u}_n = \wp(u_n)$, we can transform this equation into equation (T2) with $S(z) = 4z^3 + \alpha z + \beta$.

3.3. Relativistic Toda-type equations

We discuss in this section lattice equations of the following two classes:

$$\begin{aligned} &\ddot{u}_n = f(u_{n+1}, u_n, \dot{u}_{n+1}, \dot{u}_n) - g(u_n, u_{n-1}, \dot{u}_n, \dot{u}_{n-1}), \\ &f_{\dot{u}_{n+1}} g_{\dot{u}_{n-1}} \neq 0, \end{aligned} \tag{3.51}$$

$$\begin{aligned} &\dot{u}_n = f(u_{n+1}, u_n, v_n), \qquad \dot{v}_n = g(v_{n-1}, v_n, u_n), \\ &f_{u_{n+1}} f_{v_n} g_{v_{n-1}} g_{u_n} \neq 0, \end{aligned} \tag{3.52}$$

where partial derivatives of f, g are denoted by indices. Each of the classes is important in itself and has its own applications [10]. As will be shown below, there is a nontrivial connection between them [10]. All equations (3.51) will be Lagrangian, while systems of the form (3.52) corresponding to them will be Hamiltonian systems. For example, the relativistic Toda lattice equation [50, 51]

$$\ddot{u}_n = \frac{\dot{u}_{n+1}\dot{u}_n}{1 + \mathrm{e}^{u_n - u_{n+1}}} - \frac{\dot{u}_n \dot{u}_{n-1}}{1 + \mathrm{e}^{u_{n-1} - u_n}} \tag{3.53}$$

is of the form (3.51). Other integrable equations of the form (3.51), (3.52) have analogous algebraic properties and are called relativistic Toda-type equations for this reason. Lagrangian and Hamiltonian equations of this kind have been discussed in [8, 10, 18, 19, 27, 36, 53, 74].

In section 3.3.1, we discuss a non-point connection between the Lagrangian and Hamiltonian forms of relativistic Toda-type equations (see [10, 78]) and at the end give some useful remarks about generalized symmetries and conservation laws of the Lagrangian equations [78]. Then, in sections 3.3.2 and 3.3.3, we separately describe these Hamiltonian and Lagrangian forms and give two lists of integrable equations: (H1)–(H3) and (L1), (L2). In section 3.3.4, we point out the exact correspondence between equations of two lists and show in section 3.3.5, by constructing the master symmetries, that all of them possess generalized symmetries and conservation laws.

3.3.1. Non-point connection between Lagrangian and Hamiltonian equations and properties of Lagrangian equations. First, let us recall some well-known facts of classical mechanics. Given a Lagrangian function $L = L(u, \dot{u})$, the Euler–Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{u}} = \frac{\partial L}{\partial u}, \qquad \frac{\partial^2 L}{\partial \dot{u}^2} \neq 0. \tag{3.54}$$

If we introduce the function $v = L_{\dot{u}}$, we can express $\dot{u}$ in terms of u and v, i.e. have an invertible transformation: $(u, \dot{u}) \leftrightarrow (u, v)$. The Legendre transformation $H = v\dot{u} - L$ defines the relation between Lagrangian L and Hamilton's function $H = H(u, v)$ and leads to the equations of Hamilton:

$$\dot{u} = \frac{\partial H}{\partial v}, \qquad \dot{v} = -\frac{\partial H}{\partial u}.$$

It can be easily proved that the Euler–Lagrange equation and Hamilton's equations are equivalent. We give such a proof below in a more general case.

Let us now consider Lagrangians, depending on a field u_n living on the lattice, of the form

$$L = L(\dot{u}_n, u_{n+1}, u_n), \qquad \frac{\partial^2 L}{\partial \dot{u}_n^2} \neq 0. \tag{3.55}$$

The Euler–Lagrange equation is then defined as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{u}_n} = \frac{\partial}{\partial u_n}(1 + T^{-1})L. \tag{3.56}$$

On the right-hand side we have the formal variational derivative $\frac{\delta L}{\delta u_n}$ (see (2.205)). The relativistic Toda lattice equation (3.53) is a Lagrangian equation of the form (3.55), (3.56) [8] with Lagrangian

$$L = \dot{u}_n \log \frac{\dot{u}_n}{\mathrm{e}^{u_{n+1} - u_n} + 1}. \tag{3.57}$$

The Legendre transformation

$$H = v_n \dot{u}_n - L, \quad v_n = L_{\dot{u}_n}, \tag{3.58}$$

leads in this case to an invertible change of variables between the two sets of variables: $\{u_n, \dot{u}_n\}$ and $\{u_n, v_n\}$. The formula for v_n has the form $v_n = s(\dot{u}_n, u_{n+1}, u_n)$, while u_n remains unchanged. As $s_{\dot{u}_n} \neq 0$ due to (3.55), then $\dot{u}_n$ easily can be expressed via v_n, u_{n+1}, u_n. This is a non-point transformation because the function s depends also on u_{n+1}.

It can be easily proved that the Legendre transformation (3.58) gives the following Hamiltonian system:

$$\dot{u}_n = \frac{\delta H}{\delta v_n}, \qquad \dot{v}_n = -\frac{\delta H}{\delta u_n}, \qquad H = H(v_n, u_{n+1}, u_n), \tag{3.59}$$

$$\frac{\delta H}{\delta v_n} = \frac{\partial H}{\partial v_n}, \qquad \frac{\delta H}{\delta u_n} = \frac{\partial}{\partial u_n}(1 + T^{-1})H \tag{3.60}$$

(see equations (2.205)). Comparing (3.59) with (2.220), one can see that we have substituted h_n by H to be more closed to classical formulae.

All known integrable equations (3.51) have the Lagrangian structure (3.55), (3.56). In those cases, it is possible not only to pass to the Hamiltonian formulation (3.59) but also to transform equations (3.59) into a more simple form (3.52). This can be done, introducing an additional point transformation of the form $\hat{u}_n = a(u_n)$, $\hat{v}_n = b(u_n, v_n)$. The Hamiltonian H (or Hamiltonian density) also is simplified, and the Hamiltonian system now reads

$$\dot{u}_n = \varphi(u_n, v_n)\frac{\delta H}{\delta v_n}, \qquad \dot{v}_n = -\varphi(u_n, v_n)\frac{\delta H}{\delta u_n}, \qquad H = \Phi(u_n, v_n) + \Psi(u_{n+1}, v_n). \tag{3.61}$$

For example, in the case of the relativistic Toda lattice equation (3.53), the additional point transformation is $\hat{u}_n = \mathrm{e}^{u_n}$, $\hat{v}_n = \mathrm{e}^{v_n - u_n - 1}$. The resulting system of the form (3.52) is

$$\dot{u}_n = u_n v_n (u_{n+1} + u_n), \qquad \dot{v}_n = -u_n v_n (v_n + v_{n-1}). \tag{3.62}$$

This system, equivalent to equation (3.53), has the Hamiltonian structure (3.61) with $\varphi = u_n v_n$ and $H = u_n v_n + u_{n+1} v_n$. The invertible transformation of (3.53) into (3.62) is given by

$$\hat{u}_n = \mathrm{e}^{u_n}, \qquad \hat{v}_n = \frac{\dot{u}_n}{\mathrm{e}^{u_{n+1}} + \mathrm{e}^{u_n}}. \tag{3.63}$$

Let us now discuss how to go back from the Hamiltonian system (3.61) to the Lagrangian equation (3.55), (3.56). This is explained by the following theorem.

