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The multi-field Schrödinger lattices

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Integrable differential–difference analogues of the generalized Schrödinger equations are constructed. A one-to-one correspondence between them and the triple Jordan algebras is established.

1. As is known the nonlinear Schrödinger equation

$$i\psi_t = \psi_{xx} + 2|\psi|^2\psi, \quad \psi = \psi(t,x), \tag{1}$$

is a reduction of the integrable equation

$$u_t = u_{xx} + 2u^2v, \quad v_t = -v_{xx} - 2v^2u, \tag{2}$$
$$u = u(t,x), \quad v = v(t,x).$$

It is shown in ref. [1] that there is a Bäcklund transformation

$$\tilde{u}_x = u + \tilde{u}^2 v, \quad v_x = -\tilde{v} - v^2\tilde{u} \tag{3}$$

for eq. (2). In ref. [1] Bäcklund transformations are interpreted as infinite systems of ordinary differential equations. In particular the Bäcklund transformation (3) corresponds to the integrable lattice (i.e. having higher symmetries and local conservation laws)

$$(u_n)_x = u_{n+1} + u_n^2 v_n, \quad (v_n)_x = -v_{n-1} - v_n^2 u_n, \tag{4}$$
$$u_n = u_n(x), \quad v_n = v_n(x).$$

It is natural to call this lattice a differential–difference analogue of eq. (2). The integrable lattice (4) yields a system of ordinary differential equations which allows one to construct soliton and finite-gap solutions of eq. (2) quite easily. The approach to the construction of exact solutions of nonlinear partial differential equations which uses the existence of an integrable differential–difference analogue is discussed in refs. [1,2].

In this Letter we consider the multi-field integrable lattices

$$(u_n^i)_x = u_{n+1}^i + a_{jkm}^i u_n^j v_n^k u_n^m, \quad i = 1, \dots, N,$$
$$(v_n^i)_x = -v_{n-1}^i - \tilde{a}_{jkm}^i v_n^j u_n^k v_n^m, \quad i = 1, \dots, M, \tag{5}$$

where a_{jkm}^i, $\tilde{a}_{jkm}^i$ ($a_{jkm}^i = a_{mkj}^i$, $\tilde{a}_{jkm}^i = \tilde{a}_{mkj}^i$) are constant coefficients. In (5) and everywhere below summation over repeated indices is assumed. The lattices (5) generalize (4) and are differential–difference analogues of the multi-field Schrödinger equations

$$u_t^i = u_{xx}^i + 2a_{jkm}^i u^j v^k u^m, \quad i = 1, \dots, N,$$
$$v_t^i = -v_{xx}^i - 2\tilde{a}_{jkm}^i v^j u^k v^m, \quad i = 1, \dots M. \tag{6}$$

Equations of the form (6) were considered in refs. [3–6]. In ref. [5] wide classes of equations (6) possessing an (L, A) pair associated with Hermitian symmetric spaces have been constructed. In ref. [6] the conditions on the constants a_{jkm}^i, $\tilde{a}_{jkm}^i$ have been formulated. These conditions are necessary and sufficient for eq. (6) to have at least one higher symmetry or local conservation law of high enough order. If eq. (6) possesses a Bäcklund transformation of the form

$$\tilde{u}_x^i = u^i + A_{jkm}^i \tilde{u}^j v^k \tilde{u}^m, \quad i = 1, \dots N,$$
$$v_x^i = -\tilde{v}^i - \tilde{A}_{jkm}^i v^j \tilde{u}^k v^m, \quad i = 1, \dots M, \tag{7}$$

the conditions on the constants will be the same.

Proposition 1. Eq. (6) has the Bäcklund transformation (7) if and only if the constants a_{jkm}^i, $\tilde{a}_{jkm}^i$, A_{jkm}^i, $\tilde{A}_{jkm}^i$ satisfy the following constraints,

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$$A^i_{jkm}=a^i_{jkm}, \quad \tilde{A}^i_{jkm}=\tilde{a}^i_{jkm}, \tag{8}$$

$$a^i_{jkn}a^n_{msp}-a^i_{msn}a^n_{jkp}-a^i_{nsp}a^n_{jkm}+a^i_{mnp}\tilde{a}^n_{kjs}=0,$$
$$\tilde{a}^i_{jkn}\tilde{a}^n_{msp}-\tilde{a}^i_{msn}\tilde{a}^n_{jkp}-\tilde{a}^i_{nsp}\tilde{a}^n_{jkm}+\tilde{a}^i_{mnp}a^n_{kjs}=0. \tag{9}$$

As in the case of the scalar Bäcklund transformation (3), one can interpret (7) as a discrete equation of the form (5). Thus differential–difference analogues of the generalized Schrödinger equations (6) have been constructed. Lattices (5) with constants a^i_{jkm}, $\tilde{a}^i_{jkm}$ satisfying the identities (9) will be called many-field Schrödinger lattices.

