BÄCKLUND TRANSFORMATION AND THE CONSTRUCTION OF THE INTEGRABLE BOUNDARY-VALUE PROBLEM FOR THE EQUATION $u_{xx} - u_{tt} = e^u - e^{-2u}$.

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Abstract. Bäcklund transformation for the Bullough-Dodd-Jiber-Shabat equation $u_{xx} - u_{tt} = e^u - e^{-2u}$ is found. The construction of integrable boundary condition for this equation together with the algebro-geometric solutions satisfying it are suggested.

1. Introduction.

Most features of integrability for nonlinear differential equations are considered on a base of the inverse scattering method for solving the initial-value Cauchy problem. They are not specific for the boundary-value problems for those equations. Exception are the special forms of boundary conditions preserving the integrability. Nontrivial boundary conditions of such kind was first considered in [1] by means of r-matrix approach. In [2–5] some methods of solving the integrable boundary-value problems for nonlinear Schrodinger equation and for Sine-Gordon equation are developed. In this paper we suggest the scheme for finding the proper boundary condition of the form

$$R(e^u, u_x, u_t, u_{xt})\Big|_{x=0} = 0 \tag{1.1}$$

for the Bullough-Dodd-Jiber-Shabat equation

$$u_{xx} - u_{tt} = e^u - e^{-2u} \tag{1.2}$$

together with the algebro-geometric solutions of the boundary-value problem arisen. We also suggest the Bäcklund transformation for this equation different from that of [6].

First in section 2 we consider the mirror symmetric pairs of the algebro-geometric solutions $v(x,t)$ and $\hat{v}(x,t)$ of the equation (1.2)

$$\hat{v}(x,t) = v(-x,t) \tag{1.3}$$

and the way how to construct them. Then in section 3 we set an additional restriction for such pairs taking v and $\hat{v}$ to be bound with the Bäcklund transformations

$$v(x,t) \longrightarrow u(x,t) \longleftarrow \hat{v}(x,t) \tag{1.4}$$

via the virtual one-soliton solution $u(x,t)$ of the equation (1.2). This gives the opportunity to get several differential relationships between $v(x,t)$ and $u(x,t)$ on a boundary $x = 0$. The number of these relationships is large enough to exclude $u(x,t)$ with its derivatives and to get the boundary condition of the form (1.1) for the solution $v(x,t)$. In section 4 we managed only to reduce the exclusion problem to the system of four polynomials with three extra variables to be excluded. Being algorithmically solvable this problem though may require the use of computer methods to get the explicit form of polynomial R in (1.1).

We should note that for a long time the equation (1.2) was thought to have no auto-Bäcklund transformation (see [9–11]). In those papers authors sought Bäcklund transformation in the class of differential relationships depending on the first order derivatives of u. Our Bäcklund transformation contain the second order derivatives, therefore it is out of range of theorems proved there.

2. Mirror-symmetric finite-gap solutions.

Let Γ be an even genus Riemann surface realized as a three-fold ramified covering of the complex λ-plane with the branching points at $\lambda = 0$, at infinity and at the points $\pm\lambda_1, \dots, \pm\lambda_g$ on a real axis. Riemann surfaces of such kind are connected with the class of the algebro-geometric or so called finite-gap solutions of the equation (1.2) being the complexified spectra of Lax operators for such solutions (see [7, 8]). In order to have the same notations with [7, 8] now in sections 2 and 3 we consider the equation (1.2) in "light cone" variables where it has the form

$$u_{xt} = e^u - e^{-2u} \tag{2.1}$$

Let us consider two finite gap solutions $v(x,t)$ and $\hat{v}(x,t)$ of the equation (2.1) constructed with the help of two vectorial Baker-Achiezer functions $e(x,t,P)$ and $\hat{e}(x,t,P)$ depending on $P \in \Gamma$. Let D and $\hat{D}$ be the divisors of poles of $e(x,t,P)$ and $\hat{e}(x,t,P)$ respectively. For $v(x,t)$ and $\hat{v}(x,t)$ to obey the relationship

$$\hat{v}(x,t) = v(-t/\varkappa^2, -\varkappa^2 x) \tag{2.2}$$

(compare with (1.3) above) one should impose some extra limitations on a choice of Riemann surface Γ and divisors D and $\hat{D}$ in addition to that of [7]. Let us take Γ such that

(1) branching points $\pm\lambda_1, \dots, \pm\lambda_g$ form the set invariant under the action of the inversion $\pi : \lambda \longrightarrow -\varkappa^6/\lambda$, where $\varkappa$ is a positive constant.

