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EXPLICIT BÄCKLUND TRANSFORMATIONS FOR MULTIFIELD SCHRÖDINGER EQUATIONS. JORDAN GENERALIZATIONS OF THE TODA CHAIN

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Bäcklund transformations for multifield analogs of the nonlinear Schrödinger equation that correspond to unital Jordan algebras are found. These Bäcklund transformations are explicit invertible autotransformations and as a result they are very convenient for the construction of exact solutions. It is established that to these Bäcklund transformations there correspond integrable multifield discrete—differential equations that generalize the infinite Toda chain. A simple construction is given by means of which multifield analogs of the infinite Toda chain can be constructed from every unital Jordan algebra. New examples of such chains are given.

1. INTRODUCTION

One of the best known equations that can be integrated by the inverse scattering method is the nonlinear Schrödinger equation, which it is convenient to write as a system of two scalar equations:

$$u_t = u_{xx} - 2u^2 v, \quad v_t = -v_{xx} + 2v^2 u, \tag{1.1}$$

where $u=u(t, x)$, $v=v(t, x)$. One might suppose that all questions concerning the symmetries, conservation laws, L—A pairs, Bäcklund transformations, and other algebraic properties of this system received exhaustive answers long ago. However, new and interesting results were recently obtained. It was shown in [1] that the system (1.1), in addition to everything else, is remarkable in that it admits the autotransformation

$$\tilde{u} = u_{xx} - u^{-1}u_x^2 - u^2 v, \quad \tilde{v} = -u^{-1}, \tag{1.2}$$

which connects two solutions (u, v) and $(\tilde{u}, \tilde{v})$ of the system (1.1). We shall briefly review below for the reader necessary information about the autotransformation (1.2) (for more details, see [1—3]).

The important difference between the transformation (1.1) and the previously known Bäcklund transformations is that it is *explicit* and *invertible*. Let us explain what we mean. For comparison with (1.2), we consider the well-known classical Bäcklund transformation for (1.1):

$$(\tilde{u}+u)_x = (\tilde{u}-u)\Big[\alpha + (\tilde{u}+u)(\tilde{v}+v)\Big]^{1/2}, \quad (\tilde{v}+v)_x = (\tilde{v}-v)\Big[\alpha + (\tilde{v}+v)(\tilde{u}+u)\Big]^{1/2}. \tag{1.3}$$

If by means of (1.3) we are to construct from a known solution (u, v) of the system (1.1) a new solution $(\tilde{u}, \tilde{v})$, we must solve a system of ordinary differential equations. In the case of the transformation (1.2), we have an *explicit* expression for the new solution $(\tilde{u}, \tilde{v})$. Moreover, the transformation (1.2) enables us to construct explicitly (without solving differential equations) a certain solution (u, v) from a known solution $(\tilde{u}, \tilde{v})$. The reason for this is that the transformation is *invertible*, i.e., using the relation (1.2), we can express the variables u and v in terms of $\tilde{u}$ and $\tilde{v}$ and their derivatives with respect to x. It is readily verified that the inverse transformation has the form

$$v = \tilde{v}_{xx} - \tilde{v}^{-1}\tilde{v}_x^2 - \tilde{v}^2\tilde{u}, \quad u = -\tilde{v}^{-1}. \tag{1.4}$$

In what follows, we shall call transformations of the type (1.2) *explicit autotransformations*.

It is obvious that explicit autotransformations are very convenient for constructing exact solutions. For example, in the case of the Schrödinger equation (1.1), starting, for example, with the solution ($u_0=\varphi(t, x)$, $v_0=0$), where φ is an arbitrary solution of the heat-conduction equation $\varphi_t=\varphi_{xx}$, we can, using the autotransformation (1.2), construct an infinite family of solutions (u_k, v_k), $k\in \mathbb{N}$, the simplest of which has the form ($u_1=\varphi_{xx}-\varphi^{-1}\varphi_x^2$, $v_1=-\varphi^{-1}$). These solutions include in particular N-soliton solutions.

