LATTICE REPRESENTATIONS OF INTEGRABLE SYSTEMS

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Received 19 January 1988; accepted for publication 25 February 1988
Communicated by A.P. Fordy

We indicate the general connection between one-dimensional lattices with local symmetries and nonlinear integrable partial differential equations in 1+1 dimensions. The nonlinear chain provides a set of finite-dimensional integrable models of the corresponding PDE. The integrals of these finite-dimensional models are related in a direct way with the conserved quantities of the PDE.

1. We consider partial differential equations that can be represented as a compatibility condition of two chains,

$$\partial_x \boldsymbol{q}_n = \phi(\boldsymbol{q}_{n+1}, \boldsymbol{q}_n, \boldsymbol{q}_{n-1}) , \tag{1}$$

$$\partial_t \boldsymbol{q}_n = \psi(\boldsymbol{q}_{n+2}, \boldsymbol{q}_{n+1}, \boldsymbol{q}_n, \boldsymbol{q}_{n-1}, \boldsymbol{q}_{n-2}) . \tag{2}$$

Here $\boldsymbol{q}_n = \boldsymbol{q}_n(x, t)$ is a vector function and $n=0, \pm 1, \pm 2, \dots$. The partial differential equations related to (1), (2) can be written (see section 2) as a coupled $(\boldsymbol{u}, \boldsymbol{v})$ system,

$$\boldsymbol{u}_t = A(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{u}_{xx} + \boldsymbol{f}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}_x, \boldsymbol{v}_x) ,$$

$$\boldsymbol{v}_t = B(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{v}_{xx} + \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}_x, \boldsymbol{v}_x) , \tag{3}$$

for the vector functions

$$\boldsymbol{u} = \boldsymbol{q}_n(x, t), \quad \boldsymbol{v} = \boldsymbol{q}_{n-1}(x, t) . \tag{4}$$

In fact the right-hand sides of eqs. (3) coincide with $\psi_n = \psi(\boldsymbol{q}_{n+2}, \boldsymbol{q}_{n+1}, \dots, \boldsymbol{q}_{n-2})$ and $\psi_{n-1} = \psi(\boldsymbol{q}_{n+1}, \boldsymbol{q}_n, \dots, \boldsymbol{q}_{n-3})$, respectively. The dynamic variables $\boldsymbol{q}_{n+2}, \boldsymbol{q}_{n+1}, \boldsymbol{q}_{n-2}, \boldsymbol{q}_{n-3}$ can be expressed in terms of x-derivatives of the variables (4) by means of (1) for non-degenerated chains.

The correspondence between (3) and (1), (2) can be used in a study of the finite gap type solutions of (3). For this we should consider the periodic chains with

$$\boldsymbol{q}_{n+N} = \boldsymbol{q}_n \quad \forall n \in \mathbb{Z} . \tag{5}$$

It is clear, that in the case (5) any solution $\boldsymbol{q}_n(x, t)$, $n=1, \dots, N$, of (1), (2) defines by virtue of (4) $N-1$ interrelated solutions of the system (3). In the lattice representation approach to the integrability of partial differential equations the chains (1), (2) play the role of the L–A pair. We shall show that there exists a direct correspondence between the conservation laws as well as symmetries, of the coupled system (3) and the related chains (1), (2).

The lattice representation approach proves to be useful in the classification problem. Comparing the lists of basic scalar chains (1) and coupled equations [1]

$$u_t = u_{xx} + f(u, v, u_x, v_x) ,$$

$$-v_t = v_{xx} + g(u, v, u_x, v_x) \tag{6}$$

we obtain in section 3 the lattice representation for the two systems (6), whose integrability was an open question.

2. We now discuss the derivation of the two vector equations (3) from the compatibility condition of the chains (1) and (2). We use the common notation $h_n = h(\boldsymbol{q}_{n+m}, \boldsymbol{q}_{n+m-1}, \dots, \boldsymbol{q}_{n-p})$ for functions invariant under the shift $n \to n+1$. The chain (1) is called *non-degenerated* if the Jacobi matrices $\partial\phi_n/\partial\boldsymbol{q}_{n+1}$, $\partial\phi_n/\partial\boldsymbol{q}_{n-1}$ have non-zero determinants.

