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To a transformation theory of two-dimensional integrable systems

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Abstract

We generalise to the two-dimensional case a list of integrable Toda type lattice equations. As a result, 1 + 2 dimensional systems similar to the Davey–Stewartson coupled system are obtained together with explicit auto-Bäcklund transformations and 2D Miura type transformations.

1. Introduction

In Ref. [1] (see also Refs. [2,3]), we developed a transformation theory of the coupled Schrödinger type systems

$$\mathrm{i}u_t = u_{xx} + f(u_x, v_x, u, v),$$
$$-\mathrm{i}v_t = v_{xx} + g(u_x, v_x, u, v), \qquad (1)$$

using a close connection between systems of the class (1) and Toda type lattice equations (chains). To all the key integrable systems of the Schrödinger type (a complete list of them has been obtained in Refs. [4,5]), there correspond integrable lattice equations of the form

$$q_{nxx} = F(q_{nx}, q_{n+1}, q_n, q_{n-1}) \qquad (2)$$

(n is a discrete integer variable).

The shift transformation ($n \mapsto n+1$) in a chain generates a special Bäcklund transformation for the associated system (i.e. system (1) corresponding to this chain). That is an explicit auto-transformation which has the form of a differential substitution of the second order

$$u^* = F(u_{xx}, v_{xx}, u_x, v_x, u, v),$$
$$v^* = G(u_{xx}, v_{xx}, u_x, v_x, u, v) \qquad (3)$$

and is invertible. The inverse transformation corresponds to the backward shift ($n+1 \mapsto n$) and has the same form (3). Such a transformation brings solutions of (1) again into solutions and allows one, for example, to construct exact solutions (see e.g. Ref. [3]).

On the other hand, all integrable lattice equations of the form

$$q_{nxx} = F(q_{nx}, q_{n+1} - q_n, q_n - q_{n-1}) \qquad (4)$$

can be reduced to the Toda model by discrete Miura type transformations (the reader can find all of them in the next section). Those discrete transformations generate continuous analogs for the associated Schrödinger type systems (1). Since the Toda model corresponds to the nonlinear Schrödinger coupled

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system, resulting continuous Miura transformations reduce the associated systems (1) to the Schrödinger system.

In the present paper, we generalise this approach to the two-dimensional case. First, we construct two-dimensional analogs of known (see Refs. [6,1]) integrable chains (4) and obtain a list of lattice equations of the form

$$q_{nxy}=F(q_{nx},\, q_{ny},\, q_{n+1}-q_n,\, q_n-q_{n-1}) \qquad (5)$$

together with discrete transformations which relate equations of that list.

There are new examples in the list. Chains which are associated with 2D analogs of the degenerations of the Landau–Lifshitz model seem to be most interesting (see (22) below). In particular, the following lattice equation,

$$q_{nxy}+q_{nx}q_{ny}\left(\frac{1}{q_{n+1}-q_n}-\frac{1}{q_n-q_{n-1}}\right)=0, \qquad (6)$$

corresponds to the totally isotrophic case.

Second, we build up systems of the Davey–Stewartson type associated with chains of the form (5). We obtain those systems, using continuous analogs of discrete Miura transformations which link together 2D chains (5). We also demonstrate that all continuous 2D transformations we use (both auto-Bäcklund transformations and Miura type ones) not only are local but also admit the local prolongation to additional nonlocal dynamical variables which are characteristic features of the multi-dimensional case.

2. Discrete Miura transformations in the two-dimensional case

We start with two classes of one-dimensional chains related by Miura type transformations. The first of them consists of the following lattice equations,

$$q_{nxx}=(c_1q_{nx}^2+c_2q_{nx}+c_3)[f(q_{n+1}-q_n)-f(q_n-q_{n-1})],$$
$$f'=c_1f^2+c_4f+c_5. \qquad (7)$$

Here c_i are arbitrary constants, the function f is defined by an ordinary differential equation. One can see that the well-known (exponential) Toda model belongs to this class. The lattice equations (7) can be found in Ref. [6] in which equations of the form (2) possessing local conservation laws of a high enough order have been completely classified.