2. Before going to discuss the integrability of the Schrödinger lattices, we have to explain how to describe the constants a^i_{jkm}, $\tilde{a}^i_{jkm}$, satisfying the constraints (9). We shall recall the algebraic interpretation of integrability conditions (9) which has been given in ref. [6]. Let L be a linear space, U and V be subspaces, and $L=U\oplus V$, $\dim U=N$, $\dim V=M$. Let us provide L with the structure of a commutative triple algebra by multiplication ( ): $L\times L\times L\to L$ such that

$$(xyz)=0, \quad (\tilde{x}\tilde{y}\tilde{z})=0,$$
$$(\tilde{x}y\tilde{z})=(\tilde{z}y\tilde{x}), \quad (x\tilde{y}z)=(z\tilde{y}x), \tag{10}$$

where $x, y, z \in U$, $\tilde{x}, \tilde{y}, \tilde{z}\in V$. Let $e_1, e_2, ..., e_N$ and $\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_M$ be bases of U and V respectively. Then the multiplication ( ) is determined by the structure constants a^i_{jkm}, $\tilde{a}^i_{jkm}$ such that

$$(e_j\tilde{e}_k e_m)=a^i_{jkm}e_i, \quad (\tilde{e}_j e_k\tilde{e}_m)\tilde{a}^i_{jkm}\tilde{e}_i. \tag{11}$$

A change of bases of linear spaces U and V corresponds to the linear transformations

$$u^i=J^i_rU^r, \quad v^i=J^i_rV^r \tag{12}$$

of the variable u^i, v^i in (6).

Later on we shall be interested in nonreducible equations which cannot be reduced to the "triangle" form by means of linear transformations. In more invariant terms one can formulate the property of nonreducibility in the following way. If one can extract a closed subsystem of smaller dimension from the system (6), then (6) will be called a reducible system. In the opposite case it is a nonreducible one. Obviously the differential–difference analogues (5) of the nonreducible equations (6) are nonreducible.

Let the structure constants satisfy the constraints (9). It means that we have the following identities:

$$(x\tilde{y}(y\tilde{x}z))-(y\tilde{x}(x\tilde{y}z))-(z\tilde{x}(x\tilde{y}y))$$
$$+(y(\tilde{y}x\tilde{x})z)=0,$$
$$(\tilde{x}y(\tilde{y}x\tilde{z}))-(\tilde{y}x(\tilde{x}y\tilde{z}))-(\tilde{z}x(\tilde{x}y\tilde{y}))$$
$$+(\tilde{y}(y\tilde{x}x)\tilde{z})=0, \tag{13}$$

for any elements $x, y, z \in U$, $\tilde{x}, \tilde{y}, \tilde{z} \in V$. The triple algebras satisfying identities (10) and (13) are called Jordan pairs. They are quite familiar to the experts in algebra. The term "Jordan pairs" is motivated by the close connection of these triple algebras with Jordan algebras. One can find a detailed algebraic theory of Jordan pairs in ref. [7]. To obtain the simplest example of a Jordan pair one can choose the linear space $M_{p,q}(\mathbb{C})$ of matrices for U and V and define the multiplication by the formula

$$(x\tilde{y}z)=x\tilde{y}^{\mathrm{T}}z+z\tilde{y}^{\mathrm{T}}x,$$
$$(\tilde{x}y\tilde{z})=\tilde{x}y^{\mathrm{T}}\tilde{z}+\tilde{z}y^{\mathrm{T}}\tilde{x}, \tag{14}$$

where the superscript T means the transposition.