Theorem. Let the chains (1), (2) satisfy the compatibility condition

$$\frac{\partial\phi_n}{\partial q_{n+1}}\psi_{n+1}+\frac{\partial\phi_n}{\partial q_n}\psi_n+\frac{\partial\phi_n}{\partial q_{n-1}}\psi_{n-1}$$
$$=\sum_k\frac{\partial\psi_n}{\partial q_{n+k}}\phi_{n+k} \quad (7)$$

and the chain (1) be non-degenerated. Then there exist matrices $A_n=A(q_n, q_{n-1})$, $B_n=B(q_{n+1}, q_n)$ such that

$$\frac{\partial\psi_n}{\partial q_{n+2}}=A_n\frac{\partial\phi_n}{\partial q_{n+1}}\frac{\partial\phi_{n+1}}{\partial q_{n+2}},$$
$$\frac{\partial\psi_n}{\partial q_{n-2}}=B_n\frac{\partial\phi_n}{\partial q_{n-1}}\frac{\partial\phi_{n-1}}{\partial q_{n-2}}, \quad (8)$$

and, for any solution $q_n(x, t)$, $n=0, \pm1, ...,$ of the infinite dimensional dynamical system (1), (2), the functions (4) satisfy the partial differential equations (3). Moreover in the scalar case $A_n=\alpha$, $B_n=\beta$, where $\alpha, \beta\in\mathbb{C}$ are constants.

Proof. We denote

$$\phi'_n=\frac{\partial\phi_n}{\partial q_{n+1}},\quad \psi'_n=\frac{\partial\psi_n}{\partial q_{n+2}}.$$

It follows from (7) that

$$\phi'_n\psi'_{n+1}=\psi'_n\phi'_{n+2}. \quad (9)$$

The substitution $\psi'_n=A_n\phi'_n\phi'_{n+1}$ in (9) gives us

$$A_{n+1}=(\phi'_n)^{-1}A_n\phi'_n\Rightarrow A_n=A(q_n, q_{n-1}).$$

In the scalar case we have $A_{n+1}=A_n$, which implies $A_n=\alpha=$const. It is clear that the second formula (8) is also valid, and that $B_n=\beta=$const in the scalar case.

Now we compare the formula

$$q_{nxx}=\frac{\partial\phi_n}{\partial q_{n+1}}\phi_{n+1}+\frac{\partial\phi_n}{\partial q_n}\phi_n+\frac{\partial\phi_n}{\partial q_{n=1}}\phi_{n-1}$$

with (8) and obtain that

$$q_{nt}-A(q_n, q_{n-1})q_{nxx}=a(q_{n+1}, q_n, q_{n-1}, q_{n-2}),$$
$$q_{n-1,t}-B(q_n, q_{n-1})q_{n-1,xx}$$
$$=b(q_{n+1}, q_n, q_{n-1}, q_{n-2}). \quad (10)$$

For the non-degenerate chain (1) we have

$$q_{n+1}=P(q_n, q_{n-1}, q_{nx}),$$
$$q_{n-2}=Q(q_n, q_{n-1}, q_{n-1,x}). \quad (11)$$

We substitute these expressions into (10) and obtain the closed pair of partial differential equations (3) for q_n, q_{n-1}.

It must be noticed that the compatibility condition (7), i.e. the equality $\partial_t\partial_x q_n=\partial_x\partial_t q_n$, implies in the periodic case (5) the solvability of the finite dimensional dynamical systems (1), (2) for q_n, $n=1, ..., N$, with any prescribed initial values $q_n=q_n^0$ at $x=x^0$, $t=t^0$. The solution $q_n=q_n(x, t)$, $n=1, ..., N$, of the dynamical systems generates, by the theorem above, $N-1$ solutions of the partial differential equations (3).