This case Γ admits an involution $\pi : \lambda(\pi P) = -\varkappa^6/\lambda(P)$ commuting with the involution $\sigma : \lambda(\sigma P) = -\lambda(P)$ and with the antiinvolution $\tau : \lambda(\tau P) = \overline{\lambda(P)}$. Let us also choose D and $\hat{D}$ such that

(2) the divisor $\hat{D}$ is a map image $\hat{D} = \pi D$ of the divisor D. One can check that this condition is compatible with the previous restrictions for the choice of the divisors

$$\begin{aligned} &D + \sigma D - P_\infty - P_0 = C \\ &\hat{D} + \sigma\hat{D} - P_\infty - P_0 = C \end{aligned} \tag{2.3}$$

where $\lambda(P_\infty) = \infty$, $\lambda(P_0) = 0$ and C is the divisor of canonic class on Γ(see [7]).

Lemma 1. *Conditions* (1) *and* (2) *above are enough for the finite-gap solutions* $v(x,t)$ *and* $\hat{v}(x,t)$ *to be bound with the relationship* (2.2).

One can obtain the proof of this lemma by comparing the analytical properties (i.e. essential singularities and poles) of two Baker-Achiezer functions $\hat{e}(x,t,P)$ and $e(-t/\varkappa^2, -\varkappa^2 x, \pi P)$.

3. The differential Bäcklund transformation.

Next step is to realize the relationship (1.4). According to [8] each one-soliton solution $u(x,t)$ of the equation (2.1) on a finite-gap background originate from the appropriate one-soliton Baker-Achiezer function $\Psi(x,t,P)$. This function has two exponential singularities at P_∞ and at P_0 of the same form as $e(x,t,P)$ and poles forming the divisor $D^\circ + \Lambda + \Lambda^*$. Here D° is the divisor of degree g defining one more finite-gap solution of the equation (2.1) called the background solution in [8]. Two points Λ and Λ^* such that $\sigma\Lambda \neq \Lambda^*$ and $\lambda(\Lambda) = -\Lambda(\Lambda^*) = \lambda_0$ form the discrete spectrum of the soliton in question. Since Γ is the three-fold covering over λ-plane we can find one more point $\tilde\Lambda$ on Γ in addition to Λ and Λ^* such that $\lambda(\tilde\Lambda) = \lambda(\Lambda) = -\lambda(\Lambda^*) = \lambda_0$. In terms of $\tilde\Lambda$ we can state the new restrictions for the choice of D, $\hat D$ and D° taking the following divisors

$$D^\circ + \Lambda + \Lambda^* + \tilde\Lambda - \hat D - 3P_\infty \cong 0$$
$$D^\circ + \Lambda + \Lambda^* + \sigma\tilde\Lambda - D - 3P_\infty \cong 0 \tag{3.1}$$

to be linear equivalent to zero. This means that there are two meromorphic functions $\hat\alpha(P)$ and $\alpha(P)$ whose zeros and poles constitute the divisors (3.1). One can check that conditions (3.1) are compatible with (2.3) and the same condition for D°. From (3.1) we derive the equivalence

$$\hat D - D \cong \tilde\Lambda - \sigma\tilde\Lambda \tag{3.2}$$

which is compatible with $\hat D = \pi D$ only if $\pi\tilde\Lambda = \sigma\tilde\Lambda$. This means that $\tilde\Lambda$ is a stable point of the involution $\pi \circ \sigma$ and therefore

$$\lambda_0 = \lambda(\tilde\Lambda) = \varkappa^3 \tag{3.3}$$

This case we write (3.2) as a restriction for the choice of D

$$\pi D - D \cong \tilde\Lambda - \sigma\tilde\Lambda \tag{3.4}$$

Reality and non singularity conditions for the solutions $v(x,t)$ and $\hat v(x,t)$ in terms of divisors D and $\hat D$ (see [8]) together with (3.4) fix the position of the point $\tilde\Lambda$ on an invariant cycle α_0 of the antiinvolution τ passing trough P_∞ and P_0.