Theorem 28. *If (u_n, v_n) is a solution of the systems (3.52), (3.61) with $f_{v_n} \neq 0$ and with Hamiltonian H of the general form $H(u_{n+1}, u_n, v_n)$, then the functions y_n, z_n,*

$$y_n = u_n, \qquad z_n = \dot{u}_n = f(u_{n+1}, u_n, v_n), \tag{3.64}$$

satisfy the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial z_n} = \frac{\partial}{\partial y_n}(1 + T^{-1})L, \tag{3.65}$$

with $L(z_n, y_{n+1}, y_n)$ defined by

$$L = \psi(u_n, v_n)\dot{u}_n - H, \qquad \psi_{v_n} = 1/\varphi, \tag{3.66}$$

which is equivalent to equations (3.55), (3.56).

Proof. The invertible transformation (3.64) implies the following relations for partial derivatives:

$$\frac{\partial}{\partial v_n} = f_{v_n}\frac{\partial}{\partial z_n}, \qquad \frac{\partial}{\partial u_n} = \frac{\partial}{\partial y_n} + f_{u_n}\frac{\partial}{\partial z_n} + T^{-1}(f_{u_{n+1}})\frac{\partial}{\partial z_{n-1}}. \tag{3.67}$$

Equations (3.66), (3.67) allow us to find

$$L_{z_n} = \psi, \qquad L_{y_n} = \psi_{u_n} f - H_{u_n}, \qquad L_{y_{n+1}} = -H_{u_{n+1}}, \tag{3.68}$$

where we have used the relation $f = \varphi H_{v_n}$ which comes from equations (3.52), (3.60), (3.61). Then equation (3.65) follows from equations (3.52), (3.60), (3.61), (3.68) and $\psi_{v_n} = 1/\varphi$:

$$D_t L_{z_n} = D_t \psi = \psi_{u_n} f + \psi_{v_n} g = \psi_{u_n} f - \left(H_{u_n} + T^{-1} H_{u_{n+1}}\right) = L_{y_n} + T^{-1} L_{y_{n+1}}.$$

We can see that

$$L_{z_n z_n} = \left(\varphi f_{v_n}\right)^{-1} \neq 0, \tag{3.69}$$

and, due to (3.64), the equivalence of (3.65) and (3.55), (3.56) is obvious. □

By the point transformation $\tilde{u}_n = s(u_n)$, one can change an equation and its Lagrangian, but the form of Euler–Lagrange equation given by (3.55), (3.56) remains unchanged. On the other hand, one can introduce a new Lagrangian as follows:

$$\tilde{L} = \alpha L + \beta + \sigma(u_n)\dot{u}_n + (T - 1)\omega(u_n), \tag{3.70}$$

where $\alpha \neq 0$ and β are constants, while σ and ω are arbitrary functions. In this case, not only the Lagrangian structures (3.55), (3.56) but also the corresponding lattice equation is not changed.

If, using theorem 28, we pass from the systems (3.52), (3.61) to the Lagrangian equations (3.55), (3.56), we obtain in general an equation of the form

$$\ddot{u}_n = F(u_{n+1}, u_n, u_{n-1}, \dot{u}_{n+1}, \dot{u}_n, \dot{u}_{n-1}).$$

However, in known integrable cases, the resulting form of the Lagrangian is

$$L = L_0(\dot u_n, u_n) + \dot u_n V(u_{n+1}, u_n) + U(u_{n+1}, u_n). \tag{3.71}$$

An equation with such Lagrangian is of the class (3.51). This form of equations and Lagrangians is invariant under point transformations $\tilde u_n = s(u_n)$.

Let us consider as an example the well-known lattice system [53, 80]:

$$\dot u_n = u_{n+1} + u_n^2 v_n, \qquad \dot v_n = -v_{n-1} - v_n^2 u_n. \tag{3.72}$$

Its Hamiltonian structure (3.61) is defined by the functions $\varphi = 1$, $H = u_{n+1}v_n + \frac{1}{2}u_n^2 v_n^2$. Using theorem 28, we obtain a Lagrangian equation which, by the point transformation $\tilde u_n = \log u_n$, can be rewritten as

$$\ddot u_n = \dot u_{n+1}\,\mathrm{e}^{u_{n+1}-u_n} - \dot u_{n-1}\,\mathrm{e}^{u_n-u_{n-1}} - \mathrm{e}^{2(u_{n+1}-u_n)} + \mathrm{e}^{2(u_n-u_{n-1})}. \tag{3.73}$$

Equation (3.73) corresponds to the Lagrangian $L = (\dot u_n - \mathrm{e}^{u_{n+1}-u_n})^2$. The invertible transformation of the system (3.72) into equation (3.73) is

$$\tilde u_n = \log u_n, \qquad \tilde u_{n,t} = \frac{u_{n+1}}{u_n} + u_n v_n. \tag{3.74}$$

Let us consider now conservation laws of Lagrangian equations. In the classical case given by equation (3.54), one has the constant of motion $I_1 = \dot u L_{\dot u} - L$. Indeed, using (3.54), one can easily prove that $\mathrm{d}I_1/\mathrm{d}t = 0$. If $L_u = 0$, then $I_2 = L_{\dot u}$ is another constant of motion. Passing to the lattice equations (3.55), (3.56), we have local conservation laws instead of constants of motion.

The Hamiltonian H is always conserved density for the system (3.61), as we have shown at the very end of section 2.8. Rewriting H in terms of the variables (3.64), one is led to a conserved density for the Lagrangian equation. Using formulae (3.66), (3.68), we obtain $H = \psi\dot u_n - L = \dot u_n L_{\dot u_n} - L$, and this is a conserved density for the Lagrangian equation. Indeed, one can easily check that the following conservation law for equation (3.56) takes place:

$$D_t\big(\dot u_n L_{\dot u_n} - L\big) = (T^{-1} - 1)(\dot u_{n+1} L_{u_{n+1}}). \tag{3.75}$$

If the Lagrangian has the form $L = L(\dot u_n, u_{n+1} - u_n)$, we have another conservation law for this equation:

$$D_t L_{\dot u_n} = (1 - T^{-1}) L_{u_n}. \tag{3.76}$$

As well as in the classical case, there is for the Lagrangian equations (3.55), (3.56) the standard Noether's connection between conservation laws and symmetries. Constructing a local conservation law, starting from a generalized symmetry, is discussed in [8]. However, we are more interested here in the passage from conservation laws to symmetries [78]. So in the following, using the equivalence of Lagrangian and Hamiltonian equations, we write down a simple formula for constructing generalized symmetries.