Nonreducible lattices are the most interesting among the lattices (5). As in the case of eqs. (6) (see ref. [6]), the nonreducible lattices correspond to the simple (i.e. having no nontrivial ideals) Jordan pairs. Ref. [7] contains an exhaustive classification of simple Jordan pairs. There is a list in this work which consists of four series of arbitrarily-high-dimension algebras and of two special algebras. This permits one to obtain in explicit form all the nonreducible integrable lattices (5) and corresponding equations (6). For example, the triple algebra $L=(M_{p,q}(\mathbb{C}), M_{p,q}(\mathbb{C}))$ is a simple Jordan pair. In the case $p=1$, $q=N$ the corresponding equation (6) is the well-known vector Schrödinger equation,

$$u^i_t=u^i_{xx}+2u^iR(u,v), \quad i=1,...,N,$$
$$v^i_t=-v^i_{xx}-2v^iR(u,v), \quad i=1,...,N, \tag{15}$$
$$R(u,v)=u^1v^1+u^2v^2+...+u^Nv^N.$$

The lattice

$$(u^i_n)_x=u^i_{n+1}+u^i_nR(u_n,v_n), \quad i=1,...N,$$
$$(v^i_n)_x=-v^i_{n-1}-v^i_nR(u_n,v_n), \quad i=1,...N, \tag{16}$$

is the differential–difference analogue of (15).

3. Let us discuss the integrability of the lattices (5).

Proposition 2. Two multi-field Schrödinger lattices (5), (9) are integrable.

An integrability property can be realised as the existence of higher symmetries, local conservation laws or a recursion operator. The Schrödinger lattices connected with Jordan pairs have all these properties. Let us reproduce the recursion operator

$$L_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \tag{17}$$

satisfying the Lax equation

$$(L_n)_x = G_n L_n - L_n G_n ,$$

where

$$A_n = I_N D + 2a(j,k) u_n^j v_n^k + 2a(j,k) u_n^j (D-1)^{-1} v_n^k ,$$

$$D_n = I_M D^{-1} - 2\tilde{a}(j,k) v_n^j (D-1)^{-1} u_n^k ,$$

$$B_n = A(j,k) u_n^j u_n^k + 2A(j,k) u_n^j (D-1)^{-1} u_n^k ,$$

$$C_n = -\tilde{A}(j,k) v_n^j v_n^k - 2\tilde{A}(j,k) v_n^j (D-1)^{-1} v_n^k ,$$

$$G_n = \begin{pmatrix} I_N D + 2a(j,k) u_n^j v_n^k & A(j,k) u_n^j u_n^k \\ -\tilde{A}(j,k) v_n^j v_n^k & -I_M D^{-1} - 2\tilde{a}(j,k) v_n^j u_n^k \end{pmatrix} .$$

Here I_N and I_M are the $N \times N$ and $M \times M$ unit matrices respectively, $a(j,k)$, $\tilde{a}(j,k)$, $A(j,k)$, $\tilde{A}(j,k)$ are matrices with coefficients

$$a(j,k)^i_m = a^i_{jkm}, \quad \tilde{a}(j,k)^i_m = \tilde{a}^i_{jkm} ,$$

$$A(j,k)^i_m = a^i_{jmk}, \quad \tilde{A}(j,k)^i_m = \tilde{a}^i_{jmk} .$$

D is a shift operator such that

$$D(f(W_k, W_{k+1}, \dots, W_{k+m})) = f(W_{k+1}, W_{k+2}, \dots W_{k+m+1}) ,$$

where $W_n = (u_n^1, \dots u_n^N, v_n^1, \dots, v_n^M)^{\mathrm{T}}$. $(D-1)^{-1}$ is the inverse operator to $(D-1)$. In general $(D-1)^{-1}$ is an infinite formal series

$$(D-1)^{-1} = D^{-1} + D^{-2} + D^{-3} + \dots ,$$

though it can act on functions H_n which are represented in the form

$$H_n = (D-1)(f_n) = f_{n+1} - f_n . \tag{18}$$

If the recursion operator L_n acts on the right-hand side of the lattice (5), $(D-1)^{-1}$ will be applied only to functions of the form (18). Therefore, acting by L_n step by step on the right-hand side F_n of the Schrödinger lattice $\partial W_n/\partial x = F_n$, one can get arbitrarily many of its higher symmetries $\partial W_n/\partial t_k = L_n^k(F_n)$, $k=1, 2, 3, \dots$.