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Integrable discrete—differential equations (chains) are intimately related to explicit autotransformations. For example, (1.2) corresponds to the classical infinite Toda chain

$$(q_n)_{xx} = \exp(q_{n+1} - q_n) - \exp(q_n - q_{n-1}). \tag{1.5}$$

The chain (1.5) can be obtained from (1.2) as follows. The transformation (1.2) can be interpreted as a system of discrete—differential equations

$$u_{k+1} = u_{kxx} - u_k^{-1} u_{kx}^2 - v_k u_k^2, \qquad v_{k+1} = -u_k^{-1}, \tag{1.6}$$

where $k \in \mathbb{Z}$ is a discrete parameter. Eliminating from the system (1.6) the variables v_k, we obtain

$$(u_n)_{xx} - u_n^{-1}(u_n)_x^2 = u_{n+1} - u_n^2 u_{n-1}^{-1}, \qquad n \in \mathbb{Z}. \tag{1.7}$$

It is readily seen that the substitution $u_n = \exp q_n$ reduces the system (1.7) to the form (1.5).

The fact that the nonlinear Schrödinger equation (1.1) possesses the transformation (1.2) is not a unique phenomenon. Numerous examples of other integrable systems possessing explicit autotransformations are contained in [1]. A certain integrable discrete—differential equation is associated with each of the explicit autotransformations given in [1].

The aim of the present paper is to study the question of explicit autotransformations for multifield analogs of the nonlinear Schrödinger equation (see [4,5]) and the associated multifield analogs of the infinite Toda chain.

In [5], one of the present authors constructed a class of integrable multifield generalizations of the Schrödinger equation that are associated with Jordan pairs. We succeeded in finding explicit autotransformations for a very slightly smaller class, namely, for systems of the form

$$u_t^i = u_{xx}^i - 2a_{jkm}^i u^j v^k u^m, \qquad i = 1, \ldots, N, \quad v_t^i = -v_{xx}^i + 2a_{jkm}^i v^j u^k v^m, \quad i = 1, \ldots, N \tag{1.8}$$

(summation over repeated indices is understood), where $a_{jkm}^i \in \mathbf{C}$ are the structure constants of the triple Jordan system generated by an arbitrary Jordan algebra with unit element. For every such system (1.8), we give expressions for the explicit Bäcklund transformation and construct the corresponding integrable multifield generalization of the infinite Toda chain.

In presenting the proofs and the results, we have in a number of cases been forced to use purely algebraic concepts and methods. Therefore, the contents are illustrated by examples in which we have managed to avoid using specific algebraic terms. One such example is given below.

Example 1.1. As we were informed by V. V. Sokolov, besides the well-known vector Schrödinger equation (see [6]), it is possible to express in vector form one more of the generalizations of the nonlinear Schrödinger equation, namely,

$$U_t = U_{xx} - 4\langle U, V\rangle U + 2\langle U, U\rangle V, \quad V_t = -V_{xx} + 4\langle U, V\rangle V - 2\langle V, V\rangle U, \tag{1.9}$$

where $U = (u^1, \ldots, u^N)^T$, $V = (v^1, \ldots, v^N)^T$, and $\langle , \rangle$ is the ordinary scalar product. By direct verification we can show that

$$\tilde{U} = U_{xx} - 2\langle U, U\rangle^{-1}\langle U, U_x\rangle U_x + \langle U, U\rangle^{-1}\langle U_x, U_x\rangle U - 2\langle U, V\rangle U + \langle U, U\rangle V, \quad \tilde{V} = -\langle U, U\rangle^{-1} U \tag{1.10}$$

is an explicit Bäcklund transformation for the system (1.9). The inverse transformation has the form

$$V = \tilde{V}_{xx} - 2\langle \tilde{V}, \tilde{V}\rangle^{-1}\langle \tilde{V}, \tilde{V}_x\rangle \tilde{V}_x + \langle \tilde{V}, \tilde{V}\rangle^{-1}\langle \tilde{V}_x, \tilde{V}_x\rangle \tilde{V} - 2\langle \tilde{V}, \tilde{U}\rangle \tilde{V} + \langle \tilde{V}, \tilde{V}\rangle \tilde{U}, \quad U = -\langle \tilde{V}, \tilde{V}\rangle^{-1}\tilde{V}. \tag{1.11}$$