We next note that formulas (11) and similar formulas with high order derivatives $\partial_x^k q_n$ allow us to interchange q_n, q_{n-1}, q_{n+1}, q_{n-2}, q_{n+2}, q_{n-3}, ... with u, v, u_x, v_x, u_{xx}, v_{xx}, ... Hence any function of the dynamical variables q_n can also be represented as a local function of the variables u, v and their derivatives. We can generalize the theorem and replace (2) by the chain

$$\partial_{t_m}q_n=\psi^{(m)}(q_{n+m}, q_{n+m-1}, ..., q_{n-m}). \quad (12)$$

The formulas for $\partial_{t_m}q_n$, $\partial_{t_m}q_{n-1}$ can be rewritten as the closed (u, v) system

$$\partial_{t_m}u=U(u, v, ..., \partial_x^m u, \partial_x^m v),$$
$$\partial_{t_m}v=V(u, v, ..., \partial_x^m u, \partial_x^m v). \quad (13)$$

The block diagonal structure of the leading terms in (13) can be derived, as above in the proof of the theorem, from the compatibility condition $\partial_x\partial_{t_m}=\partial_{t_m}\partial_x$. If $\partial_t\partial_{t_m}=\partial_{t_m}\partial_t$, then formulas (12), (13) establish a direct correspondence between the higher symmetries of the system (3) [1] with the higher symmetries of the basic chain (1).

We shall now consider the connection between the conservation laws of (3) and the local conservation laws of the chains (1), (2). Let

$$h_n=h(q_{n+m}, q_{n+m-1}, ..., q_{n-m}) \quad (14)$$

be a conserved density for both chains, i.e.

$$\partial_x h_n=\rho_{n+1}-\rho_n, \quad (15)$$
$$\partial_t h_n=\sigma_{n+1}-\sigma_n. \quad (16)$$

We recall that in the periodic case (5) it follows from (15), (16) that

$$\partial_x \sum_1^N h_n = \partial_t \sum_1^N h_n = 0 .$$

The local conservation laws (15), (16) give us

$$\partial_t \rho_{n+1} - \partial_x \sigma_{n+1} = \partial_t \rho_n - \partial_x \sigma_n .$$

Consequently

$$\partial_t \rho_n - \partial_x \sigma_n = c = \text{const} . \qquad (17)$$

If $c=0$ we obtain from (17) the local conservation law

$$\partial_t \rho(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}_x, \boldsymbol{v}_x, ...) = \partial_x \sigma(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}_x, \boldsymbol{v}_x, ...)$$

for the system (3). We shall prove that $c=0$ in (17) for the chains (1), (2) which admit the reflection

$$\boldsymbol{q}_n \to \boldsymbol{q}_{-n}, \quad x \to -x, \quad t \to -t . \qquad (18)$$

Lemma 1. Let the conditions of the theorem be fulfilled and (1), (2) be invariant under the reflection (18). Then the system (3) admits the involution

$$\boldsymbol{u} \leftrightarrow \boldsymbol{v}, \quad x \to -x, \quad t \to -t , \qquad (19)$$

and for any pair of conservation laws (15), (16) with the invariant conserved density (14), i.e.

$$h(\boldsymbol{q}_m, \boldsymbol{q}_{m-1}, ..., \boldsymbol{q}_{-m}) = h(\boldsymbol{q}_{-m}, \boldsymbol{q}_{-m+1}, ..., \boldsymbol{q}_m) ,$$

we have $c=0$ in (17).

Proof. The invariance of the system (3) under involution (19) follows immediately from the definitions. From the invariance property of the functions

$$\partial_x \sum_{-n}^{n} h_k = \rho_{n+1} - \rho_n , \quad \partial_t \sum_{-n}^{n} h_k = \sigma_{n+1} - \sigma_n$$

we obtain that the reflection (18) transforms ρ_{n+1} (σ_{n+1}) into ρ_{-n} (σ_{-n}). Applying the reflection (18) to (15) we find $c=-c$, i.e. $c=0$.

We illustrate the above discussion by the conservation laws of the Volterra model

$$q_{nx} = q_n(q_{n+1} - q_{n-1}) . \qquad (20)$$

One can verify that (15) holds with $h_n = q_n$, $\rho_n = q_{n-1} q_n$ and $h_n = q_n(q_{n+1} + q_n + q_{n-1})$, $\rho_n = q_{n-1} q_n (q_{n+1} + ... + q_{n-2})$. The chain (20) admits the reflection (18) and lemma 1 implies that $\rho = uv$ and $\rho = vu_x - uv_x + 2uv(u+v)$ are the conserved densities for the system (3) related with (20). The explicit form of the system (3) for the Volterra model (20) will be given in the next section.