The following simple transformation,

$$u_n=f(q_{n+1}-q_n),\quad v_n=q_{nx}, \qquad (8)$$

maps (7) into the class

$$u_{nx}=p(u_n)(v_{n+1}-v_n),\quad v_{nx}=q(v_n)(u_n-u_{n-1}), \qquad (9)$$

with

$$p(z)=c_1z^2+c_4z+c_5,\quad q(z)=c_1z^2+c_2z+c_3.$$

This is the second class we discuss here. All the systems of the form (9) can be found, for example, in Ref. [1].

The lattice systems (9) are reduced to the Toda chain by Miura transformations. In fact, if $c_1\neq 0$, we can without loss of generality put $p(z)=z^2-b^2$ and $q(z)=z^2-a^2$. Now, either of two transformations,

$$\begin{aligned}\tilde u_n&=(u_n+b)(v_{n+1}+a),\\ \tilde v_n&=(u_n-b)(v_n-a),\\ \hat u_n&=(u_n+b)(v_n+a),\\ \hat v_n&=(u_{n-1}-b)(v_n-a),\end{aligned} \qquad (10)$$

gives the Volterra coupled system

$$u_{nx}=u_n(v_{n+1}-v_n),\quad v_{nx}=v_n(u_n-u_{n-1}) \qquad (11)$$

(the renumeration $u_n=w_{2n}$, $v_n=w_{2n-1}$ allows one to pass to the usual form of the Volterra equation in terms of w_k). In its turn, the system (11) is reduced to the polynomial Toda chain

$$u_{nx}=u_n(v_{n+1}-v_n),\quad v_{nx}=u_n-u_{n-1}. \qquad (12)$$

Transformations in this case are

$$\tilde u_n=u_nv_{n+1},\quad \tilde v_n=u_n+v_n,$$
$$\hat u_n=u_nv_n,\quad \hat v_n=u_{n-1}+v_n. \qquad (13)$$

We are now going to construct two-dimensional local generalisations for all above transformations except for (13) (a generalisation would not be local in this case, see the conclusions). The first three of them

$$u_n=\exp(q_{n+1}-q_n),\quad v_n=q_{ny}, \qquad (14)$$

$$\tilde u_n=q_{nx},\quad \tilde v_n=q_n-q_{n-1}, \qquad (15)$$

$$\hat{u}_n = q_{nx}, \quad \hat{v}_n = \exp(q_n - q_{n-1}), \tag{16}$$

are direct analogs of (8). These link the chains

$$q_{nxy} = \exp(q_{n+1} - q_n) - \exp(q_n - q_{n-1}), \tag{17}$$

$$q_{nxy} = q_{nx}(q_{n+1} - 2q_n + q_{n-1}), \tag{18}$$

$$q_{nxy} = q_{nx}[\exp(q_{n+1} - q_n) - \exp(q_n - q_{n-1})] \tag{19}$$

and the well-known 2D Toda and Volterra lattice systems,

$$u_{ny} = u_n(v_{n+1} - v_n), \quad v_{nx} = u_n - u_{n-1}, \tag{20}$$

$$u_{ny} = u_n(v_{n+1} - v_n), \quad v_{nx} = v_n(u_n - u_{n-1}) \tag{21}$$

(cf. (12) and (11)). The chains (17) and (18) are reduced to (20) by (14) and (15), respectively. The relation between (19) and (21) is (16).

From our point of view the most interesting two-dimensional chains are of the form

$$q_{nxy} = (q_{nx} + a)(q_{ny} - a)[f(q_{n+1} - q_n) - f(q_n - q_{n-1})],$$
$$f' = f^2 - b^2, \tag{22}$$

where a and b are arbitrary constants. There is a relationship between (22) and the 2D Volterra coupled system (21), though it is not so simple as the above transformations. In fact, introducing

$$u_n = f(q_{n+1} - q_n), \quad v_n = q_{nx}, \quad w_n = q_{ny}, \tag{23}$$

we are led to chains of the form

$$u_{nx} = (u_n^2 - b^2)(v_{n+1} - v_n),$$
$$u_{ny} = (u_n^2 - b^2)(w_{n+1} - w_n),$$
$$v_{ny} = w_{nx} = (v_n + a)(w_n - a)(u_n - u_{n-1}), \tag{24}$$

which generalise the modified Volterra systems (9) with $c_1 \neq 0$.