The vector generalization of the infinite Toda chain corresponding to (1.11) takes the form

$$U_{nxx} = 2\langle U_n, U_n\rangle^{-1}\langle U_n, U_{nx}\rangle U_{nx} - \langle U_n, U_n\rangle^{-1}\langle U_{nx}, U_{nx}\rangle U_n$$

$$+ U_{n+1} + \langle U_n, U_n\rangle\langle U_{n-1}, U_{n-1}\rangle^{-1} U_{n-1} - 2\langle U_{n-1}, U_{n-1}\rangle^{-1}\langle U_n, U_{n-1}\rangle U_n. \tag{1.12}$$

The system of discrete—differential equations (1.12) has an infinite series of higher symmetries and conservation laws (see below). ■

2. MULTIFIELD NONLINEAR SCHRÖDINGER EQUATIONS ASSOCIATED WITH UNITAL JORDAN ALGEBRAS (*J—S* SYSTEMS)

In this section we shall describe more rigorously and in more detail than in the Introduction the class of systems that we consider, namely, we give a construction by means of which every Jordan algebra with unit element can be associated with a system (1.8). We then discuss which of our systems form the subclass of the larger class of integrable multifield analogs of the nonlinear Schrödinger equation that were studied in [5].

We recall the definition of a Jordan algebra (for more details about Jordan algebras, see [7—9]).

Definition 2.1. *A finite-dimensional commutative algebra J is called a Jordan algebra if the multiplication, which we shall denote by (,), satisfies the identity*

$$\Big(((x \circ x) \circ z) \circ x\Big) = ((x \circ x) \circ (z \circ x)). \ \blacksquare \tag{2.1}$$

Suppose that in the algebra J we have chosen a basis $e_1, e_2, \ldots, e_N$. Multiplication in J can be defined by specifying a set of structure constants c^i_{jk}:

$$(e_j \circ e_k) = c^i_{jk} e_i. \tag{2.2}$$

Let c^i_{jk} be the structure constants of some Jordan algebra J. We determine the constants a^i_{jkm} of the system (1.8) as follows:

$$a^i_{jkm} = c^i_{jr} c^r_{km} + c^i_{mr} c^r_{kj} - c^i_{kr} c^r_{jm}. \tag{2.3}$$

Thus, with every Jordan algebra J we have associated a certain system (1.8). We shall assume that systems related by linear transformations,

$$u^i = M^i_k \tilde{u}^k, \quad v^i = M^i_k \tilde{v}^k, \quad i = 1, 2, \ldots, N, \quad \det M \neq 0, \tag{2.4}$$

are equivalent. It is obvious that the established correspondence does not depend on the choice of the basis, since transition to a new basis in the Jordan algebra J corresponds to a linear transformation (2.4).

In what follows, we shall assume that the Jordan algebra J has a unit element, which we shall denote by e. We shall call the generalizations of the nonlinear Schrödinger equations (1.8) constructed as above on the basis of a Jordan algebra with unit element J—S systems.

It is convenient to use more compact and invariant [with respect to the transformations (2.4)] vector forms of expression of the systems (1.8). Let $U=u^i e_i$, $V=v^i e_i$. Then in terms of the multiplication in the Jordan algebra J, the system (1.8) can be expressed as

$$U_t = U_{xx} - 4((U \circ V) \circ U) + 2((U \circ U) \circ V), \qquad V_t = -V_{xx} + 4((V \circ U) \circ V) - 2((V \circ V) \circ U). \tag{2.5}$$

A more elegant and succinct expression is obtained if for any three elements $x, y, z \in J$ we define the ternary operation known as the triple Jordan product (see [8,9]):

$$\{x \circ y \circ z\} = ((x \circ y) \circ z) + ((z \circ y) \circ x) - (y \circ (x \circ z)) \tag{2.6}$$

[cf. (2.3), which is the expression of (2.8) in the structure constants]. In terms of the ternary Jordan product, the system (1.8) takes the form

$$U_t = U_{xx} - 2\{U \circ V \circ U\}, \qquad V_t = -V_{xx} + 2\{V \circ U \circ V\}. \tag{2.7}$$

We now give examples that should convince the readers that most of the examples of multifield generalizations of the nonlinear Schrödinger equation known to them from other sources are J—S systems.