In addition to the formulas (8) we have

Lemma 2. Let the scalar chains (1), (2) with nonzero $\partial\phi_n/\partial q_{n+1}$, $\partial\psi_n/\partial q_{n+2}$ satisfy the compatibility relation (7). Then there exist $\alpha \in \mathbb{C}$ and a function $\rho_n = \rho(q_{n+1}, q_n, q_{n-1}, q_{n-2})$ such that the following formulas hold,

$$\partial_x \ln \frac{\partial \phi_n}{\partial q_{n+1}} = \rho_{n+1} - \rho_n , \qquad (21)$$

$$\frac{\partial \psi_n}{\partial q_{n+2}} = \alpha \frac{\partial \phi_n}{\partial q_{n+1}} \frac{\partial \phi_{n+1}}{\partial q_{n+2}} , \qquad (22)$$

$$\frac{\partial \psi_n}{\partial q_{n+1}} = \alpha \frac{\partial \phi_n}{\partial q_{n+1}} \left(\rho_{n+1} + \rho_n + \frac{\partial \phi_{n+1}}{\partial q_{n+1}} + \frac{\partial \phi_n}{\partial q_n} \right) . \qquad (23)$$

Proof. Formula (22) was obtained in the proof of the theorem. Differentiating eq. (7) with respect to q_{n+2} we find that

$$\alpha \left(\partial_x (\ln \phi'_n \phi'_{n+1}) + \frac{\partial \phi_{n+2}}{\partial q_{n+2}} - \frac{\partial \phi_n}{\partial q_n} \right)$$

$$= (\phi'_{n+1})^{-1} \frac{\partial \psi_{n+1}}{\partial q_{n+2}} - (\phi'_n)^{-1} \frac{\partial \psi_n}{\partial q_{n+1}} ,$$

where $\phi'_n \equiv \partial\phi_n/\partial q_{n+1}$. This relation is equivalent to formulas (21), (22).

Formulas (21)–(23) suggest the algorithm which defines the chain (2) in terms of the basic chain (1). For example, in the case

$$q_{nx} = q_n(q_{n+1} - q_n) \qquad (24)$$

the above formulas give us

$$q_{nt} = q_n q_{n+1}(q_{n+2} - q_n) = q_{nxx} + 2q_n q_{nx} .$$

This is the well-known Burgers equation. The substitution $q_n = \tilde{q}_{n+1}/\tilde{q}_n$ connects (24) with the linear chain $\tilde{q}_{nx} = \tilde{q}_{n+1}$.

3. The compatibility condition (7) puts strong restrictions on the choice of the basic chains (1). For example, in the scalar case the conservation law (21) must be valid for (1) [1]. We do not discuss here the classification of all the scalar chains (1) for which relation (7) is solvable. It is an open problem as yet. Our consideration is restricted to chains with high order conservation laws and we go on to the following list of the basic chains [1]:

$$q_{nx}=P(q_n)(q_{n+1}-q_{n-1}),\quad P'''=0\,, \tag{25}$$

$$q_{nx}=Q(q_n)[(q_{n+1}-q_n)^{-1}+(q_n-q_{n-1})^{-1}],\quad Q^{(4)}=0\,, \tag{26}$$

$$q_{nx}=(q_{n+1}-q_{n-1})^{-1}(R_n+\epsilon\sqrt{r_n r_{n+1}}),\quad \epsilon=0,1\,. \tag{27}$$

In the last formula

$$R(u,v,w)=(\alpha v^2+2\beta v+\gamma)uw+(\beta v^2+\mu v+\delta)(u+w)+\gamma v^2+2\delta v+\nu$$

and

$$R_n=R(q_{n+1},q_n,q_{n-1})\,,\quad r_n=r(q_{n-1},q_n)=R(q_{n-1},q_n,q_{n-1})\,. \tag{28}$$