Let us consider the Hamiltonian systems (3.61) and Lagrangian equations (3.55), (3.56) related by theorem 28. This is the general case, as Hamiltonian H has the general form $H(u_{n+1}, u_n, v_n)$. Let p be a conserved density of the Lagrangian equation,

$$p = p\big(\dot u_{n+i_1}, \dot u_{n+i_1-1}, \ldots, \dot u_{n+i_2}, u_{n+j_1}, u_{n+j_1-1}, \ldots, u_{n+j_2}\big), \tag{3.77}$$

where $i_1 \geqslant i_2$, $j_1 \geqslant j_2$. Using the invertible transformation (3.64), we can pass to a density $\hat p$ of the corresponding Hamiltonian system which depends of the variables u_{n+i}, v_{n+j}. By changing p, we only replace $\dot u_{n+i}$ by the functions $f(u_{n+1+i}, u_{n+i}, v_{n+i})$. As it has been shown in section 2.8 in (2.224), the generalized symmetry of the Hamiltonian system can be obtained as

$$u_{n,\tau} = \varphi \frac{\delta \hat{p}}{\delta v_n}, \qquad v_{n,\tau} = -\varphi \frac{\delta \hat{p}}{\delta u_n}. \tag{3.78}$$

If we return to the variables u_{n+j} and $\dot{u}_{n+i}$, i.e. to the Lagrangian equation (3.55), (3.56), we will have instead of (3.78) two formulae of the form $u_{n,\tau} = G$, $\dot{u}_{n,\tau} = \hat{G}$ expressed in terms of the conserved density (3.77) and Lagrangian L. The second equation follows from the first one, as $\hat{G} = D_t G$, and we only rewrite the first equation in order to obtain a generalized symmetry of the Lagrangian equation. Using equations (3.67), (3.69), we have in terms of the variables (3.64),

$$y_{n,\tau} = u_{n,\tau} = \varphi \frac{\partial}{\partial v_n} \sum_i T^i \hat{p} = \varphi f_{v_n} \frac{\partial}{\partial z_n} \sum_i T^i p = \left(L_{z_n z_n}\right)^{-1} \frac{\delta p}{\delta z_n}.$$

So, we are led to the following generalized symmetry:

$$u_{n,\tau} = \frac{1}{L_{\dot{u}_n \dot{u}_n}} \frac{\delta p}{\delta \dot{u}_n}, \qquad \frac{\delta p}{\delta \dot{u}_n} = \frac{\partial}{\partial \dot{u}_n} \sum_{i=-i_1}^{-i_2} T^i p. \tag{3.79}$$

We can formulate the obtained results in the following theorem.

Theorem 29. *The Lagrangian equations (3.55), (3.56) always possess the local conservation law (3.75). If $L = L(\dot{u}_n, u_{n+1} - u_n)$, these equations also have the conservation law (3.76). If the function (3.77) is a conserved density of these Lagrangian equations, then equation (3.79) is the generalized symmetry.*

In the case of the standard conservation laws (3.75), (3.76), formula (3.79) gives the trivial Lie point symmetries: $u_{n,\tau} = \dot{u}_n$ and $u_{n,\tau} = 1$. A nontrivial example will be presented in the case of the relativistic Toda lattice equation (3.53). This equation has two conserved densities, such that

$$\hat{p}_n = \dot{u}_{n+1}\dot{u}_n \phi(u_{n+1} - u_n) + \frac{\dot{u}_n^2}{2}, \qquad \phi(z) = \frac{1}{1 + \mathrm{e}^{-z}}, \tag{3.80}$$

$$\check{p}_n = \frac{(1 + \mathrm{e}^{u_{n+1} - u_n})(1 + \mathrm{e}^{u_n - u_{n-1}})}{\dot{u}_n}. \tag{3.81}$$

In this case, we have $L_{\dot{u}_n \dot{u}_n} = 1/\dot{u}_n$, for the Lagrangian (3.57), and easily obtain the following generalized symmetries:

$$u_{n,\tau_1} = \dot{u}_{n+1}\dot{u}_n \phi(u_{n+1} - u_n) + \dot{u}_n \dot{u}_{n-1} \phi(u_n - u_{n-1}) + \dot{u}_n^2, \tag{3.82}$$

$$u_{n,\tau_2} = -\check{p}_n. \tag{3.83}$$

3.3.2. Hamiltonian form of relativistic lattice equations. Let us discuss the relativistic lattice equations which are Hamiltonian and belong to the class (3.52). This class, unlike equations (2.1), (2.195), is not so convenient for the generalized symmetry method because of the existence of 'nonstandard' generalized symmetries and integrability conditions.

More precisely, let us consider the systems (3.52) which are Hamiltonian with respect to the structure (3.61). In this case, we may use the existence of generalized symmetries only. Higher order conservation laws imply no additional conditions, as one can see in sections 2.6 and 2.8. The generalized symmetries of (3.52) split into two different classes.

If we consider generalized symmetries of the order $m \geqslant 2$:

$$u_{n,\tau} = F(u_{n+m}, v_{n+m}, u_{n+m-1}, v_{n+m-1}, \ldots),$$
$$v_{n,\tau} = G(u_{n+m}, v_{n+m}, u_{n+m-1}, v_{n+m-1}, \ldots), \tag{3.84}$$

and investigate the compatibility condition of the systems (3.52), (3.84), we easily can show that $F_{v_{n+m}} = G_{u_{n+m}} = 0$ and then derive the first integrability condition. In the case $F_{u_{n+m}} \neq 0$, there is the following condition:

$$D_t \log f_{u_{n+1}} = (T-1)q_n, \tag{3.85}$$

where the function q_n is unknown. This is the standard integrability condition, similar to (3.19). One can use such conditions in an effective way for carrying out the classification of integrable cases.

In the case $G_{v_{n+m}} \neq 0$, we obtain a nonstandard condition:

$$D_t p_n = (T^m - 1)g_{v_n}, \tag{3.86}$$

where p_n is the unknown function. Equation (3.86) can be written in the same form as (3.85), i.e. in the form of the usual conservation law, as

$$T^m - 1 = (T-1)(T^{m-1} + T^{m-2} + \cdots + 1).$$

However, it depends on the order m of the symmetry and has an undefined conserved density p_n. Up to now, the problem of how to study and use such integrability conditions has not been solved. For this reason, we will consider some special symmetries (see definition 10 below) to avoid the appearance of nonstandard conditions similar to (3.86).

So, we introduce an additional requirement and pass to a particular case. In such a case, we will have no exhaustive classification unlike the case of the Volterra- and Toda-type equations. The generalized symmetry method will be used here only to find a number of new integrable equations.

Definition 10. *A special symmetry of order m is a generalized symmetry of the form (3.84) with $F_{v_{n+m}} = G_{u_{n+m}} = G_{v_{n+m}} = G_{v_{n+m-1}} = 0$, i.e.*

$$u_{n,\tau} = F(u_{n+m}, u_{n+m-1}, v_{n+m-1}, \ldots),$$
$$v_{n,\tau} = G(u_{n+m-1}, u_{n+m-2}, v_{n+m-2}, \ldots), \tag{3.87}$$

where $F_{u_{n+m}} \neq 0$.

In the following we consider systems of the form (3.52) with the Hamiltonian structure (3.61) which possess some higher order special symmetries. The classification is carried out up to point transformations of the form

$$\tilde{u}_n = \nu(u_n), \qquad \tilde{v}_n = \eta(v_n), \qquad \tilde{t} = ct, \tag{3.88}$$

where c is a constant. These transformations do not change the given structure of systems and their special symmetries.

Theorem 30. *If a system (3.52) has a special symmetry (3.87) of order $m \geqslant 4$, then it satisfies the following conditions:*

$$D_t p_n^{(i)} = (T-1)q_n^{(i)}, \qquad i = 1, 2, 3,$$
$$p_n^{(1)} = \log f_{u_{n+1}}, \qquad p_n^{(2)} = q_n^{(1)} + f_{u_n},$$
$$p_n^{(3)} = q_n^{(2)} + \tfrac{1}{2}\left(p_n^{(2)}\right)^2 + f_{v_n} g_{u_n}. \tag{3.89}$$

The integrable systems (3.52) are Hamiltonian, and conditions (3.89) provide us with both low order conservation laws and generalized symmetries (see equations (3.78)). Let us write down a list of systems which satisfy conditions (3.89). Here the coefficients c_j of the polynomials $r = r(u_n, v_n)$ and α, β are arbitrary constants. We omit again index n for simplicity.