Example 2.1. The simplest example of a Jordan algebra is $J_{\mathrm{Mat}(N,N)}$, the algebra of $N \times N$ matrices with Jordan multiplication

$$(x \circ y) = \frac{1}{2}(xy + yx), \tag{2.8}$$

where xy is the ordinary product of matrices. It is obvious that the unit element in this Jordan algebra is the unit matrix. The ternary multiplication operation (2.6) is given by

$$\{x \circ y \circ z\} = \frac{1}{2}(xyz + zyx).$$

The J—S system corresponding to $J_{\mathrm{Mat}(N,N)}$ is the well-known matrix nonlinear Schrödinger equation

$$U_t = U_{xx} - 2UVU, \qquad V_t = -V_{xx} + 2VUV, \tag{2.9}$$

where $U(t, x)$ and $V(t, x($ are $N \times N$ matrices. ■

The generalization of the considered example to the case of an arbitrary associative algebra makes it possible to construct a large class of J—S systems. Let A be an arbitrary associative algebra with unit element in which the multiplication is denoted by $(x*y)$. Defining the new multiplication

$$(x \circ y) = \frac{1}{2}((x * y) + (y * x)), \tag{2.10}$$

we obtain a new algebra $A^{(+)}$, which will be a Jordan algebra with unit element.

Example 2.2. The system (1.9) in Example 1.1 is a *J—S* system. We consider the Jordan algebra V_N that is obtained by equipping the *N*-dimensional vector space *V* with the multiplication

$$(x \circ y) = \langle e, x\rangle y + \langle e, y\rangle x - \langle x, y\rangle e, \tag{2.11}$$

where $e=(1, 0, \ldots, 0)^T$, and $\langle,\rangle$ is the ordinary scalar product. It is obvious that the vector *e* is the unit element of the algebra V_N. It is readily verified that the ternary multiplication operator (2.6) in the considered case has the form

$$\{x \circ y \circ z\} = \langle Ay, x\rangle z + \langle Ay, z\rangle x - \langle x, z\rangle Ay,$$

where *A* is the linear operator whose matrix is given by $A=2e\otimes e-\langle e, e\rangle E$, where *E* in the unit matrix. The corresponding *J—S* system is

$$U_t = U_{xx} - 4\langle U, AV\rangle U + 2\langle U, U\rangle AV, \quad V_t = -V_{xx} + 4\langle V, AU\rangle V - 2\langle V, V\rangle AU. \tag{2.12}$$

The system (1.9) is obtained from (2.12) as a result of the linear transformation (2.4), where $M=\mathrm{diag}(1, i, \ldots, i)$. ■

The class of *J—S* systems that we have constructed does not exhaust the multifield generalizations of the nonlinear Schrödinger equation (2.1) of the form

$$u^i_t = u^i_{xx} - 2a^i_{jkm}u^j u^k v^m, \qquad i = 1, 2, \ldots, N, \quad v^i_t = -v^i_{xx} + 2\tilde{a}^i_{jkm}v^j v^k u^m, \quad i = 1, 2, \ldots, M, \tag{2.13}$$

where a^i_{jkm} and $\tilde{a}^i_{jkm}$ are constants. Systems of such form were considered in [4] (in the approach associated with *L—A* pairs) and in [5] (in the approach associated with higher symmetries, conservation laws, and the recursion operator). It was shown in [5] that the system (2.13) has a higher symmetry or nondegenerate higher conservation law if and only if a^i_{jkm} and $\tilde{a}^i_{jkm}$ are the structure constants of a Jordan pair (with regard to Jordan pairs, see [10]). The *J—S* systems considered in this paper are a special case of such systems that is still fairly rich in examples. In terms of [5], they correspond to Jordan pairs (*J*, *J*) generated by a Jordan algebra with unit element.

Thus, *J—S* systems are integrable; they have an infinite algebra of higher symmetries and an infinite series of local conservation laws. For example, the simplest higher symmetry of the *J—S* system (2.7) has the form

$$U_\tau = U_{xxx} - 6\{U \circ V \circ U_x\}, \qquad V_\tau = V_{xxx} - 6\{V \circ U \circ V_x\}, \tag{2.14}$$

and the three simplest conservation laws in the canonical series can be expressed as follows (see [5]):

$$\rho_1 = \omega(U, V), \tag{2.15}$$

$$\rho_2 = \omega(U, V_x) - \omega(U_x, V), \tag{2.16}$$

$$\rho_3 = \omega(U_x, V_x) + \omega(\{U \circ V \circ U\}, V), \tag{2.17}$$

where ω is the bilinear form whose components are defined by the formula $\omega_{kj}=a^m_{kjm}$.