In contrast to (20) and (21), the lattice systems (24) are, in a sense, nonlocal. However, the usual structure is restored on the next step. Namely, using the following analogs of the Miura type transformations (10) (any of them)

$$\tilde{u}_n = (u_n + b)(v_{n+1} + a),$$
$$\tilde{v}dn = (u_n - b)(w_n - a),$$
$$\hat{u}_n = (u_n + b)(v_n + a),$$
$$\hat{v}_n = (u_{n-1} - b)(w_n - a), \tag{25}$$

we obtain (21).

One of two compositions of (23) and (25) will be the following transformation,

$$\hat{u}_n = -\frac{q_{nx}}{q_{n+1} - q_n}, \quad \hat{v}_n = -\frac{q_{ny}}{q_n - q_{n-1}}, \tag{26}$$

if we restrict ourselves to the simplest case $a = b = 0$, $f(z) = -1/z$. This local transformation reduces the chain (6) of Introduction to the 2D Volterra coupled system (21) (cf. Ref. [8], where (6) and (26) have also arisen).

It is clear that up to point transformations there are several main one-dimensional cases among (7) and (9), and the chains (17)–(22), (24) generalise all of them.

3. Auxiliary linear problems

The two-dimensional lattice equations presented in the previous section are closely connected to the Laplace–Darboux transformations of a second order linear PDE with two independent variables which can be written in the form

$$L\psi \equiv \partial_x \partial_y \psi + a\partial_x \psi + b\partial_y \psi + c\psi = 0 \tag{27}$$

(coefficients a, b, c depend on x and y). Not only the lattice equations arise in a natural way but also corresponding L–A pairs and transformations relating these equations. We explain in this section how to derive the 2D Toda and Volterra systems (20) and (21) which play here the key role. If one consider some different Laplace–Darboux transformations, one could in principle be led to different (or even new) integrable equations.

The Laplace–Darboux transformations

$$A: \psi \to \hat{\psi}, \quad L \to \hat{L},$$

are defined by differential operators of the form

$$A = \alpha\partial_x + \beta\partial_y + \gamma \tag{28}$$

and transform solutions of (27) as follows: $\hat{\psi} = A\psi$. These operators have to satisfy the following operator relationship,

$$\hat{L}A - \hat{A}L \equiv \sum a_{ij}\partial_x^i \partial_y^j = 0, \tag{29}$$

where

$$\hat{L} = \partial_x\partial_y + \hat{a}\partial_x + \hat{b}\partial_y + \hat{c},$$

$$\hat{A} = \hat{\alpha}\partial_x + \hat{\beta}\partial_y + \hat{\gamma}.$$

Equating the coefficients a_{ij} to zero, we can find constraints for the coefficients of A and $\hat{A}$. For example, $a_{21} = \alpha - \hat{\alpha}$ and $a_{12} = \beta - \hat{\beta}$, therefore

$$\alpha = \hat{\alpha}, \quad \beta = \hat{\beta}.$$

Also, Eq. (29) allows us to know how the coefficients of L are transformed.

The zeroth order Laplace–Darboux transformation is defined by the operator of multiplication by a function γ and is called the gauge transformation. In this case $A = \hat{A} = \gamma$. The gauge transformation has two invariants

$$r = c - ab - b_y, \quad \rho = c - ab - a_x \tag{30}$$

(r and ρ are not changed under its action, i.e. $\hat{r} = r$ and $\hat{\rho} = \rho$).

A classical example of the first order transformation goes back to Laplace and is defined by

$$\begin{aligned} L_1A_2 - A_1L_2 &\equiv (\partial_x\partial_y + a_1\partial_x + b_1\partial_y + c_1)(\partial_x + b_2) \\ &\quad - (\partial_x + b_1)(\partial_x\partial_y + a_2\partial_x + b_2\partial_y + c_2) = 0. \end{aligned} \tag{31}$$

Relation (31) is equivalent to

$$a_1 = a_2, \quad b_1 = b_2 - (\log r_2)_x, \quad \rho_1 = r_2, \tag{32}$$

where r_i and ρ_i are the invariants (30) corresponding to the operators L_i. Formulae (32) allow us to express the coefficients of L_1 via the coefficients of L_2 and vice versa. The iteration of this transformation (31) which will be called the Laplace transformation generates a set of operators L_i ($i = 0, \pm 1, \pm 2, \ldots$).