List of relativistic lattice equations in Hamiltonian form

$$\dot{u} = u_1 + u^2 v + \alpha u, \qquad -\dot{v} = v_{-1} + v^2 u + \alpha v \tag{H1}$$

$$\begin{aligned} &\dot{u} = r(u_1 - u + \alpha r_v) + \beta r_v, \qquad -\dot{v} = r(v_{-1} - v + \alpha r_u) + \beta r_u \\ &r = c_1 uv + c_2 u + c_3 v + c_4, \qquad r_u r_v \neq 0 \end{aligned} \tag{H2}$$

$$\dot{u} = \frac{2r}{u_1 - v} + r_v + \alpha u + \beta, \qquad -\dot{v} = \frac{2r}{v_{-1} - u} + r_u - \alpha v - \beta \tag{H3}$$

$$\begin{aligned} &\textit{Case 1}: \quad \alpha = 0, \qquad r = c_1(u-v)^2 + c_2(u-v) + c_3 \\ &\textit{Case 2}: \quad \beta = 0, \qquad r = c_1 u^2 + c_2 v^2 + c_3 uv \\ &\textit{Case 3}: \quad \alpha = \beta = 0, \\ &r = c_1 u^2 v^2 + c_2 uv(u+v) + c_3(u^2 + v^2) + c_4 uv + c_5(u+v) + c_6 \end{aligned}$$

Equation (H1) has the Hamiltonian structure (3.61) given by

$$\varphi = 1, \qquad H = u_{n+1} v_n + \tfrac{1}{2} u_n^2 v_n^2 + \alpha u_n v_n.$$

In the case of equation (H2), one has

$$\varphi = r, \qquad H = (u_{n+1} - u_n) v_n + \alpha r + \beta \log r,$$

with $r = r(u_n, v_n)$ specified above. Analogously for equation (H3) we have

$$\varphi = r, \qquad H = \log r - 2\log(u_{n+1} - v_n) + \sigma(u_n, v_n),$$

where $r = r(u_n, v_n)$ again, and $\sigma = 0$ in case 3. In case 2, the function σ is defined by two compatible PDE: $\sigma_{v_n} = \alpha u_n / r$, $\sigma_{u_n} = -\alpha v_n / r$. In case 1, both functions r and σ depend on $u_n - v_n$, and σ is given by the ODE: $\sigma'(z) = -\beta / r(z)$.

Theorem 31. *A system (3.52) with the Hamiltonian structure (3.61) satisfies conditions (3.89) if and only if it can be transformed by a point transformation (3.88) into one of the systems (H1)–(H3).*

Theorems 30, 31 and the list of integrable systems (H1)–(H3), together with the master symmetries and some applications, can be found in [77] and the review [10]. Partly, the list (H1)–(H3) has been published earlier in [53]. Bäcklund auto-transformations and L–A pairs for some of the systems (H1)–(H3) have been constructed in [12]. Schlesinger-type auto-transformations are presented in [78].

All equations of the Volterra, Toda and relativistic Toda type, we present in this review, generate Bäcklund auto-transformations for nonlinear Schrödinger-type equations [7, 12, 35, 52, 53]. The systems (H1)–(H3) are closely connected with such well-known equations as the Ablowitz–Ladick and Sklyanin lattices and allow one to construct a list of integrable systems of hyperbolic equations similar to the Pohlmeyer–Lund–Regge system [10]. On the other hand, the systems (H1), (H2) give a simple polynomial representation for some well-known relativistic Toda-type equations presented in section 3.3.4, including the relativistic Toda lattice equation itself.

The classification result given by theorem 31 is weaker than in the case of Volterra- and Toda-type equations because integrable systems (3.52), (3.61) may not have only special symmetries but also different generalized symmetries. For example, the simplest such symmetry of the Hamiltonian form (3.62) of the relativistic Toda lattice equation (3.53) is

$$u_{n,\tau} = \frac{u_n + u_{n-1}}{v_n u_{n-1}}, \qquad v_{n,\tau} = -\frac{v_{n+1} + v_n}{v_{n+1} u_n}.$$

3.3.3. Lagrangian form of relativistic lattice equations. The class of equations (3.51) is analogous to (3.52) from the viewpoint of the generalized symmetry method. In this case, there are two types of generalized symmetries and integrability conditions. Equations (3.82), (3.83) exemplify two different kinds of symmetries for the relativistic Toda lattice equation (3.53).

The list of equations (L1), (L2) presented below has been obtained in [7] (see also [8, 60]). The integrability conditions were not derived there, and a simpler scheme of the generalized symmetry method was used. As in the case of systems (3.52), we obtain only a list of new integrable equations, with no exhaustive classification.

Let us briefly discuss this scheme. If we use the existence of only one generalized symmetry of a simple fixed form, we also can obtain, in principle, a list of integrable equations. It is assumed in [7] that equations (3.51) possess symmetries of the form

$$u_{n,\tau} = f(u_{n+1}, u_n, \dot{u}_{n+1}, \dot{u}_n) + g(u_n, u_{n-1}, \dot{u}_n, \dot{u}_{n-1}), \tag{3.90}$$

where the functions f, g are the same as in (3.51). The relativistic Toda lattice equation (3.53) has also a symmetry of this kind, namely (3.82). Such symmetry can be expressed always as a nonlinear Schrödinger-type system in terms of $u = u_{n+1}$ and $v = u_n$:

$$u_\tau = u_{tt} + 2g(u, v, u_t, v_t), \qquad v_\tau = -v_{tt} + 2f(u, v, u_t, v_t). \tag{3.91}$$

In order to do so, one rewrites the symmetry (3.90), using equation (3.51), on their common solutions.

One uses an additional condition that the system (3.91) must be integrable as well. As it is known [41], the system (3.91) possessing a higher order conservation law must satisfy the following integrability condition:

$$g_{u_t} - f_{v_t} \in \operatorname{Im} D_t. \tag{3.92}$$

Here D_t is the total derivative with respect to t, and thus (3.92) reads $g_{u_t} - f_{v_t} = D_t s(u, v) = s_u u_t + s_v v_t$. Returning to the variables $u_{n+j}, \dot{u}_{n+j}$, we pass to the relation

$$T\frac{\partial g}{\partial \dot{u}_n} - \frac{\partial f}{\partial \dot{u}_n} = \frac{\partial s}{\partial u_{n+1}}\dot{u}_{n+1} + \frac{\partial s}{\partial u_n}\dot{u}_n, \qquad s = s(u_{n+1}, u_n), \tag{3.93}$$

in terms of the functions f, g given in equation (3.51). So, apart from the existence of a symmetry of the form (3.90), we obtain the condition that there must exist a function s satisfying (3.93).

Using these two conditions, we can write down a list of two equations with many arbitrary constants. Here coefficients of the polynomials P, Q, r are arbitrary constants, and the functions a, b are defined by a system of ODE. The function S is a 4th degree polynomial, as it is the discriminator of r. Index n is omitted.

List of relativistic lattice equations in Lagrangian form

$$\begin{aligned} &\ddot{u} = P(\dot{u})(\dot{u}_1 a(u_1 - u) - \dot{u}_{-1} a(u - u_{-1}) + b(u_1 - u) - b(u - u_{-1})) \\ &P(z) = \varepsilon z^2 + \alpha z + \beta, \qquad Q(z) = \varepsilon z^2 + \gamma z + \delta \\ &a' = aQ'(b) - \alpha a^2, \qquad b' = Q(b) - \beta a^2 \end{aligned} \tag{L1}$$

$$2\ddot{u} = (\dot{u}^2 - S(u))\left(\frac{\partial s_1/\partial u - \dot{u}_1}{s_1} + \frac{\partial s/\partial u + \dot{u}_{-1}}{s}\right) + S'(u)$$
$$r(x, y) = c_1x^2y^2 + c_2xy(x + y) + c_3(x^2 + y^2) + c_4xy + c_5(x + y) + c_6 \qquad \text{(L2)}$$
$$S(x) = r_y^2 - 2rr_{yy}, \qquad s_1 = r(u_1, u), \qquad s = r(u, u_{-1})$$

As $f_{\dot{u}_{n+1}} \neq 0$ from equation (3.51), one can easily check that $Pa \neq 0$. It is not easy to solve the system of ODE for the functions a, b. That is why we write down in section 3.3.4 equations (L1) in a more explicit and detailed form. To solve the system for $a(z)$ and $b(z)$, we introduce the function $I = Q(b)/a + \beta a - \alpha b$. By direct calculation one can check that $\frac{\mathrm{d}I}{\mathrm{d}z} = 0$.