3. EXPLICIT AUTOTRANSFORMATIONS FOR *J—S* SYSTEMS

In this section, we give for every *J—S* system an explicit autotransformation that is a natural generalization of the scalar autotransformation (1.2). We then discuss the question of this transformation's being an autotransformation for the higher symmetries and densities of the conservation laws of *J—S* systems.

Roughly speaking, to write down an explicit autotransformation for the *J—S* system (2.7) it is sufficient to replace in (1.2) the products of the functions *u* and *v* by the triple Jordan products of the vectors *U* and *V*, and u^{-1} by the element of the Jordan algebra that is the inverse of the vector *U*. But if this is to be truly done, we must at least recall the definition of the inverse element and give an explicit expression for it. More detailed information about inverse elements of a Jordan algebra can be found in [8,9].

Definition 3.1. *Let J be a Jordan algebra with unit element e. An element $x\in J$ is said to be invertible if there exists an* *element y such that*

$$(x \circ y) = e, \qquad ((x \circ x) \circ y) = x.$$

The element y is called the inverse of x and denoted x^{-1}. ■

Definition 3.1 can be reformulated in a more constructive form that will enable us to write down an explicit expression for the inverse element. In the standard manner, we introduce the operators of multiplication by the element x in the Jordan algebra J:

$$L(x) : J \to J, \qquad L(x)y = (x \circ y). \tag{3.1}$$

We define the linear operators $P(x, y)$, $P(x)$: $J \to J$,

$$P(x, y) = L(x)L(y) + L(y)L(x) - L((x \circ y)), \tag{3.2}$$

$$P(x) = P(x, x) = 2L(x)^2 - L((x \circ x)). \tag{3.3}$$

In accordance with [9], the element x of a Jordan algebra is invertible in the sense of Definition 3.1 if and only if the linear operator $P(x)$ is invertible. At the same time, the inverse element is unique and given by the expression

$$x^{-1} = P(x)^{-1}x. \tag{3.4}$$

The reader should not be alarmed by the apparent complexity of the expressions for the inverse element. It is readily verified that for a chosen basis the components of the matrices of the linear operators $P(e_k, e_j)$ are determined by the formula

$$(P(e_k, e_m))^i_j = a^i_{kjm}, \tag{3.5}$$

where a^i_{kjm} are the constants on the right-hand side of the system (1.8). In a number of cases, the inverse element in a Jordan algebra takes a fairly simple form, indeed as one would expect on intuitive grounds.

Example 3.1. In the case of the Jordan algebra $J_{\mathrm{Mat}(N,N)}$ in Example 2.1, whose elements are the $N \times N$ matrices, the inverse element in the sense of Definition 2.2 is precisely the inverse matrix. For the Jordan algebras V_N in Example 2.2, all elements x for which $\langle x, x \rangle \neq 0$ are invertible. The element that is the inverse of x has the form

$$x^{-1} = \langle x, x \rangle^{-1} Ax. \ ■ \tag{3.6}$$

THEOREM 3.1. *Every J—S system* (2.7) *admits the explicit autotransformation*

$$\tilde{U} = U_{xx} - \left\{U_x \circ P(U)^{-1}U \circ U_x\right\} - \{U \circ V \circ U\}, \quad \tilde{V} = -P(U)^{-1}U. \ ■. \tag{3.7}$$

Scheme of Proof. We first show that the transformation (3.7) is invertible, i.e., the vectors U and V can be expressed in terms of the vectors $\tilde{U}$ and $\tilde{V}$ and their derivatives with respect to x. Using the fact that if z is an invertible element of a Jordan algebra then z^{-1} os also invertible and $(z^{-1})^{-1} = z$ (see [9]), we find from the second relation in (3.7) that

$$U = -P(\tilde{V})^{-1}\tilde{V}. \tag{3.8}$$

Substituting the expression found for U in the first of the relations (3.7), we find that

$$\tilde{U} = \left(P(\tilde{V})^{-1}\tilde{V}\right)_{xx} - \left\{P(\tilde{V})^{-1}\tilde{V} \circ \tilde{V} \circ P(\tilde{V})^{-1}\tilde{V}\right\}$$

$$-\left\{\left(P(\tilde{V})^{-1}\tilde{V}\right)_x \circ P\left(P(\tilde{V})^{-1}\tilde{V}\right)^{-1} P(\tilde{V})^{-1}\tilde{V} \circ \left(P(\tilde{V})\right). \tag{3.9}$$