We can find the chains (2) related to (25)–(27) by direct calculations, which are based on (21)–(23) and on the invariance property under the reflection (18). In the most intricate case (27), (28) we obtain, for example, that for $\epsilon=0$

$$q_{nt}=\frac{r_{n+1}r_n}{(q_{n+1}-q_{n-1})^2}\left(\frac{1}{q_n-q_{n+2}}+\frac{1}{q_{n-2}-q_n}\right), \tag{29}$$

and for $\epsilon=1$

$$q_{nt}=\phi_{n+1}\frac{\partial\phi_n}{\partial q_{n+1}}-\phi_{n-1}\frac{\partial\phi_n}{\partial q_{n-1}}+\tfrac{1}{2}\phi_n\frac{\sqrt{r_{n+1}r_n}}{q_{n+1}-q_{n-1}}\frac{\partial}{\partial q_n}\ln\frac{r_{n+1}}{r_n}\,. \tag{30}$$

The calculations simplify if we make use of the hamiltonian structure. In the case (25) the high order chains (12) can be obtained via the formula

$$\partial_{t_m}q_n=P(q_n)\left(P(q_{n+1})\frac{\partial}{\partial q_{n+1}}-P(q_{n-1})\frac{\partial}{\partial q_{n-1}}\right)H^{(m)}\,. \tag{31}$$

Here the "hamiltonian"

$$H^{(m)}=\sum_{n=-N}^{N}h_n^{(m)},\quad N\gg 1\,, \tag{32}$$

is constructed from the conserved densities $h_n^{(m)}$. The chain (2) corresponds to $h_n=q_{n+1}q_n+R(q_n)$ and

$$h_n=q_{n+1}q_n\quad (P=\alpha q^2+\gamma)\,,\quad h_n=q_{n+1}q_n+\tfrac{1}{2}q_n^2\quad (P=q)\,. \tag{33}$$

We now want to write down the list of systems (3) corresponding to (25)–(27). It follows from section 2 (see the theorem and lemma 1) that all the systems (3) have the form (6) and admit the involution (19). We shall write down only the first equation from the pair (6),

$$u_t=u_{xx}+f(u,v,u_x,v_x)\,. \tag{34}$$

The second one can be reconstructed by virtue of (19). We recall that actually (34) and (2) coincide after interchanging q_n, q_{n-1} with u, v and so on. Eqs. (34), which correspond to (25)–(27), can be written down in the following form,

$$u_t=u_{xx}+[2P(u)v+\beta u^2]_x \tag{35}$$

$$u_t=u_{xx}+2u_x^2(u-v)^{-1}+2[r(u,v)u_x-Q(u)v_x](v-u)^{-2}\,, \tag{36}$$

$$u_t=u_{xx}-(u_x^2+A)[2v_x/r+(\ln r)_u]+\tfrac{1}{2}A'\,. \tag{37}$$

$$u_t=u_{xx}-(u_x^2+A)v_x/r-(\ln r)_u u_x^2+\frac{r_u r_v-r r_{uv}}{r}u_x\,. \tag{38}$$

Here (35) corresponds to (25) with the same polynomial $P(q)=\alpha q^2+\beta q+\gamma$ (see (31)–(33)). In the next equation (36), corresponding to (26) with the same Q, the polynomial $r(u,v)$ is constrained by the conditions

$$r(u,v)=r(v,u),\quad r_{uu}=r_{vv}=\tfrac{4}{3}\,.$$

The last two equations with $A=A(u)=\frac{1}{2}rr_{vv}-\frac{1}{4}r_v^2$ are related to (27) ((37) ⇔ (29), (38) ⇔ (30)). The polynomial $r(u, v)$ is defined by (28).

The general explicit integrability conditions for systems (6) were discussed in ref. [1] (naturally these conditions hold for (35)–(38)). However, the practical integration method is still beyond the scope of the theory presented in ref. [1] (see ref. [2]). The required complement is provided by the latice representation approach, and by lucky coincidence the last two systems (6), the integrability of which was an open question, coincide with (37), (38).