Theorem 32. *The complete list of equations (3.51), possessing a generalized symmetry of the form (3.90) and satisfying condition (3.93), consists, up to point transformations (3.25), of equations of the form (L1), (L2).*

Any equation of the complete list can be expressed in the Lagrangian form (3.55), (3.56) [8]. In the case of equation (L1), the Lagrangian is given by

$$L = R(\dot{u}_n) - \dot{u}_n A(u_{n+1} - u_n) - B(u_{n+1} - u_n), \tag{3.94}$$
$$R''(z) = 1/P(z), \qquad A'(z) = a(z), \qquad B'(z) = b(z). \tag{3.95}$$

The functions R, A, B are not completely defined. However, equation (L1) is well-defined by this Lagrangian because changing the Lagrangian according to the formula (3.70), we do not change the corresponding lattice equation.

The Lagrangian for equation (L2) reads

$$L = \log\frac{r(u_{n+1}, u_n)}{\dot{u}_n^2 - S(u_n)} + \dot{u}_n(A(u_{n+1}, u_n) + B(\dot{u}_n, u_n)), \tag{3.96}$$
$$A_{u_{n+1}} = \frac{1}{r(u_{n+1}, u_n)}, \qquad B_{\dot{u}_n} = \frac{2}{\dot{u}_n^2 - S(u_n)}. \tag{3.97}$$

The functions A, B are defined up to arbitrary functions of u_n. Those arbitrary functions do not arise in the equation due to formula (3.70), i.e. the equation with such a Lagrangian is also well defined.

Bäcklund auto-transformations for equations of the form (L1) can be found in [4, 8], and L–A pairs for equations (L1) are presented in [9]. Schlesinger-type auto-transformations for all equations (L1), (L2) have been constructed in [78]. The local master symmetries for equations (L1), (L2) are presented in [10].

3.3.4. Relations between the presented lists of relativistic equations. Here we discuss the precise correspondence between the relativistic lattice equations of two lists (H1)–(H3) and (L1), (L2) related by non-point invertible transformations. Such transformations allow one to rewrite generalized symmetries, conservation laws and solutions. These two lists have been obtained independently, and later the equivalence between them has been observed in [10]. This is probably the first nontrivial application of non-point invertible transformations to integrable equations.

First of all we write down a detailed and explicit form of equations (L1). Index n is omitted here again. The coefficients μ and ν are arbitrary constants. The exact correspondence and detailed lists of equations are given below in accordance with [78].

Detailed form of Lagrangian equations (L1)

$$\ddot{u} = \dot{u}_1 \, e^{u_1 - u} - \dot{u}_{-1} \, e^{u - u_{-1}} - e^{2(u_1 - u)} + e^{2(u - u_{-1})} \tag{Ld1}$$

$$\ddot{u} = \dot{u} \left(\frac{\dot{u}_1}{u_1 - u} - \frac{\dot{u}_{-1}}{u - u_{-1}} + u_1 - 2u + u_{-1} \right) \tag{Ld2}$$

$$\ddot{u} = \dot{u} \left(\frac{\dot{u}_1}{1 + \mu \, e^{u - u_1}} - \frac{\dot{u}_{-1}}{1 + \mu \, e^{u_{-1} - u}} - \nu (e^{u_1 - u} - e^{u - u_{-1}}) \right) \tag{Ld3}$$

$$\ddot{u} = \dot{u}(1 - \dot{u}) \left(\frac{\dot{u}_1}{u_1 - u} - \frac{\dot{u}_{-1}}{u - u_{-1}} \right) \tag{Ld4}$$

$$\ddot{u} = \dot{u}(\dot{u} - \mu) \left(\frac{\dot{u}_1}{e^{u_1 - u} + \mu} - \frac{\dot{u}_{-1}}{e^{u - u_{-1}} + \mu} \right). \tag{Ld5}$$

Equations (Ld1)–(Ld5) are obtained by solving the system of ODE for the functions a, b given in equation (L1) and applying simple n- and t-dependent point transformations. Using point transformations of the form (3.38), one can transform any equation (L1) into the form (L2), or into one of the equations (Ld1)–(Ld5), or into the equation

$$\ddot{u}_n = \frac{\dot{u}_{n+1} \dot{u}_n}{u_{n+1} - u_n} - \frac{\dot{u}_n \dot{u}_{n-1}}{u_n - u_{n-1}}. \tag{3.98}$$

The last equation (cf (Ld2)) is trivial. Indeed, the invertible transformation $w_n = \frac{\dot{u}_n}{u_n - u_{n-1}}$, with u_n unchanged, allows one to rewrite equation (3.98) as the system

$$\dot{w}_n = w_n (w_{n+1} - w_n), \qquad \dot{u}_n = w_n (u_n - u_{n-1}). \tag{3.99}$$

The first part of equations (3.99) is the linearizable discrete Burgers equation (see (2.142)–(2.144)), while the second one is a linear equation for u_n. Generalized symmetries for (3.98) can be constructed, if necessary, using the master symmetry (3.111), (3.112) with $b_n = 0$ of equation (Ld2) presented in section 3.3.5.

We consider below equations (L2) and (Ld1)–(Ld5) only. The Lagrangians for equations (Ld1)–(Ld5) will not be written down because they can be easily obtained, using formulae (3.94), (3.95) together with (L1).

Using n- and t-dependent point transformations (3.100), we can not only reduce the number of arbitrary constants in the systems (H1)–(H3) but also rewrite these systems so that the equivalence relation between the Lagrangian and Hamiltonian forms will become simpler and clearer. Here we present a list of systems corresponding to equations (Ld1)–(Ld5). Index n is omitted; μ and ν are arbitrary constants. We have in the list particular modified cases of (H1)–(H3), that is why we write down also the functions φ and H defining the Hamiltonian structure (3.61).