Since [cf. (2.6) and (3.2)]

$$\{x \circ y \circ z\} = P(x, z)y \tag{3.10}$$

and in the case of a Jordan algebra with unit element

$$\det P\left(P(\tilde{V})^{-1}\tilde{V}\right) \not\equiv 0,$$

we can use the relation (3.9) to express the vector V in terms of the vectors $\tilde{U}$, $\tilde{V}$, $\tilde{V}_x$, $\tilde{V}_{xx}$. Making some manipulations of the relation (3.9) using the Jordan identity (2.1), we can obtain a compact explicit expression for the vector V:

$$V = \tilde{V}_{xx} - \left\{\tilde{V}_x \circ P(\tilde{V})^{-1}\tilde{V} \circ \tilde{V}_x\right\} - \{\tilde{V} \circ \tilde{U} \circ \tilde{V}\}. \tag{3.11}$$

It remains to show that (3.7) is indeed an autotransformation of the system (2.7). Substituting in the system

$$\tilde{U}_t = \tilde{U}_{xx} - 2\{\tilde{U} \circ \tilde{V} \circ \tilde{U}\}, \quad \tilde{V}_t = -\tilde{V}_{xx} + 2\{\tilde{V} \circ \tilde{U} \circ \tilde{V}\}$$

the values of $\tilde{U}$ and $\tilde{V}$ from the expressions (3.7) and replacing the derivatives with respect to t using the system (2.7), we obtain a set of relations. It can be verified that these relations are satisfied identically by virtue of the Jordan identity (2.1).

This operation is not as trivial as in the case of the scalar nonlinear Schrödinger equation (1.1). To implement it, we must essentially use not only the Jordan identity (2.1) but also some not too obvious consequences of it and also the fact that the J—S systems that we consider correspond to Jordan algebras with unit element. We omit this part of the proof, since it has a purely algebraic nature and will hardly be of interest for the readers. Those that wish to can fill the gap by using the consequences of (2.1) given in [8,9]. ■

Example 3.2. In the case of the matrix Schrödinger equation (2.9) in Example 2.1, the explicit autotransformation has the form

$$\tilde{U} = U_{xx} - U_x U^{-1} U_x - U V U, \quad \tilde{V} = -U^{-1}. \blacksquare \tag{3.12}$$

We shall now show what happens to the higher symmetries and conservation laws of J—S systems under the transformations (3.7).

Proposition 3.1. *The transformation* (3.7) *is an autotransformation for the higher symmetries of the J—S systems.*

Scheme of Proof. Since the transformation (3.7) is invertible, it can be applied to any system. It is obvious that if we act with an invertible transformation on a symmetry of a J—S system we obtain some symmetry. Using the homogeneity of the higher symmetries of a J—S system (see [5]) and the transformation (3.7), we can show that the obtained symmetry is identical to the original one. ■

Proposition 3.2. *Under the action of the transformation* (3.7), *the densities of the local conservation laws of J—S systems go over into equivalent densities.*

Scheme of Proof. Let ρ be the density of a local conservation law. By $\tilde{\rho}$ we denote the function ρ in which the arguments U, V, U_x, V_x, ... are replaced by $\tilde{U}$, $\tilde{V}$, $\tilde{U}_x$, $\tilde{V}_x$, ..., respectively. For obvious reasons, the function $\tilde{\rho}$ in which the change of variables (3.7) is made is also a density of a local conservation law. It is necessary to show that $\tilde{\rho}$ differs from ρ by a total derivative with respect to x of some function. In other words, this means that there exists a function h which depends on a finite number of variables in the set U, V, U_x, V_x, ... and is such that

$$\tilde{\rho} - \rho = h_x. \tag{3.13}$$

Using the fact that (see [5]) the densities of the conservation laws of J—S systems are homogeneous polynomials in the variables u^i, v^i and their derivatives with respect to x that depend quadratically on the higher derivatives, one can prove formula (3.13). ■