Let us recall that in the case of lattices a local conservation law is a relationship of the form $(\rho_n) = (D-1)(\sigma_n)$, where ρ_n, σ_n are functions depending on a finite number of variables u_j^i, v_j^i. The integrable Schrödinger lattices possess an infinite series of local conservation laws. Conserved densities $\rho_n(1)$, $\rho_n(2)$, $\rho_n(3)$, ... may be constructed through the formula $\rho_n(k) = \mathrm{tr}(\mathrm{res}(L_n^k))$, where

$$\mathrm{res}(a_m D^m + a_{m-1} D^{m-1} + \dots + a_1 D + a_0 + a_{-1} D^{-1} + \dots) \stackrel{\text{def}}{=} a_0 .$$

4. Let us choose Hamiltonian lattices among the Schrödinger ones (5). They can be written in the form

$$(U_n)_x = R^{\mathrm{T}} \delta\rho/\delta V_n, \quad (V_n)_x = -R \delta\rho/\delta U_n , \tag{19}$$

where $U_n = (u_n^1, \dots, u_n^N)^{\mathrm{T}}$, $V_n = (v_n^1, \dots, v_n^N)^{\mathrm{T}}$, R is a constant matrix, $\delta\rho/\delta W_n = (\delta\rho/\delta w_n^1, \dots \delta\rho/\delta w_n^N)^{\mathrm{T}}$,

$$\frac{\delta\rho}{\delta w_n^j} \stackrel{\text{def}}{=} \sum_{i=n-m}^{n} \frac{\partial D^i(\rho)}{\partial w_n^j} ,$$

$$\rho = \rho(W_0, W_1, \dots, W_m) .$$

In the periodic case

$$U_n = U_{n+K}, \quad V_n = V_{n+K}$$

the lattice (19) will be a finite Hamiltonian system of ordinary differential equations of the form

$$\begin{pmatrix} p_x^1 \\ \vdots \\ p_x^r \end{pmatrix} = S \begin{pmatrix} \partial H/\partial p^1 \\ \vdots \\ \partial H/\partial p^r \end{pmatrix} , \tag{20}$$

where S is a constant antisymmetric matrix, $r = 2KN$. The Hamiltonian H is determined with the help of the formula

$$H=\sum_{i=1}^{K} D^i(\rho)\ .$$

It is not hard to verify that eqs. (6) are Hamiltonian together with their Hamiltonian differential–difference analogues. Therefore, when we construct finite-gap solutions of the Hamiltonian equations (6), we can solve the corresponding Hamiltonian systems of ordinary differential equations (20).

As in the case of eqs. (6) (see ref. [6]), it is easy to formulate simple conditions which are sufficient for the lattice (5) to be a Hamiltonian one. It is sufficient to assume that $N=M$ and the lattice has a conserved density of the form

$$h_n=Q_{ij}u^i_{n+1}v^j_n+P_{ijkm}u^i_n u^j_n v^k_n v^m_n\ , \tag{21}$$

where $Q=(Q_{ij})$ is a nondegenerate matrix. The function (21) is a conserved density if and only if the matrix Q satisfies the condition

$$Q_{ir}\tilde{a}^r_{kjm}-Q_{rm}a^r_{ikj}=0\ . \tag{22}$$

The Schrödinger lattice (5) satisfying this condition may be represented as (19) where $R=Q^{-1}$,

$$\rho=h_0=Q_{ij}u^i_1 v^j_0+\tfrac{1}{2}Q_{rm}a^r_{ikj}u^i_0 u^j_0 v^k_0 v^m_0\ .$$

From the algebraic point of view the relationship (22) means that (Q_{ij}) is the matrix of an invariant bilinear form defined on a triple algebra L. It is known (see ref. [17]) that any Jordan pair L possesses canonical invariant bilinear forms. The structure constants a^i_{jkm}, $\tilde{a}^i_{jkm}$ define matrices of these bilinear forms by the formulae

$$Q_{ij}=\mathrm{tr}(a(i,j))\ ,\quad \tilde{Q}_{ij}=\mathrm{tr}(\tilde{a}(j,i))\ . \tag{23}$$

Invariant bilinear forms are nondegenerate in the case of simple Jordan pairs L corresponding to nonreducible lattices. Therefore all the nonreducible integrable lattices (5) are Hamiltonian ones.

5. It is known (see ref. [8]) that if an integrable equation has a Bäcklund transformation, one may usually construct a new integrable equation. The equations are connected with each other by a differential substitution which is a Miura type transformation.