Some of Hamiltonian systems in detailed form

$$\begin{aligned} &\dot{u} = e^{u_1 - u} + e^{u - v}, && \dot{v} = e^{v - v_{-1}} + e^{u - v} \\ &\varphi = -e^{v - u}, && H = e^{u_1 - v} + \tfrac{1}{2} e^{2(u - v)} \end{aligned} \tag{Hd1}$$

$$\begin{aligned} &\dot{u} = (u_1 - u)(u - v), && \dot{v} = (v - v_{-1})(u - v) \\ &\varphi = v - u, && H = v(u - u_1) \end{aligned} \tag{Hd2}$$

$$\dot{u} = (e^{u_1-u} + \mu)(e^{u-v} + \nu), \qquad \dot{v} = (e^{v-v_{-1}} + \mu)(e^{u-v} + \nu)$$
$$\varphi = -(1 + \nu\, e^{v-u}), \qquad H = e^{u_1-v} + \mu\, e^{u-v} \tag{Hd3}$$

$$\dot{u} = \frac{u_1 - u}{u_1 - v}, \qquad \dot{v} = \frac{v - v_{-1}}{u - v_{-1}}$$
$$\varphi = v - u, \qquad H = \log \frac{u - v}{u_1 - v} \tag{Hd4}$$

$$\dot{u} = \frac{e^{u_1-u} + \mu}{e^{u_1-v} + 1}, \qquad \dot{v} = \frac{e^{v-v_{-1}} + \mu}{e^{u-v_{-1}} + 1}$$
$$\varphi = \mu - e^{v-u}, \qquad H = \log \frac{1 + e^{u_1-v}}{1 - \mu\, e^{u-v}} \tag{Hd5}$$

It can be proved that, using point transformations of the form

$$\tilde{u}_n = \kappa_n(t, u_n), \qquad \tilde{v}_n = \eta_n(t, v_n), \qquad \tilde{t} = \theta(t), \tag{3.100}$$

one can transform any of the equations (H1), (H2) and (H3), cases 1, 2 into (H3), case 3 or into one of the equations (Hd1)–(Hd5). So, instead of (H1)–(H3), we consider below equations (Hd1)–(Hd5) and (H3), case 3. The precise correspondence between Lagrangian equations and Hamiltonian systems is

$$(Hd_i) \sim (Ld_i), \qquad i = 1, 2, 3, 4, 5,$$
$$(H_3), \qquad \text{case } 3 \sim (L_2). \tag{3.101}$$

If one starts from any of the Hamiltonian systems (Hd1)–(Hd5) and (H3), case 3 and passes to a Lagrangian equation in accordance with theorem 28, one obtains the equation shown by (3.101), and no additional point transformation is necessary.

According to theorem 28, u_n remains unchanged, and a relation between $\dot{u}_n$ and v_n is given by the first equation of the Hamiltonian system. All those relations can be easily inverted except for the first equation of the system (H3), case 3. However, this equation can be rewritten as

$$\dot{u}_n = \frac{2r(u_{n+1}, u_n)}{u_{n+1} - v_n} - \frac{\partial r(u_{n+1}, u_n)}{\partial u_{n+1}},$$

and then the function v_n can be easily expressed in terms of $\dot{u}_n, u_n, u_{n+1}$.

One can pass from any Hamiltonian system (H3), case 3 and (Hd1)–(Hd5) to the corresponding Lagrangian equation in the following way. One differentiates the first equation of the Hamiltonian system with respect to the time t, then excludes $\dot{v}_n$, using the second equation, and then eliminates v_{n+j}, using the first one. That gives for u_n exactly the Lagrangian equation (3.101). A Lagrangian constructed by (3.66) will coincide with the Lagrangian, given above for all equations (Ld1)–(Ld5), (L2), up to formula (3.70).

It should be remarked that the correspondence (3.101) between the two lists of equations provides a simple polynomial representation of the relativistic Toda-type equations (Ld1)–(Ld3), including the relativistic Toda lattice (3.53) itself. In fact, using the transformation $\hat{u}_n = e^{u_n}$, $\hat{v}_n = e^{-v_n}$, one transforms the system (Hd1) into (3.72) and (Hd3) into the following:

$$\dot{u}_n = (u_{n+1} + \mu u_n)(u_n v_n + \nu),$$
$$\dot{v}_n = -(v_{n-1} + \mu v_n)(u_n v_n + \nu). \tag{3.102}$$

The systems (3.72), (3.102) have a polynomial form as well as (Hd2). The invertible transformations of equations (Ld1)–(Ld3) into the polynomial systems (3.72), (Hd2), (3.102), respectively, read

$$\hat{u}_n = \mathrm{e}^{u_n}, \qquad \hat{v}_n = \frac{\dot{u}_n - \mathrm{e}^{u_{n+1}-u_n}}{\mathrm{e}^{u_n}},$$
$$\hat{u}_n = u_n, \qquad \hat{v}_n = u_n + \frac{\dot{u}_n}{u_n - u_{n+1}},$$
$$\hat{u}_n = \mathrm{e}^{u_n}, \qquad \hat{v}_n = \frac{\dot{u}_n}{\mathrm{e}^{u_{n+1}} + \mu \mathrm{e}^{u_n}} - \nu\, \mathrm{e}^{-u_n}.$$

3.3.5. Master symmetries for relativistic lattice equations. Here we show the integrability of relativistic Toda-type equations using master symmetries and simple non-invertible transformations. We have demonstrated in the previous sections the equivalence between Hamiltonian and Lagrangian equations. Generalized symmetries, conservation laws and master symmetries can be rewritten from one equivalent equation to another. So, we can simplify the explanation why all relativistic equations are integrable, considering the Hamiltonian or Lagrangian form only. According to (3.101), we will construct generalized symmetries and conservation laws only for the Lagrangian equations (Ld1)–(Ld5) and Hamiltonian system (H3), case 3.

First of all we write down the master symmetries[2] for equations (Ld1)–Ld5) [10]; some of master symmetries for relativistic lattice equations can be found in [47, 80]. Equations (Ld4), (Ld5) are of the form (L1) with $b = 0$ and can be expressed as systems in terms of u_n and $v_n = \dot{u}_n$:

$$\begin{aligned} \dot{u}_n &= v_n, \\ \dot{v}_n &= P(v_n)(v_{n+1}a_n - v_{n-1}a_{n-1}), \qquad a_n = a(u_{n+1} - u_n), \end{aligned} \tag{3.103}$$

where the functions $P(z)$ and $a(z)$ are given by the table

$$\begin{array}{llll} P = z(1-z) & a = \dfrac{1}{z} & \text{for} & \text{(Ld4)} \\ P = z(z-\mu) & a = \dfrac{1}{\mathrm{e}^z + \mu} & \text{for} & \text{(Ld5)}. \end{array} \tag{3.104}$$

Local master symmetries for these systems are

$$\begin{aligned} u_{n,\tau} &= n v_n, \\ v_{n,\tau} &= P(v_n)((n+1)v_{n+1}a_n - (n-1)v_{n-1}a_{n-1} + \lambda), \end{aligned} \tag{3.105}$$

where

$$\begin{array}{lll} \lambda = 0 & \text{for} & \text{(Ld4)} \\ \lambda = -1 & \text{for} & \text{(Ld5)}. \end{array} \tag{3.106}$$

In the case of equations (Ld1)–(Ld3), we use the non-invertible transformation

$$w_n = u_{n+1} - u_n, \qquad v_n = \dot{u}_n \tag{3.107}$$

to write them in the form

$$\begin{aligned} \dot{w}_n &= v_{n+1} - v_n, \\ \dot{v}_n &= P(v_n)(v_{n+1}a_n - v_{n-1}a_{n-1} + b_n - b_{n-1}), \end{aligned} \tag{3.108}$$

[2] The master symmetries (3.105), (3.111) presented below are taken from [10]. However, those master symmetries have been rewritten, using point and non-point invertible transformations, in order to construct generalized symmetries and conservation laws in an easier way. We also correct some essential misprints contained in [10].

where

$$a_n = a(w_n), \qquad b_n = b(w_n) \tag{3.109}$$

(see equation (L1)). The functions $P(z), a(z), b(z)$ are defined as

$$\begin{array}{lllll} P = 1 & a = \mathrm{e}^z & b = -\mathrm{e}^{2z} & \text{for} & \text{(Ld1)} \\ P = z & a = z^{-1} & b = z & \text{for} & \text{(Ld2)} \\ P = z & a = \dfrac{1}{1+\mu\,\mathrm{e}^{-z}} & b = -\nu\,\mathrm{e}^z & \text{for} & \text{(Ld3).} \end{array} \tag{3.110}$$