It is still an open question whether the integrable systems of [5] that do not belong to the class of J—S systems admit explicit autotransformations. We expect the answer to be in the negative. For example, we have not succeeded in finding an explicit autotransformation for the well-known vector Schrödinger equation (see [6])

$$U_t = U_{xx} - 2\langle U, V\rangle U, \quad V_t = -V_{xx} + 2\langle U, V\rangle V, \tag{3.14}$$

where $U=(u^1, \ldots, u^N)^T$, $V=(v^1, \ldots, v^N)^T$, and $\langle , \rangle$ is the ordinary scalar product. We note that nevertheless any integrable system of the form (2.13) in [5] has some implicit Bäcklund transformation (see [11]).

4. JORDAN GENERALIZATIONS OF THE TODA CHAIN (J—S SYSTEMS)

This section is devoted to multifield generalizations of the infinite Toda chain.

In exactly the same way as we proceeded with the autotransformation (1.2) in Sec. 1, we interpret the transformation (3.7) as a system of discrete—differential equations

$$U_{n+1} = U_{nxx} - \left\{U_{nx} \circ P(U_n)^{-1} U_n \circ U_{nx}\right\} - \left\{U_n \circ V_n \circ U_n\right\}, \quad V_{n+1} = -P(U_n)^{-1} U_n. \tag{4.1}$$

Eliminating V_n from (4.1), we obtain the multifield discrete—differential equation

$$U_{nxx} = \left\{U_{nx} \circ P(U_n)^{-1} U_n \circ U_{nx}\right\} + U_{n+1} - \left\{U_n \circ P(U_{n-1})^{-1} U_{n-1} \circ U_n\right\}. \tag{4.2}$$

THEOREM 4.1. *To every Jordan algebra with unit element there corresponds a system of discrete—differential equations* (4.2) *that has an infinite series of higher symmetries and local conservation laws.* ■

Systems of the form (4.2) are Jordan generalizations of the infinite Toda chain (1.7) [it is readily seen that (1.7) is obtained from (4.2) in the case of a trivial one-dimensional Jordan algebra]. We shall call them *J—T* systems.

Example 4.1. The *J—T* system corresponding to the Jordan algebra $J_{\mathrm{Mat}(N,N)}$ has the form

$$U_{nxx} - U_{nx} U_n^{-1} U_{nx} = U_{n+1} - U_n U_{n-1}^{-1} U_n, \tag{4.3}$$

or, equivalently,

$$(U_{nx} U_n^{-1})_x = U_{n+1} U_n^{-1} - U_n U_{n-1}^{-1}. \tag{4.4}$$

It is none other than the well-known matrix Toda chain. ■

One further example of a *J—T* system is given in Example 1.1 [see Eq. (1.12)].

Scheme of Proof of Theorem 4.1. As we have already said, every *J—S* system possesses both an infinite series of higher symmetries and an infinite series of local conservation laws. On the other hand, from the symmetries and conservation laws of the *J—S* system one can construct the symmetries and conservation laws of the corresponding *J—T* system.

In the construction of the symmetries, we use the fact that a symmetry

$$U_\tau = F(U, V, U_x, V_x, \ldots), \quad V_\tau = G(U, V, U_x, V_x, \ldots) \tag{4.5}$$

of the *J—S* system (2.7) is invariant with respect to the transformation (3.7) (see Proposition 3.1). This is equivalent to the fact that the multifield chain

$$U_{n\tau} = F(U_n, V_n, U_{nx}, V_{nx}, \ldots), \quad V_{n\tau} = G(U_n, V_n, U_{nx}, V_{nx}, \ldots), \tag{4.6}$$

which is obtained from (4.5) by going over to the variables U_n, V_n, is a symmetry of the chain (4.1). The symmetry of the *J—T* system (4.2) can be obtained from (4.6) if we express the variables V_n, V_{nx}, V_{nxx}, ... in terms of the variables U_n, U_{nx}, U_{nxx}, ..., using the formula

$$V_n = -P(U_{n-1})^{-1} U_{n-1} \tag{4.7}$$

[see (4.1)]. Finally, we note that the higher derivatives U_{nxx}, U_{nxxx}, ... can be eliminated by means of (4.2). As a result, from the symmetry of the *J—S* system we construct the symmetry $U_{n\tau}=H_n$ of the corresponding *J—T* system, the right-hand side H_n of the symmetry depending only on the variables

$$U_n, U_{nx}, U_{n\pm1}, U_{n\pm1,x}, U_{n\pm2}, U_{n\pm2,x}, \ldots. \tag{4.8}$$