The Laplace transformation is closely connected to the 2D Toda model. Evidently, all the operators L_i have the same coefficients $a_i = a$. The gauge transformation enables one to make $a_i = 0$ for all i, and then $L_i = \partial_x\partial_y + b_i\partial_y + c_i$. Now, Eqs. (32) give

$$c_{ix} = c_i(b_{i+1} - b_i), \quad b_{iy} = c_i - c_{i-1}, \tag{33}$$

i.e. we are led to the 2D polynomial Toda chain (20) ((33) coincides with (20) if we change $c_i \to u_n$, $b_i \to v_n$, $x \leftrightarrow y$).

The operators L_i satisfy not only (31) but also the operator relation

$$L_i(c_i)^{-1}\partial_y = \partial_y(c_{i-1})^{-1}L_{i-1}. \tag{34}$$

One can verify the equivalence of (31) and (34) by direct calculation. The last relation reflects the invertibility of the Laplace transformation and gives rise a new operator: $B_i = (c_i)^{-1}\partial_y$. It is easy to check that the pair of the operators A_i and B_i generates the auxillary linear problem for the 2D Toda chain.

First, the system of equations

$$A_i\psi_i = \psi_{i-1}, \quad B_i\psi_i = -\psi_{i+1} \tag{35}$$

rewritten as a pair of equations for ψ_i and ψ_{i+1} immediately implies that $L_k\psi_k = 0$ ($k = i, i+1$). Thus, one can consider (35) just as a representation of the linear equation (27) in the form of a system of two first order equations.

On the other hand, introducing the shift operator T, we can rewrite the system (35) as follows,

$$(\partial_x + b_i - T^{-1})\psi_i = 0, \quad (\partial_y + c_iT)\psi_i = 0. \tag{36}$$

The consistency condition for (36) will be the following commutator relation,

$$[\partial_x + b_i - T^{-1}, \partial_y + c_iT] = 0. \tag{37}$$

So, we obtain the L–A pair of the 2D Toda system (33) found in Ref. [9]. We will use this L–A pair for the construction of higher symmetries.

The two-dimensional Volterra chain corresponds to the case $c_i = 0$ for all i and can be obtained by the gauge transformation. Let us introduce a set of solutions $\{\varphi_i\}$ of the linear equation (27) linked together by the Laplace transformation, i.e. $L_i\varphi_i = 0$, $A_i\varphi_i = \varphi_{i-1}$, where

$$L_i = \partial_x\partial_y + b_i\partial_y + c_i, \quad A_i = \partial_x + b_i.$$

Using the gauge transformation, we construct new operators $\tilde{L}_i = \varphi_i^{-1}L_i\varphi_i$ and new ψ-functions $\tilde{\psi}_i = \psi_i/\varphi_i$. It is clear that $\tilde{L}_i\tilde{\psi}_i = 0$, and the corresponding Laplace–Darboux transformation is defined by $\tilde{A}_i = \varphi_{i-1}^{-1}A_i\varphi_i$.

The new operators have the form $\tilde{L}_i = \partial_x\partial_y + \tilde{a}_i\partial_x + \tilde{b}_i\partial_y$, where

$$\tilde{a}_i = \varphi_{iy}/\varphi_i, \quad \tilde{b}_i = \varphi_{i-1}/\varphi_i. \tag{38}$$

It is easy to see that the functions φ_i satisfy the equations

$$\varphi_{ix} = \varphi_{i-1} - b_i\varphi_i, \quad \varphi_{iy} = -c_i\varphi_{i+1}. \tag{39}$$

Using these equations together with Eqs. (33) of the 2D Toda chain, we can check that $\tilde{a}_i$ and $\tilde{b}_i$ satisfy the 2D Volterra chain

$$\tilde{a}_{ix} = \tilde{a}_i\left(\tilde{b}_{i+1} - \tilde{b}_i\right), \quad \tilde{b}_{iy} = -\tilde{b}_i\left(\tilde{a}_i - \tilde{a}_{i-1}\right). \tag{40}$$

The chain (40) coincides with (21) if we change $\tilde{a}_i \to -u_n$, $\tilde{b}_i \to v_n$, $x \leftrightarrow y$.