Lemma 2. *For the above defined meromorphic functions $\alpha(P)$ and $\hat\alpha(P)$ normalized by the condition*

$$\alpha(P) \sim \hat\alpha(P) \sim \lambda(P) \text{ as } P \to P_\infty$$

their values at $P = P_0$

$$\alpha(P_0) = -\lambda(\tilde\Lambda) = -\lambda_0 \qquad \hat\alpha(P_0) = -\lambda(\sigma\tilde\Lambda) = \lambda_0$$

are defined by the discrete spectrum $\lambda = \lambda_0 = \varkappa$ of the intermediate soliton solution $u(x,t)$ in (1.4).

Proof. Considering zeros and poles for meromorphic function $\alpha(P)\alpha(\sigma P)$ and its normalizing condition we get

$$\alpha(P)\alpha(\sigma P) = -\frac{\omega^\circ(P)}{\omega(P)}\left[\lambda^2(P) - \lambda_0^2\right] \tag{3.5}$$

where $\omega(P)$, $\hat{\omega}(P)$ and $\omega^{\circ}(P)$ are abelian differential of the third kind defined by the divisors (2.3) and $D^{\circ}+\sigma D^{\circ}-P_{\infty}-P_0$ of canonic class on Γ (see [7]). Substituting $P=P_0$ in (3.5) gives $\alpha^2(P_0)=\lambda_0^2$. The sign of $\alpha(P_0)$ then is defined by considering zeros and poles on a cycle α_0 for the function $\alpha(P)$ which is real valued on that cycle. For $\hat{\alpha}(P_0)$ proof is quite similar. □

Now using $\alpha(P)$ and $\hat{\alpha}(P)$ we slightly modify the one soliton Baker-Achiezer function $\Psi(P)$ to get the functions $\alpha(P)\Psi(P)$ and $\hat{\alpha}(P)\Psi(P)$ whose poles form the divisors $3P_{\infty}-D$ and $3P_{\infty}-\hat{D}$. Comparing the analytical properties of $\hat{\alpha}(P)\Psi(P)$ and $e(P)$ we get the matrix relationship

$$\alpha(P)\Psi(P)=[\mathcal{C}\lambda(P)+\mathcal{D}]\,e(P) \tag{3.6}$$

where $\mathcal{C}$ is the upper-triangular matrix with unities in the diagonal and $\mathcal{D}$ is the lower-triangular matrix with the following values in the diagonal

$$\mathcal{D}_{11}=\alpha(P_0)e^{v-u}\qquad \mathcal{D}_{22}=\alpha(P_0)e^{u-v}\qquad \mathcal{D}_{11}=\alpha(P_0) \tag{3.7}$$

Similarly for the function $\hat{\alpha}(P)\Psi(P)$ we have

$$\hat{\alpha}(P)\Psi(P)=\left[\hat{\mathcal{C}}\lambda(P)+\hat{\mathcal{D}}\right]e(P) \tag{3.8}$$

with the matrices of the similar shape and with diagonal elements in the matrix $\hat{\mathcal{D}}$ being of the form

$$\hat{\mathcal{D}}_{11}=\hat{\alpha}(P_0)e^{\hat{v}-u}\qquad \hat{\mathcal{D}}_{22}=\hat{\alpha}(P_0)e^{u-\hat{v}}\qquad \hat{\mathcal{D}}_{11}=\hat{\alpha}(P_0) \tag{3.9}$$

Lemma 3. *Equivalence* (3.4) *is enough for the finite gap solutions of the equation* (2.1) *corresponding to the divisors D and πD to be bound with the one soliton solution of this equation via the Bäcklund transformations* (1.4).