The master symmetries for the systems (3.108)–(3.110) are given by

$$\begin{aligned} w_{n,\tau} &= (n+k)v_{n+1} - (n-k+1)v_n + c_n, \\ v_{n,\tau} &= P(v_n)((n+k)(v_{n+1}a_n + b_n) - (n-k)(v_{n-1}a_{n-1} + b_{n-1})) + \lambda v_n^2, \end{aligned} \tag{3.111}$$

with the constants k, λ and function c_n defined in the following table:

$$\begin{array}{llll} \text{(Ld1)}: & k = 2 & \lambda = 1 & c_n = -2a_n \\ \text{(Ld2)}: & k = 2 & \lambda = 0 & c_n = b_n^2 \\ \text{(Ld3)}: & k = \frac{3}{2} & \lambda = 1 & c_n = b_n - \mu\nu. \end{array} \tag{3.112}$$

In the case of (Ld3), this is the master symmetry only if $\mu = 0$ or $\nu = 0$. If $\mu\nu \neq 0$, we introduce both into the system (3.108) and its master symmetry (3.111) a dependence on the time τ of the master symmetry, so that

$$\nu = \nu(\tau), \qquad \frac{\mathrm{d}\nu}{\mathrm{d}\tau} = \mu\nu^2 \tag{3.113}$$

(see about time-dependent master symmetries at the end of section 2.7).

We construct conserved densities for the systems (3.103), (3.108), using formula (2.189), and need a starting density. Let us write down three simple densities for equations (Ld1)–(Ld5), namely

$$p_n^- = \int \frac{\mathrm{d}v_n}{P(v_n)} - \int a_n\,\mathrm{d}w_n, \qquad p_n^+ = \int \frac{v_n\,\mathrm{d}v_n}{P(v_n)} + \int b_n\,\mathrm{d}w_n, \tag{3.114}$$

$$p_n^0 = \log P(v_n) + \log a_n, \tag{3.115}$$

in terms of the functions v_n, w_n, a_n, b_n defined by (3.107), (3.109). The functions P, a, b are given by equations (3.104) with $b = 0$ and (3.110). These formulae also provide with conserved densities any of the equations (L1).

In fact, equations (L1) are Lagrangian and have, according to (3.75), (3.76), the conserved densities $p_n^- = L_{\dot{u}_n}$, $p_n^+ = \dot{u}_n L_{\dot{u}_n} - L$ in terms of the functions R, A, B given by equations (3.95). Taking into account that $(zR'(z) - R(z))' = z/P(z)$, we obtain formulae (3.114). The function (3.115) is another form of the densities (3.114), also used below, and it can be obtained as the linear combination:

$$p_n^0 = c_1 p_n^- + c_2 p_n^+ + c_3(u_{n+1} - u_n) + c_4,$$

with some constant coefficients c_j.

Equations (Ld4), (Ld5) are equivalent to the systems (3.103), (3.104), and the formulae (3.114), (3.115), written in terms of v_n and $w_n = u_{n+1} - u_n$, give conserved densities for these systems. The function p_n^- cannot be a starting density for the master symmetry (3.105), (3.106), as $D_\tau p_n^- \sim 0$. The next density p_n^+ can be taken as a starting one, and on the first step we obtain

$$D_\tau p_n^+ \sim v_n(v_{n+1}a_n + \lambda). \tag{3.116}$$

This function with $v_n = \dot{u}_n$ is a new conserved density for equations (Ld4), (Ld5).

In the case of (Ld1)–(Ld3), transformation (3.107) is not invertible, and we have to check directly that (3.114), (3.115) considered as the functions of v_n, w_n are conserved densities for the system (3.108), (3.109) too. The density p_n^+ can be used again as a starting point. Using the master symmetry (3.111), (3.112), one obtains a new conserved density for the systems (3.108)–(3.110):

$$D_\tau p_n^+ \sim (2k-1)(v_n v_{n+1} a_n + (v_n + v_{n+1}) b_n) + \frac{\lambda v_n^3}{P(v_n)} + b_n c_n + \frac{\partial b_n}{\partial \tau}. \qquad (3.117)$$

Here $\frac{\partial b_n}{\partial \tau} = 0$ in all cases, except for (Ld3) with $\mu\nu \neq 0$, in which

$$b_n = -\nu\, \mathrm{e}^{w_n}, \qquad \frac{\partial b_n}{\partial \tau} = -\mu\nu^2\, \mathrm{e}^{w_n}$$

due to (3.113). Now we can construct a conserved density for equations (Ld1)–(Ld3), using transformation (3.107), namely, replacing v_n, w_n in (3.117) by the functions $\dot u_n$, $u_{n+1} - u_n$.

So, using the master symmetries (3.105), (3.106) and (3.111), (3.112), we construct conserved densities for the systems (3.103), (3.104) and (3.108)–(3.110) and then rewrite those conserved densities for equations (Ld1)–(Ld5). As these equations are Lagrangian, we can then use formula (3.79) in order to obtain generalized symmetries. The Lagrangians for equations (Ld1)–(Ld5) have the form (3.94), (3.95), hence $L_{\dot u_n \dot u_n} = 1/P(\dot u_n)$. For example, starting from the density (3.116), we obtain for equations (Ld4), (Ld5) the following symmetry:

$$u_{n,\tau} = P(\dot u_n)(\dot u_{n+1} a(u_{n+1} - u_n) + \dot u_{n-1} a(u_n - u_{n-1}) + \lambda).$$

Example of relativistic Toda lattice (3.53). We discuss this important example in detail. The relativistic Toda lattice equation is included in equation (Ld3) by choosing $\mu = 1$, $\nu = 0$. Its representation (L1) is defined, according to (3.110), by $P = z$, $b = 0$, $a = \phi(z) = \frac{1}{1+\mathrm{e}^{-z}}$. The corresponding system of the forms (3.108), (3.109) is

$$\begin{aligned} \dot w_n &= v_{n+1} - v_n, \\ \dot v_n &= v_n v_{n+1} \phi(w_n) - v_n v_{n-1} \phi(w_{n-1}), \end{aligned} \qquad (3.118)$$

and the master symmetries (3.111), (3.112) take the form

$$\begin{aligned} w_{n,\tau} &= \left(n + \tfrac{3}{2}\right) v_{n+1} - \left(n - \tfrac{1}{2}\right) v_n, \\ v_{n,\tau} &= \left(n + \tfrac{3}{2}\right) v_n v_{n+1} \phi(w_n) - \left(n - \tfrac{3}{2}\right) v_n v_{n-1} \phi(w_{n-1}) + v_n^2. \end{aligned} \qquad (3.119)$$

We will discuss here only conserved densities for the system (3.118), as conserved densities for the relativistic Toda lattice are constructed by transformation (3.107), and corresponding generalized symmetries are obtained, using formula (3.79) with $L_{\dot u_n \dot u_n} = 1/\dot u_n$ (see equation (3.57)).