Using (39), one can invert the formulae (38) and obtain a transformation which will reduce the 2D Volterra chain to the 2D Toda chain and will be a two-dimensional analog of (13) (see the conclusions). On the other hand, if we pass in (35) to the new ψ-functions, we are led to new operators

$$\tilde{A}_i = \tilde{b}_i^{-1}\partial_x + 1, \quad \tilde{B}_i = -\left(\tilde{a}_i^{-1}\partial_y + 1\right)$$

and, as in the previous case, obtain the L–A pair for the 2D Volterra chain (40),

$$\left[\partial_x + \tilde{b}_i(1 - T^{-1}),\ \partial_y - \tilde{a}_i(T-1)\right] = 0. \tag{41}$$

It should be remarked that in the case of the periodic closure $L_{i+N} = L_i$, the relations (37) and (41) define zero curvature representations for corresponding hyperbolic systems. The spectral parameter can be introduced by the Bloch periodicity property of the ψ-function: $\psi_{i+N} = \nu\psi_i$. More precisely, the replacement in (37) and (41) of the shift operator T by the $N \times N$ matrix $(T_{i,j})$, such that $T_{i,i+1} = 1$, $T_{N,1} = \nu$, and all other elements are zeros, gives the standard (matrix) zero curvature representations.

4. Sketch of classification of Davey–Stewartson type systems

In this Section, we (following Refs. [2,1]) construct the associated systems for the two-dimensional chains of Section 2 which will be integrable 1 + 2 dimensional systems similar to the Davey–Stewartson coupled system. Systems associated with the 2D Toda and Volterra chains are derived, using corresponding L–A pairs. For the construction of the others, continuous analogs of discrete Miura type transformations of Section 2 are used. In all the cases, the above 2D chains generate explicit auto-Bäcklund transformations for corresponding associated systems.

As it follows from what has been said above, the 2D polynomial Toda chain (20) has the representation

$$\left[\partial_x + u_nT,\ \partial_y + v_n - T^{-1}\right] = 0 \tag{42}$$

(cf. (37)). A more or less standard scheme allows us to derive higher symmetries.

We can introduce formal series of the form

$$W_n^{(1)} = u_nT - V_n - \tilde{V}_nT^{-1} - \dots,$$

$$W_n^{(2)} = -T^{-1} + v_n - U_nT - \tilde{U}_nT^2 - \dots$$

in negative and positive powers of the shift operator T, respectively, satisfying the Lax relations

$$W_{nx}^{(i)} = \left[W_n^{(i)},\ u_nT\right], \quad W^{(i)}dn_y = \left[W_n^{(i)},\ v_n - T^{-1}\right]. \tag{43}$$

The coefficients $V_n, \tilde{V}_n, U_n, \tilde{U}_n, \dots$ of these series will be called the nonlocal variables. For example, the first of them are defined by the equations

$$V_{ny} = v_{nx}, \quad u_n(V_{n+1} - V_n) = u_{nx},$$

$$U_{nx} = u_{ny}, \quad U_n - U_{n-1} = v_{ny}, \tag{44}$$

which easily can be obtained by (43). The right hand sides of higher symmetries will be expressed in terms of u_n, v_n and the nonlocal variables.

Higher symmetries are constructed, using powers of the series $W_n^{(i)}$, and we need the notations

$$\left(W_n^{(i)}\right)^k = \dots + r_n^{(1,i,k)}T + r_n^{(0,i,k)} + r_n^{(-1,i,k)}T^{-1} + \dots.$$

There are the following simple formulae for the higher symmetries (cf. Ref. [7]),

$$u_{nx_k} = -r_{nx}^{(1,1,k)}, \quad v_{nx_k} = r_{n-1}^{(1,1,k)} - r_n^{(1,1,k)},$$

$$u_{ny_k} = u_n\left(r_{n+1}^{(0,2,k)} - r_n^{(0,2,k)}\right), \quad v_{ny_k} = r_{ny}^{(0,2,k)}. \tag{45}$$