Matrix equations (3.6) and (3.8) are the integral form of transformations (1.4). Using the equations

$$\begin{aligned}
&\partial_x e(P)=\mathbf{L}(v)e(P) &&\partial_t e(P)=\mathbf{A}(v)e(P)\\
&\partial_x \Psi(P)=\mathbf{L}(u)\Psi(P) &&\partial_t \Psi(P)=\mathbf{A}(u)\Psi(P)
\end{aligned}$$

of the direct spectral problem with Lax operators $\mathbf{L}$ and $\mathbf{A}$ one can obtain the following equations for the matrix $\mathcal{B}=\mathcal{C}\lambda+\mathcal{D}$

$$\partial_x\mathcal{B}=\mathbf{L}(u)\mathcal{B}-\mathcal{B}\mathbf{L}(v)\qquad \partial_t\mathcal{B}=\mathbf{A}(u)\mathcal{B}-\mathcal{B}\mathbf{A}(v) \tag{3.10}$$

Taking into account the special form of the matrices $\mathcal{C}$ and $\mathcal{D}$ and (3.7) from (3.10)we derive the further specialization for $\mathcal{B}$

$$\mathcal{B}=\begin{bmatrix}\lambda+be^{v-u} & p\lambda-\lambda(u_x-v_x) & q\lambda e^{-u}\\ (q-b(u_t-v_t))e^{u+v} & \lambda+be^{u-v} & \lambda p\\ p & qe^{-v} & \lambda+b\end{bmatrix}$$

where $b=\alpha(P_0)=-\lambda_0=-\varkappa^3$. Matrix $\hat{\mathcal{B}}=\hat{\mathcal{C}}+\hat{\mathcal{D}}$ has the similar structure derived with use of (3.9). Substituting the specialized form of matrix $\mathcal{B}$ into equation (3.10) one can extract a lot of differential equations part of which forming the differential form of Bäcklund transformation for the equation (2.1) and others being their consequences. For the sake of brevity we introduce the following notations:

$$\begin{aligned} &U = (u+v)/2 && U = (u-v)/2 \\ &h = e^{-U}\cosh(V) - e^{2U} && R = V_x V_t - h^2 \end{aligned} \tag{3.11}$$

For $p(x,t)$ and $q(x,t)$ then we have

$$\begin{aligned} p &= 2e^{2U}R^{-1}(hV_x - bV_t^2) \\ q &= 2e^{2U}R^{-1}(bhV_t - V_x^2) \end{aligned} \tag{3.12}$$

Moreover these functions are connected with $U(x,t)$ and $V(x,t)$ via the differential equations

$$\begin{aligned} &p_t = 2e^U\sinh(V) && p_x = U_x p - e^{-U} q\sinh(V) \\ &q_x = 2be^U\sinh(V) && q_t = U_t - e^U p\sinh(V) \end{aligned} \tag{3.13}$$

Theorem 1. *The differential form of the Bäcklund transformation for the equation* (2.1) *consist of three equations*

$$b^{-1}V_x^3 + bV_t^3 + e^{-2U}R^2 - 3hR - 2h^3 = 0 \tag{3.14}$$

$$\begin{aligned} V_{xx} + 2U_x V_x &= \frac{3}{2}p_x \\ V_{tt} + 2U_t V_t &= \frac{3}{2}b^{-1}q_t \end{aligned} \tag{3.15}$$

being after the substitution (3.13), (3.12) *and* (3.11) *in* (3.14) *and* (3.15) *the set of differential equalities in* $u(x,t)$, $v(x,t)$ *consistent with the equation* (2.1) *and closed in the sense that it has no functionally independent differential consequences with the first and second order derivatives.*

Though the Bäcklund transformation (3.14), (3.15) is derived on a base of finite-gap solutions, it is applicable to arbitrary solutions of the equation (2.1) since the the proof of the theorem 1 require nothing but direct calculations.

4. On the extraction of the integrable boundary condition.

Let us consider the pair of Bäcklund transformations (1.4) together with the non-local equality (2.2). Note that (2.2) is local when restricted on a moving boundary $x = -t/\varkappa^2$. To deal with the stable boundary $x = 0$ we should introduce new variables by substituting $\varkappa^{-1}(x-t)/2$ and $\varkappa(x+t)/2$ for x and t respectively. On doing that we find that the equation (2.1) takes its initial form (1.2) and (2.2) becomes (1.3). As a consequence of (1.3) we obtain the equalities

$$\begin{aligned} &\hat{v} = v && \hat{v}_x = -v_x && \hat{v}_t = v_t \\ &\hat{v}_{xt} = -v_{xt} && \hat{v}_{xx} = v_{xx} && \hat{v}_{tt} = v_{tt} \end{aligned} \tag{4.1}$$

