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ON BÄCKLUND TRANSFORMATIONS FOR INTEGRABLE EVOLUTION EQUATIONS

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An equation of the form

$$u_t = F(u, u_1, \dots, u_n), \quad u: \mathbf{R}^2 \to \mathbf{C}, \quad u_i \overset{\text{def}}{=} d^i u/dx^i, \tag{1}$$

is said to be *formally integrable* if the relation

$$dL/dt = [F_*, L] \tag{2}$$

is solvable, where $F_* \overset{\text{def}}{=} \sum_0^n (\partial F/\partial u_i) D^i$, $u_0 \overset{\text{def}}{=} u$, and $D \overset{\text{def}}{=} d/dx$. We specify what is meant by the solvability of (2). Following [1], we require that there exist a formal series $L = \sum_{-\infty}^1 a_i D^i$, $a_i = a_i(u, \dots, u_{k_i})$, satisfying (2) if u is taken to be any solution of (1).

It is not hard to verify (see [2]) that the functions res L^i, $i \in \mathbf{N}$, where $\operatorname{res} \sum b_i D^i \overset{\text{def}}{=} b_{-1}$, are densities of conservation laws for a formally integrable equation (1) (since all or almost all these conservation laws can turn out to be trivial, this does not imply that every formally integrable equation has an infinite series of conservation laws). The collection of orders of the conservation laws with densities res L^i is an important characteristic of a formally integrable equation (if a conservation law is trivial, we agree to regard its order as -1).

It was shown in [2] that any equation of the form (1) having an infinite series of local conservation laws is formally integrable. Moreover, all equations of the form $u_t = u_3 + f(u, u_1, u_2)$ with an infinite series of conservation laws were classified in [2] to within substitutions $u \to \varphi(u)$. In this note we classify these equations to within Bäcklund transformations

$$v = \Phi(u, u_1, \dots, u_m). \tag{3}$$

Equations $u_t = F(u, \dots)$ and $v_t = G(v, \dots)$ are said to be *connected by the transformation* (3) if for any solution u of the equation $u_t = F$ the function $v = \Phi(u, \dots, u_m)$ is a solution of the equation $v_t = G(v)$; the number m is called *the order of the transformation*. Introduction of the potential $v = u_1$ is the simplest transformation of this kind. It can be shown that Table 1 contains all the different equations from the list in [2] to within substitutions of the form $u \to \varphi(u)$ and to within the introduction of a potential. The orders of the conservation laws with densities res L^1, res L^3 and res L^5 are given for each equation in this table (it was shown in [2] that the densities res L^{2i} give trivial conservation laws for the equations in Table 1).

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TABLE 1

Nº	Equation	Order of the law of conservation with density		
		res L	res L^3	res L^5
I	$u_t = u_3 + c_1 u_1$	-1	-1	-1
II	$u_t = u_3 + uu_1 + c_1 u_1$	0	0	1
III	$u_t = u_3 - \frac{1}{6} u^2 u_1 + c_1 u_1$	0	1	2
IV	$u_t = u_3 - \frac{1}{8} u_1^3 + (c_1 e^{u/2} + c_2 e^{-u/2})^2 u_1 + c_3 u_1$	1	2	3
V	$u_t = u_3 - \frac{3}{2} \frac{u_1 u_2^2}{u_1^2 + 1} - \frac{3}{2} \wp(u)(u_1^2 + 1) u_1 + c_1 u_1$, where $(\wp')^2 = 4\wp^3 - g_2 \wp - g_3$	2	3	4
VI	$u_t = u_3 - \frac{3}{2} \frac{u_2^2}{u_1} + \frac{c_1}{u_1} - \frac{3}{2} \wp(u) u_1^3 + c_2 u_1$	2	3	4

It is known (see [3]) that if the equations $u_t = F$ and $v_t = G$ are connected by equation (3), and the equation $v_t = G$ is formally integrable, then the equation $u_t = F$ is also formally integrable. Moreover, the series L and $\tilde{L}$ satisfying the relations $L_t = [F_*, L]$ and $\tilde{L}_t = [G_*, \tilde{L}]$ are connected by the formula

$$L = \Phi_*^{-1} \tilde{L}\big|_{v=\Phi} \Phi_* .$$

The following assertion can be deduced from this formula.

THEOREM 1. 1) *If the conservation law with density* res $\tilde{L}^i$, $i \in \mathbf{N}$, *is trivial, then the law with density* res L^i *is trivial.*

2) *If the law with density* res $\tilde{L}^i$ *has order* k, *then the law with density* res L^i *has order* $k + m$, *where* m *is the order of the transformation.*

Starting from this theorem, we easily see which equations in Table 1 can be connected by a transformation (3) and what its order is. The explicit form of the transformations can be found by direct computations.