In the case $k=2$,

$$r_n^{(1,1,2)} = -u_n(V_{n+1}+V_n),$$

$$r_n^{(0,2,2)} = v_n^2 + U_n + U_{n-1},$$

and we are led to two simplest higher symmetries of (20) which can be written as follows:

$$u_{nx_2} = u_{nxx} + 2(u_nV_n)_x,$$

$$v_{nx_2} = -v_{nxx} + (V_n^2)_y + 2u_{nx}, \tag{46}$$

$$u_{ny_2} = u_{nyy} + 2(u_nv_n)_y,$$

$$v_{ny_2} = -v_{nyy} + (v_n^2 + 2U_n)_y. \tag{47}$$

One can see that (46) and (47) are infinite sets of the same (1 + 2)-dimensional systems which generalise the well-known dispersive water wave system,

$$u_t = u_{xx} + 2(uV)_x,$$

$$v_t = -v_{xx} + (V^2)_y + 2u_x, \quad V_y = v_x, \tag{48}$$

$$u_\tau = u_{yy} + 2(uv)_y,$$

$$v_\tau = -v_{yy} + (v^2 + 2U)_y, \quad U_x = u_y. \tag{49}$$

The shift transformation $(u_n, v_n) \to (u_{n+1}, v_{n+1})$ in the 2D Toda chain (20) gives us the following explicit and invertible transformation,

$$v^* = v + (\log u)_y, \quad u^* = u + v_x^*. \tag{50}$$

If we use (44), we can expand this transformation to the nonlocal variables,

$$V^* = V + (\log u)_x, \quad U^* = U + v_y^*. \tag{51}$$

Formulae (50) and (51) define the explicit auto-Bäcklund transformation for both systems (48) and (49). It brings solutions of (48) and (49) into solutions and enables one to construct exact solutions.

In the case of the 2D Volterra chain (21), the operator representation is

$$[\partial_x + u_n(T-1), \partial_y + v_n(1-T^{-1})] = 0 \tag{52}$$

(cf. (41)). The chain is symmetrical: it is invariant under the involution $x \leftrightarrow -y$, $u_n \leftrightarrow v_{-n}$. For this reason, one needs formulae for the construction of only one set of higher symmetries; the second set can be obtained, using the involution. As in the previous case, we introduce

$$W_n^{(1)} = u_nT - (u_n + V_n) - \tilde{V}_nT^{-1} - \dots$$

satisfying Lax relations similar to (43). Higher symmetries can be constructed as follows,

$$u_{nx_k} = -g_{nx}^{(k)}, \quad v_{nx_k} = v_n(g_{n-1}^{(k)} - g_n^{(k)}),$$

$$g_n^{(k)} = \sum_{j=1}^{k} r_n^{(j,1,k)}. \tag{53}$$

If $k=2$, then $g_n^{(2)} = -u_n^2 - u_n(V_{n+1}+V_n)$, and one obtains a simplest higher symmetry. The corresponding (1 + 2)-dimensional system has the form

$$u_t = u_{xx} + (u^2 + 2uV)_x,$$

$$v_t = -v_{xx} + (V^2)_y + 2(uv)_x, \quad V_y = v_x, \tag{54}$$

and generalises an integrable system found in Ref. [10]. It is invariant under the transformation

$$v^* = v + (\log u)_y, \quad u^* = u + (\log v^*)_x,$$

$$V^* = V + (\log u)_x. \tag{55}$$

The second system associated to (21) can be constructed, using the involution $x \leftrightarrow -y$, $t \leftrightarrow -\tau$, $u \leftrightarrow v$, $U \leftrightarrow V$. We are led to one simpler (1 + 2)-dimensional system

$$u_\tau = u_{yy} + (U^2)_x + 2(vu)_y,$$

$$v_\tau = -v_{yy} + (v^2 + 2vU)_y, \quad U_x = u_y, \tag{56}$$

which coincides with (54) if $t=\tau$, $x=y$. In order to have an auto-transformation not only for (54) but also for (56), we should add to (55) the following formula,

$$U^* = U + (\log v^*)_y. \tag{57}$$

It should be remarked that, in the one-dimensional case, the auto-Bäcklund transformations (50), (51) and (55), (57) have also been obtained for the first time in Ref. [10] and, as in this paper, using the Toda and Volterra chains.

Now, let us pass to systems associated with the chains (17)–(19), (22). All of them can be derived, using Miura type transformations, and will be reduced to (48), (49), (54) or (56). A system corresponding to the exponential Toda chain (17) will not be written down because it is the well-known Davey–Stewartson coupled system (there is an example together with the explicit auto-Bäcklund transformation in Ref. [3]).