Proposition 3. Every equation (6), (9) is connected by means of the differential substitution

$$u^i=U^i_x-a^i_{jkm}U^jV^kU^m\ ,\quad v^i=V^i\ , \tag{24}$$

with the integrable equation

$$\begin{aligned}U^i_t&=U^i_{xx}-2a^i_{jkm}U^jV^k_xU^m\\&\quad-2a^i_{jkm}\tilde{a}^k_{pqr}U^jV^pU^qV^rU^m\ ,\\V^i_t&=-V^i_{xx}-2\tilde{a}^i_{jkm}V^jU^k_xV^m\\&\quad+2\tilde{a}^i_{jkm}a^k_{pqr}V^jU^pV^qU^rV^m\ .\end{aligned} \tag{25}$$

Let us show how the substitution (24) is generated by the Bäcklund transformation (7). Let u^i and v^i, $\tilde{u}^i$ and $\tilde{v}^i$ satisfy the generalized Schrödinger equation (6), (9) and they are linked together by the transformation (7). If we substitute

$$\begin{aligned}&\tilde{u}^i=U^i\ ,\quad v^i=V^i\ ,\\&u^i=U^i_x-a^i_{jkm}U^jV^kU^m\ ,\\&\tilde{v}^i=-V^i_x-\tilde{a}^i_{jkm}V^jU^kV^m\end{aligned} \tag{26}$$

in the equations

$$\begin{aligned}&\tilde{u}^i_t=\tilde{u}^i_{xx}+2a^i_{jkm}\tilde{u}^j\tilde{v}^k\tilde{u}^m\ ,\\&v^i_t=-v^i_{xx}-2\tilde{a}^i_{jkm}v^ju^kv^m\ ,\end{aligned}$$

we see that $U^i=\tilde{u}^i$, $V^i=v^i$ satisfy eq. (25). Note that the last two relationships (26) are another form of the Bäcklund transformation (7).

It is easy to see from (26) that eqs. (6), (9) and (25), (9) are connected with each other not only by the differential substitution (24) but also by

$$\tilde{u}^i=U^i\ ,\quad \tilde{v}^i=-V^i_x-\tilde{a}^i_{jkm}V^jU^kV^m\ .$$

Note that a substitution of the form (24) connecting a "scalar" equation of the form (25) with eq. (2) has been obtained in ref. [9] in a similar way.

6. The algebraic nature of the relationships for constants which define the integrable generalized Schrödinger equations (6) and their differential–difference analogues (5) is not unique. For example it has been shown in refs. [10,11] that multi-field generalizations of the scalar Burgers and Korteweg–de Vries equations are closely connected with left-symmetric and Jordan algebras. In the future we hope to find algebraic structures which determine multi-field generalizations of many other "scalar" integrable equations of the form

$$u_t = u_{xx} + f(u, v, u_x, v_x) ,$$

$$v_t = -v_{xx} + g(u, v, u_x, v_x)$$

(for an exhaustive list see refs. [12,9]) and to construct their differential–difference analogues.

References

[1] A.B. Shabat and R.I. Yamilov, Algebra Anal. 2 (1990) 183.
[2] A.B. Shabat and R.I. Yamilov, Phys. Lett. A 130 (1988) 271.
[3] M.J. Ablowitz, Stud. Appl. Math. 108 (1978) 17.
[4] A.V. Jiber, Teor. Mat. Fiz. 52 (1982) 405.
[5] A. Fordy and P. Kulish, Commun. Math. Phys. 89 (1983) 427.
[6] S.I. Svinolupov, Commun. Math. Phys. (1991), in press.
[7] O. Loos, Lecture notes in mathematics, Vol. 460 (Springer, Berlin, 1975).
[8] R.I. Yamilov, Teor. Mat. Fiz. 85 (1990) 368.
[9] A.V. Mikhailov, A.B. Shabat and R.I. Yamilov, Commun. Math. Phys. 115 (1988) 1.
[10] S.I. Svinolupov, Phys. Lett. A 135 (1989) 32.
[11] S.I. Svinolupov, Teor. Mat. Fiz. 87 (1991) 101.
[12] A.V. Mikhailov, A.B. Shabat and R.I. Yamilov, Usp. Mat. Nauk 42 (1987) 3.