The list of basic chains (25)–(27) allow us to connect any scalar chain (1) possessing the higher local conservation law (15) with one of the basic chains by some substitutions [1,3]. The systems (6) related to these chains are connected with (35)–(38) by the transformations described in refs. [1,2]. The lattice representation suggests additional links between integrable systems. For example the well-known systems

$$\tilde{u}_t = \tilde{u}_{xx} - 2\tilde{u}_x^2(\tilde{u}-\tilde{v})^{-1},$$
$$-\tilde{v}_t = \tilde{v}_{xx} + 2\tilde{v}_x^2(\tilde{u}-\tilde{v})^{-1}, \quad (39)$$

$$u_t = u_{xx} - 2u_x^2 v_x, \quad -v_t = v_{xx} + 2v_x^2 u_x \quad (40)$$

are connected with one and the same chain

$$q_{nx} = (q_{n+1} - q_{n-1})^{-1}. \quad (41)$$

The system (40) is a special case of (37) with $R=r=1$ and is related to (27), (29) in the usual way, $u=q_n$, $v=q_{n-1}$. For (39) $\tilde{u}=q_n$, $\tilde{v}=q_{n-2}$. This second choice of dynamical variables is admitted for any chain $q_{nx}=\phi(q_{n+1}-q_{n-1})$.

A hamiltonian chain derived by Ablowitz and Ladik [4] provides an example of a non-degenerate integrable vector chain. We have

$$q_{nx} = r_n(q_{n+1}+q_{n-1}), \quad -p_{nx} = r_n(p_{n+1}+p_{n-1}),$$
$$q_{nt} = r_n \partial H/\partial p_n, \quad -p_{nt} = r_n \partial H/\partial q_n, \quad (42)$$

where

$$r_n = 1 + q_n p_n, \quad H = \sum_{-N}^{N} h_n, \quad N \gg 1$$

and

$$h_n = \alpha(r_{n+1} q_{n+2} p_n + \tfrac{1}{2} q_{n+1}^2 p_n^2)$$
$$+\gamma(r_{n+1} q_n p_{n+2} + \tfrac{1}{2} q_n^2 p_{n+1}^2). \quad (43)$$

The change (4) of the dynamical variables

$$q_n = q, \quad q_{n-1} = q', \quad p_n = p, \quad p_{n-1} = p'$$

leads to the system of equations (3) which is invariant under the two involutions

$$q \leftrightarrow p', \quad p \leftrightarrow q', \quad x \to -x, \quad t \to -t,$$
$$q \leftrightarrow p, \quad q' \leftrightarrow p', \quad x \to -x, \quad t \to -t, \quad \alpha \leftrightarrow \gamma. \quad (44)$$

The first one of eqs. (3) has the form

$$q_t = \alpha q_{xx} + 2\alpha \rho q_x + (\gamma - \alpha) r q'_x - (\alpha+\gamma) q q' p_x$$
$$-(\alpha+\gamma) r(\rho q' + r' q),$$
$$r = 1 + qp, \quad r' = 1 + q'p', \quad \rho = qp' - pq'. \quad (45)$$

One can reconstruct the other ones by the substitutions (44).

The general simple idea how to connect the partial differential equations and the chains (1), (2) is not restricted to non-degenerate chains (1) [5]. A good example here is provided by the Toda lattice representation of the nonlinear Schrödinger coupled equations

$$u_t = u_{xx} + 2u^2 v, \quad -v_t = v_{xx} + 2v^2 u. \quad (46)$$

In this case we have

$$u = \exp(q_n), \quad v = \exp(-q_{n-1}),$$
$$q_{nx} = p_n, \quad p_{nx} = \exp(q_{n+1} - q_n) - \exp(q_n - q_{n-1}),$$
$$q_{nt} = \partial H/\partial p_n, \quad p_{nt} = -\partial H/\partial q_n, \quad H = \sum h_n,$$
$$h_n = \tfrac{1}{3} p_n^3 + (p_{n+1} + p_n) \exp(q_{n+1} - q_n). \quad (47)$$

The lattice (1), (2) can be considered as a special kind of Bäcklund transformation for the system (3). For example the above formulas (47) define the well-known Bäcklund transformation for the nonlinear Schrödinger equation (46) [6].

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