At first, let us discuss a system associated with the chain (18). It has been shown in Ref. [1] that the functions $r_n = q_n$, $s_n = -q_{n-1}$ satisfy in the one-dimensional case a Schrödinger-type system of the form (1). We take the same functions, but intend to obtain a (1 + 2)-dimensional system.

It is clear that the discrete transformation (15) which reduces (18) to the 2D polynomial Toda chain (20) can be rewritten in the form of a continuous one,

$$u_n = r_{nx}, \quad v_n = r_n + s_n. \tag{58}$$

For the nonlocal variables U_n, V_n (see (44)), one has $U_{nx} = r_{nxy}$, $V_{ny} = (r_n + s_n)_x$, and the following formulae can be added to (58),

$$U_n = r_{ny}, \quad V_n = R_n. \tag{59}$$

Here R_n is the nonlocal variable for a system we are going to write down: $R_{ny} = (r_n + s_n)_x$.

Using the transformation (58), (59) and the systems (46) and (47), we easily find the time evolution of r_n, s_n. In the second case (if we use (47)), we cannot define the time evolution by differential polynomials of r_n, s_n, R_n and should introduce an additional nonlocal variable S_n: $S_{nx} = (s_n r_{nx} - r_n s_{nx})_y$. As a result, the following two systems arise,

$$r_t = r_{xx} + 2Rr_x, \quad s_t = -s_{xx} + 2Rs_x,$$
$$R_y = (r+s)_x, \tag{60}$$

$$r_\tau = r_{yy} + (r^2 + rs)_y + S,$$
$$s_\tau = -s_{yy} + (s^2 + rs)_y - S, \quad S_x = (sr_x - rs_x)_y \tag{61}$$

(we do not write n and change $x_2 \to t$, $y_2 \to \tau$). According to (58) and (59), the new systems (60) and (61) are reduced by the transformation

$$u = r_x, \quad v = r + s,$$
$$U = r_y, \quad V = R \tag{62}$$

to (48) and (49), respectively. Both systems (60) and (61) are two-dimensional analogs of an integrable system found in Ref. [11].

In order to obtain the explicit auto-Bäcklund transformation which corresponds to the shift in the chain (18), one should express r_{n+1}, s_{n+1} in terms of r_n, s_n, using (18) and the formulae $r_n = q_n$, $s_n = -q_{n-1}$. That transformation can easily be expanded to the nonlocal variables R_n, S_n (one can rewrite the functions $(R_{n+1} - R_n)_y$ and $(S_{n+1} - S_n)_x$ in terms of r_n, s_n and then integrate them w.r.t. y and x). The auto-transformation is of the form

$$s^* = -r, \quad r^* + s^* = r + s + (\log r_x)_y,$$
$$R^* = R + (\log r_x)_x,$$
$$S^* = S + \left[2r_y - r(\log r_x)_y\right]_y.$$

It can be checked by direct calculation that (60) and (61) are invariant under this transformation.

In two other cases, new systems and transformations are obtained in a similar way. In the case of the chain (19), we use the discrete transformation (16), the systems (54), (56) associated with the 2D Volterra chain (21) and introduce new dynamical variables as follows: $r_n = \exp q_n$, $s_n = \exp(-q_{n-1})$. Resulting systems and transformations have the form

$$r_t = r_{xx} + 2Rr_x, \quad s_t = -s_{xx} + 2Rs_x, \quad R_y = (rs)_x, \tag{63}$$

$$r_\tau = r_{yy} + r[(rs)_y + S],$$
$$s_\tau = -s_{yy} + s[(rs)_y - S], \quad S_x = (sr_x - rs_x)_y, \tag{64}$$

$$u = (\log r)_x, \quad v = rs,$$
$$U = (\log r)_y, \quad V = R, \tag{65}$$

$$s^* = r^{-1}, \quad r^* s^* = rs + (\log(r_x/r))_y,$$
$$R^* = R + (\log(r_x/r))_x,$$
$$S^* = S + \left[2rs + (\log(r_x r))_y\right]_y. \tag{66}$$

Systems (63) and (64) are generalisations of the well-known nonlinear derivative Schrödinger coupled system [12]. The transformation (65) reduces them, correspondingly, to (54) and (56). The formulae (66) define the auto-Bäcklund transformation for each of them.