By direct calculation one can check that the following functions are conserved densities for the system (3.118):

$$\begin{aligned} p_n^0 - p_n^- &= w_n, \\ p_n^0 &= \log v_n + \log \phi(w_n), \\ p_n^+ &= v_n, \\ \hat p_n &= v_n v_{n+1} \phi(w_n) + \frac{1}{2} v_n^2, \\ \hat p_n^\diamond &= v_n v_{n+1} v_{n+2} \phi(w_n) \phi(w_{n+1}) + (v_n + v_{n+1}) v_n v_{n+1} \phi(w_n) + \frac{1}{3} v_n^3, \\ \check p_n &= \frac{1}{v_n}(1 + \mathrm{e}^{w_n})(1 + \mathrm{e}^{w_{n-1}}), \\ \check p_n^\diamond &= \check p_n \check p_{n+1} \phi(w_n) + \frac{1}{2} \check p_n^2. \end{aligned}$$

The conserved densities (P1)–(P3) come from equations (3.114), (3.115), while (P4), (P6) are taken from (3.80), (3.81). The densities (P1), (P3) are written down up to some nonessential constants of integration. The density (P1) becomes trivial when we pass to equation (3.53): $w_n = (T-1)u_n \sim 0$, but it is not trivial at the level of the systems (3.118) and helps us to see the complete picture.

Using the ODE $\phi' = \phi - \phi^2$, it is easy to check that the operator D_τ corresponding to the master symmetry (3.119) acts on (P1)–(P4) as

$$D_\tau w_n \sim p_n^+, \qquad D_\tau p_n^0 \sim 2p_n^+, \qquad D_\tau p_n^+ \sim 2\hat{p}_n, \qquad D_\tau \hat{p}_n \sim 3\hat{p}_n^\diamond.$$

So, starting from (P1) or (P2), one can construct a new density (P5) as well as infinitely many densities with polynomial dependence on the variables v_{n+j}. The system (3.118) also has an infinite hierarchy of conserved densities with rational dependence on v_{n+j} which are similar to (P6), (P7); however, the master symmetry (3.119) does not help to construct them. In fact

$$D_\tau \check{p}_n = (T^{-1}-1)\left(\frac{2n+1}{2}(1+\mathrm{e}^{w_n})\right) \sim 0, \qquad D_\tau \check{p}_n^\diamond \sim -\check{p}_n.$$

All relativistic lattice equations have a second hierarchy of conservation laws and generalized symmetries, but it is an open problem to construct it.

Let us discuss now the Hamiltonian system (H3), case 3, namely

$$\dot{u}_n = \frac{2r}{u_{n+1}-v_n} + r_{v_n}, \qquad \dot{v}_n = \frac{2r}{u_n - v_{n-1}} - r_{u_n}, \tag{3.120}$$

where r is defined as

$$r = r(u_n, v_n) = r(v_n, u_n), \qquad \frac{\partial^3 r}{\partial u_n^3} = 0. \tag{3.121}$$

The local master symmetry in this case has a very simple form

$$u_{n,\tau} = n\dot{u}_n, \qquad v_{n,\tau} = (n-1)\dot{v}_n, \tag{3.122}$$

but coefficients of the polynomial r depend on τ. The dependence on τ is given by

$$r_\tau = r r_{u_n v_n} - r_{u_n} r_{v_n}, \tag{3.123}$$

exactly as in the case of (3.32), (3.33). This master symmetry can be found in [10]. It can be rewritten as a master symmetry of the Lagrangian equation (L2), equivalent to the system (3.120), (3.121), which is also given in [10].

Generalized symmetries can be obtained by this master symmetry or, as the system (3.120), (3.121) is Hamiltonian, using the formula (3.78). One can construct a hierarchy of conserved densities, starting from the density $p_n^{(1)}$ given by (3.89). One can check that, up to some nonessential constants,

$$p_n^{(1)} = \log r - 2\log(u_{n+1} - v_n), \tag{3.124}$$

$$p_n^{(2)} = -\frac{4r}{(u_{n+1}-v_n)(u_n - v_{n-1})} + \frac{2r_{u_n}}{u_{n+1}-v_n} - \frac{2r_{v_n}}{u_n - v_{n-1}} + r_{u_n v_n} \tag{3.125}$$

(see (3.89)). Applying the operator D_τ, we obtain

$$D_\tau p_n^{(1)} = p_n^{(2)} + (1-T)\frac{2n\dot{u}_n}{u_n - v_{n-1}} + \frac{r_\tau - r r_{u_n v_n} + r_{u_n} r_{v_n}}{r},$$

i.e $D_\tau p_n^{(1)} \sim p_n^{(2)}$ only if equation (3.123) is satisfied. Two simplest conserved densities of the second hierarchy are the following functions, symmetrical to (3.124), (3.125):

$$\tilde{p}_n^{(1)} = \log r - 2\log(v_{n+1} - u_n),$$

$$\tilde{p}_n^{(2)} = -\frac{4r}{(v_{n+1}-u_n)(v_n - u_{n-1})} + \frac{2r_{v_n}}{v_{n+1}-u_n} - \frac{2r_{u_n}}{v_n - u_{n-1}} + r_{u_n v_n}.$$

4. Conclusions

In this review, we have presented the discrete version of the generalized symmetry method suitable for the classification of certain classes of integrable equations and for testing given equations for integrability. In addition to the general theory presented in section 2, we have discussed in section 3 some of the most interesting results obtained, using this method.

In particular, in sections 3.1 and 3.2, we have considered the case of Volterra- and Toda-type equations, in which there are exhaustive classification results for equations possessing higher order generalized symmetries and conservation laws. In section 3.3, in the case of relativistic Toda-type equations, one only obtains two lists of new integrable lattice equations. In all cases we are able not only to obtain lists of equations, using necessary conditions for the integrability, but also to show the integrability for all obtained equations, constructing hierarchies of generalized symmetries and conservation laws. We do that using local master symmetries, non-invertible transformations of the Miura type and invertible non-point transformations together with the Hamiltonian and Lagrangian structures.

In an attempt to keep this review reasonably short, we have left out some interesting and important topics. Among them we have not included a discussion of lattice equations depending explicitly on the discrete spatial variable n and on the time t. The generalized symmetry method for n- and t-dependent equations of the Volterra- and Toda-type has been developed in [32, 33, 79]. Some interesting examples of n- and t-dependent integrable equations are presented in [10, 12, 32].

Also left out is the area of multi-component integrable lattice equations, i.e. systems of arbitrarily many equations. In this case, one starts from an equation and its generalized symmetry of a fixed form, but with arbitrary constants, and reduces the classification problem to the algebraic one for well-known non-associative structures like the Jordan algebra. Integrable multi-component generalizations for some Volterra-, Toda- and relativistic Toda-type equations have been considered in [11, 25, 63, 64]. The recursion operators, master symmetries and L–A pairs for those equations can be found in [1, 11, 63].

An area also not covered in the present review is the study of purely discrete equations. In this case it is difficult to use the existence of generalized symmetries for the classification. One can classify integrable equations by considering different properties as, for example, the existence of Bäcklund auto-transformations [3–6]. Two lists of purely discrete equations of the Toda- and relativistic Toda-type have been obtained in [3, 4]; examples of this kind are also contained in [26, 57–59].

Here we do not consider the construction and investigation of the solutions of obtained systems. Equations obtained as a result of the classification procedure are integrable by the inverse scattering method. If necessary, one can construct an L–A pair and investigate in this way completely an equation of the list. For example, in the case of relativistic Toda-type lattice equations which are discussed in section 3.3, L–A pairs for the key resulting equations have been constructed in [9, 12], and Bäcklund auto-transformations have been found in [8, 12, 78].

Many open problems remain in this research area. For example, in the important case of relativistic Toda-type equations, there are not only no exhaustive classification results, such as for Volterra- and Toda-type equations, but also no integrability conditions suitable for testing a given equation for integrability. It is unclear how to study multi-dimensional lattice equations similar to the two-dimensional Toda lattice. In paper [54], one can find a small list of integrable multi-dimensional lattice equations. Different ways of developing the generalized symmetry method in the case of partial differential equations are discussed in [10, 37, 43]. Ideas of these papers could be possibly used in the discrete case.

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