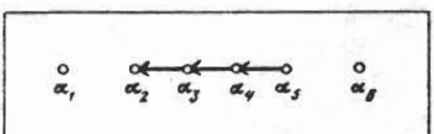

FIGURE 1

Figure 1 represents a graph whose vertices correspond to the equations in Table 1. Two vertices α_i and α_j are joined by an arrow pointing to α_j if there exists a first-order transformation mapping the solutions of the equation with index i into solutions of the equation with index j.

The transformations corresponding to the arrows of the graph have the form

$$\alpha_2 \leftarrow \alpha_3, \quad v = u_1 - u^2/6; \quad \alpha_3 \leftarrow \alpha_4, \quad v = 3u_1/2 + c_1 e^{u/2} + c_2 e^{-u/2};$$
$$\alpha_4 \leftarrow \alpha_5,$$
$$v = \ln\left(\wp(u) + e_1/2 + (\wp(u) - e_2)^{1/2}(\wp(u) - e_3)^{1/2}\right) + 2\ln\left(u_1 + (u_1^2 + 1)^{1/2}\right),$$

where the e_i, $i = 1, 2, 3$, are the roots of the polynomial defining the function $\wp(u)$ in Table 1. These transformations are not unique. The nonuniqueness is due to the fact that equations III–V are invariant under certain substitutions of the form $u \to \varphi(u)$. It can be verified that *all* the transformations of the form (3) connecting equations in Table 1 are compositions of the first-order transformations indicated above and substitutions $u \to \varphi(u)$ admitted by the equations. More precisely, this assertion is valid if we require that a transformation from the equation with index i into the equation with index j exist for all values of the constants appearing in the equation with index i (here, of course, the transformation itself and the constants in the equation with index j depend on these constants).

As is clear from Figure 1, equation VI does not in general reduce to any of the remaining equations in Table 1. This equation first appeared in papers of Krichever and Novikov (see [4]) in connection with algebraic solutions of the Kadomtsev-Petviashvili equation. A representation of zero curvature was determined for equation VI in [4], and it was noted that this equation has an infinite series of conservation laws. It can be shown that the substitution $u \to 2\wp^{-1}(u)$ reduces VI to the form

$$u_t = u_3 - (3/2)(u_2^2/u_1) + (c_1/4)(h(u)/u_1), \tag{4}$$

where $h = 4u^3 - g_2 u - g_3$, with g_2 and g_3 constants determining the function $\wp$ in Table 1. In the case when h has a multiple root it is possible by a linear fractional substitution of u to lower the degree of h, and then by the transformation

$$v = -3u_3/u_1 + (3/2)(u_2^2/u_1^2) - c_1 h/4u_1^2 \tag{5}$$

to reduce (4) to the Korteweg-de Vries (KdV) equation $v_t = v_3 + vv_1$. In the general case there is no transformation of the form (3) which connects these two equations. However, it can be shown that (4) is a reduction of the system

$$v_t = v_3 + vv_1 - 12c_1 u_1, \qquad u_t = -2u_3 - u_1 v. \tag{6}$$

Namely, if u is a solution of (4), then (u, v), where v is defined by (5), is a solution of (6). We remark that (6) is one of the generalized KdV systems connected with a Kac-Moody algebra of type $B_2^{(1)}$ (see [5] and [6]). The unknown u is easily eliminated from (6). Here an equation of the form $v_{tt} = H(v, v_1, \ldots, v_6, v_t)$ connected with (4) by the transformation (5) is obtained for v.

In [7] there is a list of formally integrable discrete equations of the form $du_n/dt = F(u_{n-1}, u_n, u_{n+1})$ that includes the equation

$$du_n/dt = h(u_n)\left(1/(u_{n+1} - u_n) + 1/(u_n - u_{n-1})\right), \quad h(u_n) = 4u_n^3 - g_2 u_n - g_3, \tag{7}$$

whose properties call to mind equation (4). Namely, if h has a multiple root, then by a linear fractional substitution of u_n we can lower the degree of h, and then by a transformation

$$v_n = -h(u_n)/((u_{n+1} - u_n)(u_n - u_{n-1}))$$

reduce (7) to the KdV difference equation $dv_n/dt = v_n(v_{n+1} - v_{n-1})$. In the general case (7) is a reduction of the system

$$dv_n/dt = v_n(v_{n+1} - v_{n-1}) - 4v_n(u_{n+1} - u_{n-1}), \quad du_n/dt = -v_n(u_{n+1} - u_{n-1}).$$

We remark that after eliminating the unknown u_n from this system we obtain the equation

$$d^2w_n/dt^2 = e^{(w_{n+2}+w_{n+1})} - e^{(w_{n+1}+w_n)} - e^{(w_n+w_{n-1})} + e^{(w_{n-1}+w_{n-2})},$$

in $w_n = \ln v_n$, and this equation is directly connected with the Toda chain.

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