Note that by a local conservation law of the chain (4.2) what we mean precisely is a relation of the form

$$(r_n)_x = s_{n+1} - s_n. \tag{4.9}$$

Here, r_n and s_n are functions of a finite number of the variables (4.8), and r_n is differentiated by virtue of (4.2). In the construction of the local conservation laws of the *J—T* system, we use the circumstance that for any density ρ of a local conservation law of the *J—S* system a relation of the form (3.13) holds (see Proposition 3.2). Going over to the variables U_n, V_n and eliminating V_n, V_{nx}, V_{nxx}, ... by means of (4.7), we obtain from (3.13) the local conservation law (4.9) for the *J—T* system (4.2). It is clear that as a result of these operations the density ρ goes over into the function s_n, and the function h into the density r_n. ■

Note that we can also go in the opposite direction. Namely, from the higher symmetries and local conservation laws of the *J—T* system, we can construct the symmetries and conservation laws of the corresponding *J—T* system (see [1,12]).

Example 4.2. For obvious reasons, we have the right, when constructing the simplest higher symmetry of the *J—T* system (4.2), to use as initial symmetry the *J—S* system (2.7). Thus, one of the symmetries of the *J—T* system (4.2) has the form

$$U_{nt} = \left\{U_{nx} \circ P(U_n)^{-1} U_n \circ U_{nx}\right\} + U_{n+1} + \left\{U_n \circ P(U_{n-1})^{-1} U_{n-1} \circ U_n\right\}. \tag{4.10}$$

In the case of the matrix Toda chain (4.4), it becomes the chain

$$U_{nt} = U_{nx}U_n^{-1}U_{nx} + U_{n+1} + U_nU_{n-1}^{-1}U_n.$$

The readers can write down one further symmetry of the J—T system independently, taking as basis the simplest higher symmetry (2.14) of the J—S system (2.7). ■

Example 4.3. In the case of the matrix nonlinear Schrödinger equation (2.9), the first of the densities (2.15)—(2.17) has the form $\rho_1 = \mathrm{tr}(UV)$. To this density there corresponds the function $h_1 = -\mathrm{tr}(U_xU^{-1})$ [found directly from the relation (3.13)]. The described scheme leads to the conservation law (4.9) of the matrix Toda chain (4.4) with $r_n^1 = \mathrm{tr}(U_{nx}U_n^{-1})$, $s_n^1 = \mathrm{tr}(U_nU_{n-1}^{-1})$. This conservation law can be readily obtained directly from (4.4). The answer is not so obvious if the density $\rho_2 = \mathrm{tr}(UV_x)$ is used. In this case

$$r_n^2 = \mathrm{tr}\left[\frac{1}{2}(U_{nx}U_n^{-1})^2 + U_nU_{n-1}^{-1}\right], \qquad s_n^2 = -\,\mathrm{tr}\left[U_n(U_{n-1}^{-1})_x\right].$$

In the case of the vector J—S system (1.9), the first two densities in (2.15)—(2.17) can be expressed as follows: $\rho_1 = \langle U, V\rangle$, $\rho_2 = \langle U, V_x\rangle$. To them there correspond the conservation laws (4.9) of the vector chain (1.12) with

$$r_n^1 = \langle U_n, U_n\rangle^{-1}\langle U_n, U_{nx}\rangle, \quad s_n^1 = \langle U_{n-1}, U_{n-1}\rangle^{-1}\langle U_n, U_{n-1}\rangle,$$

$$r_n^2 = \langle U_{n-1}, U_{n-1}\rangle^{-1}\langle U_n, U_{n-1}\rangle + \langle U_n, U_n\rangle^{-2}\langle U_n, U_{nx}\rangle^2 - \frac{1}{2}\langle U_n, U_n\rangle^{-1}|U_{nx}|^2,$$

$$s_n^2 = -\left\langle U_n, (\langle U_{n-1}, U_{n-1}\rangle^{-1}U_{n-1})_x\right\rangle. \ ■$$

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