The last example is closely connected to the degenerations of the Landau–Lifshitz model. If we start with the chain (22), use the transformations (23), (25) reducing it to the 2D Volterra chain, and introduce new variables r_n, s_n as in the case of the chain (18), we are led to the following (1 + 2)-dimensional system of equations,

$$r_t = r_{xx} + 2(r_x^2 - a^2)f(r+s) + 4abr_x$$
$$+ 2(r_x + a)R,$$

$$-s_t = s_{xx} + 2(s_x^2 - a^2)f(r+s) - 4abs_x - 2(s_x - a)R,$$
$$R_y = (f^2 - b^2)\big[(r_y - a)(s_x + a) - (r_x - a)(s_y + a)\big],$$
$$f' = f^2 - b^2, \tag{67}$$

where a, b are arbitrary constants. Another system associated with (22) can be obtained by the involution: $x \leftrightarrow -y$, $t \leftrightarrow -\tau$, $r \leftrightarrow -s$, $R \leftrightarrow S$. The Miura type transformation

$$u = [b + f(r+s)](a + r_x),$$
$$v = [b - f(r+s)](a + s_y),$$
$$U = S + [b + f(r+s)](a + r_y),$$
$$V = R + [b - f(r+s)](a + s_x)$$

reduces the associated systems to (54) and (56), respectively.

Note that in one-dimensional case ($x = y$, $R = 0$) these systems will be the degenerations of the Landau–Lifshitz model rewritten by the stereographic projection. The case $a = b = 0$, $f(z) = -z^{-1}$ corresponds to the isotrophic Heisenberg model, the case $a = b = 1$, $f(z) = -\tanh z$ corresponds to the anisotrophic one. If $a = 1$, $b = 0$, $f(z) = -z^{-1}$, then one has one more known integrable system (see e.g. Refs. [4,5]).

5. Conclusions

In the previous section, we have constructed for each of the chains (18)–(22) exactly two associated systems which exemplify integrable (1 + 2)-dimensional systems of equations. It is clear that any linear combination of such a pair of associated systems will be integrable too. This linear combination admits the same explicit auto-Bäcklund transformation we have written down for the corresponding two systems. In the case when there is a transformation which reduces both the associated systems to (48), (49), (54) or (56), the linear combination of these systems will be reduced by the same transformation to the same linear combination of the systems (48) and (49) or (54) and (56).

It is known that the Miura type transformations are not local in the case of the Kadomtsev–Petviashvili type equations (see e.g. Refs. [13,14]). The transformation

$$\tilde{u}_n = u_n v_{n+1}, \quad \tilde{v}_n = \partial_x^{-1}\partial_y u_n + v_n,$$

which reduces the 2D Volterra chain (21) to the 2D Toda chain (20) is of the same kind. It is interesting that, unlike those, all the 2D discrete transformations of Section 2 are local. The continuous Miura type transformations of Section 4 as well as the auto-Bäcklund transformations not only are local as themselves but also admit the local prolongation to additional nonlocal variables.

In the one-dimensional case, there is a complete list of integrable chains of the form (2) obtained in Ref. [6]. It consists, up to point transformations, of the chains (7), the chain

$$q_{nxx} = \exp(q_{n+1} - 2q_n + q_{n-1}), \tag{68}$$

and one more exceptional lattice equation which generates the explicit auto-Bäcklund transformation for the Landau–Lifshitz model and has a form different from (4). If we restrict ourselves to chains of the form (4), the list of equations (7) and (68) becomes complete. We have succeeded to build up two-dimensional generalisations for all the integrable chains (4); a 2D analog of (68) can easily be added to the chains of Section 2 and will have the form

$$q_{nxy} = \exp(q_{n+1} - 2q_n + q_{n-1})$$

(it is reduced to (17) by the obvious transformation $\tilde{q}_n = q_{n+1} - q_n$).

Schrödinger type systems (1) associated with the integrable chains (4) cover the key integrable cases except for the Landau–Lifshitz model [1]. Thus, one can hope that the (1 + 2)-dimensional systems constructed in Section 4 exhaust, in a sense, a list of Davey–Stewartson type systems. Of cause, the problem of the generalisation of the Landau–Lifshitz model remains open.

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