which hold on a boundary $x = 0$. Using the same notations (3.11) except for

$$R = v_x^2 - V_t^2 - h^2 \qquad \hat{R} = U_x^2 - V_t^2 - h^2 \tag{4.2}$$

we can rewrite the equation (3.14) for the changed variables x, t

$$\begin{aligned} &2V_x^3 + 6V_x V_t^2 + e^{-2U} R^2 - 3hR - 2h^3 = 0\\ &2U_x^3 + 6U_x V_t^2 + e^{-2U} \hat{R}^2 - 3h\hat{R} - 2h^3 = 0 \end{aligned} \tag{4.3}$$

The second of these equations correspond to the second transformation from (1.4). Combining the equations (3.15) and (3.16) and changing variables x, t in them for new ones above we obtain the following two differential equations

$$\begin{aligned} &V_{xx} + V_{tt} + 2U_x V_x + 2U_t V_t = 3he^U \sinh(V) R^{-1} V_x +\\ &+ 3he^{2U}(U_x V_x + U_t V_t) R^{-1} + 3e^U \sinh(V)(V_x^2 + V_t^2) R^{-1} +\\ &+ 3e^{2U}(U_x V_x^2 + U_x V_t^2 - 2U_t V_x V_t) R^{-1} \end{aligned} \tag{4.4}$$

$$\begin{aligned} &V_{xt} + 2U_x V_t + 2U_t V_x = -3he^U \sinh(V) R^{-1} V_t +\\ &+ 3he^{2U}(U_x V_t + U_t V_x) R^{-1} + 6e^U \sinh(V) R^{-1} V_x V_t +\\ &+ 3e^{2U}(U_t V_x^2 + U_t V_t^2 - 2U_x V_x V_t) R^{-1} \end{aligned} \tag{4.5}$$

It is worth to note that these equations do not contain the constant parameter $b = -\varkappa^3 = -\lambda_0$ as well as the equations (4.4) (see also (3.3) above). The second Bäcklund transformation from (1.4) via (4.1) give rise to the next two equations

$$\begin{aligned} &V_{xx} + V_{tt} + 2U_x V_x + 2U_t V_t = -3he^U \sinh(V) \hat{R}^{-1} U_x +\\ &+ 3he^{2U}(U_x V_x + U_t V_t) \hat{R}^{-1} + 3e^U \sinh(V)(U_x^2 + V_t^2) \hat{R}^{-1} -\\ &- 3e^{2U}(V_x U_x^2 + V_x V_t^2 - 2U_t U_x V_t) \hat{R}^{-1} \end{aligned} \tag{4.6}$$

$$\begin{aligned} &2U_t + 2V_x V_t + 2U_t U_x = 3he^U \sinh(V) \hat{R}^{-1} V_t +\\ &+ 3he^{2U}(V_x V_t + U_t U_x) \hat{R}^{-1} + 6e^U \sinh(V) \hat{R}^{-1} U_x V_t -\\ &- 3e^{2U}(U_t U_x^2 + U_x V_t^2 - 2U_x V_x V_t) \hat{R}^{-1} \end{aligned} \tag{4.7}$$

which are also independent of $b = -\varkappa^3 = -\lambda_0$. Now by subtracting (4.4) from (4.6) and by subtracting (4.5) from (4.7) we obtain two equations containing the only derivative of the second order $v_{xt} = U_{xt} - V_{xt}$. These two equations together with (4.4) may be written in the form of four polynomial equations in the following independent variables

$$v_{xt}, v_x, v_t, e^v, u_x, u_t, e^u$$

Excluding last three variables one can get the integrable boundary condition of the form (1.1) which is automatically fulfilled for the finite-gap solutions $v(x, t)$ of the equation (1.2) constructed above

$$R(e^v, v_x, v_t, v_{xt})\Big|_{x=0} = 0$$

Note that polynomial R here is different from that of (3.11) and (4.2). It is not yet known to us. The only obstacle that prevented us from getting the polynomial R in the explicit form is the enormous amount of calculations needed to complete this task. We suppose this task may be of interest as the touchstone for testing the computer algorithms of polynomial algebra.

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