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SYMMETRIES OF NONLINEAR CHAINS

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ABSTRACT. A direct relationship between nonlinear chains of the Toda type and partial differential equations possessing higher-order symmetries is discussed. For a given partial differential equation, a chain is determined (up to a change in notation) by an invertible transformation

$$u(x, t) \to v(x, t) = V(u(x, t), u_x(x, t), u_{xx}(x, t), \dots,),$$

which takes solutions of the equation again into solutions. The invertibility of this transformation plays an important role in the general theory developed in the paper, and we term the corresponding chains regular. A table at the end of the article lists the key equations generalizing the Schrödinger equation with a cubic nonlinearity, together with the invertible differential substitutions, expressed in the form of nonlinear chains, which these equations admit.

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1. Introduction

1.1. Examples of chains from the inverse scattering method. The following general scheme can be used to construct finite-dimensional dynamical systems describing the behavior of finite-band potentials. Consider a linear spectral problem expressed as a first-order vector equation

$$\Phi_x = U(x, \lambda)\Phi. \tag{1.1}$$

The explicit formulas of the dressing method (see, e.g., [1]) lead to a chain of potentials $U_n = U_n(x, \lambda)$, $n \in \mathbb{Z}$, related by the formulas

$$\Phi_{n,x} = U_n\Phi_n, \qquad \Phi_{n+1} = W_n(\lambda)\Phi_n, \tag{2.1}$$

$$W_{n,x} = U_{n+1}W_n(\lambda) - W_n(\lambda)U_n, \tag{1.3}$$

where the matrix functions $W_n(\lambda) \equiv W_n(x, \lambda)$ are polynomials in the spectral parameter. As shown in the examples below (see also [2]–[4]), the latter condition enables us to eliminate the potentials U_n, U_{n+1} from (1.3) and obtain an infinite nonlinear chain for the coefficients of the polynomial $W_n(\lambda)$ in powers of λ. This infinite chain of equations becomes a *finite dynamical system* if we impose the periodicity condition

$$U_{n+N} = U_n \tag{1.4}$$

and write

$$\overset{(N)}{W} = W_{n+N-1}, \dots, W_{n+1}W_n. \tag{1.5}$$

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We then find from (1.3), (1.4) that

$$d\overset{(N)}{W}/dx = U_{n+N}\overset{(N)}{W} - \overset{(N)}{W}U_n = [U_n, \overset{(N)}{W}]. \tag{1.6}$$

An obvious consequence of (1.6) is the formula

$$\frac{d}{dx}\operatorname{trace}[\overset{(N)}{W}(x, \lambda)] = 0, \tag{1.7}$$

which gives a first integral, polynomial in the spectral parameter, of the finite-dimensional dynamical system (1.3), (1.4).

As a first example we consider how to apply the above general scheme to the linear Schrödinger equation

$$\psi_{xx} + (u(x) + \lambda)\psi = 0. \tag{1.8}$$

In this case in (1.1) we have

$$U(x, \lambda) = \begin{pmatrix} 0 & 1 \\ -u(x) - \lambda & 0 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix},$$

and a straightforward analysis shows that equations (1.3) have the very simple polynomial solution

$$W_n(\lambda) = \begin{pmatrix} q_n & 1 \\ q_n^2 - \lambda + \alpha_n & g_n \end{pmatrix} \qquad (\det W_n = \lambda - \alpha_n, \quad \alpha_n \in \mathbb{C}) \tag{1.9}$$

and that the corresponding chain of equations has the following form:

$$\frac{d}{dx}(q_{n+1} + q_n) = q_{n+1}^2 - q_n^2 + \alpha_{n+1} - \alpha_n, \qquad n \in \mathbb{Z}. \tag{1.10}$$

The formula

$$u_n = q_{nx} - q_n^2 - \alpha_n \tag{1.11}$$

for recovering the potential in the Schrödinger equation (1.8) can be used to relate (1.10) to the chain of classical transformations of the discrete spectrum of the Schrödinger equation which are used to construct multi-soliton potentials. Of greater interest, however, is the fact that the periodic closure condition

$$q_{n+N} = q_n, \qquad \alpha_{n+N} = \alpha_n \tag{1.12}$$

for N odd enables one to pass from the under-determined system of equations (1.10) to a finite-dimensional dynamical system

$$\begin{gathered} \frac{d}{dx}(T + E)\mathbf{q} = (T - E)\mathbf{q}^2, \\ (T + E)^{-1} = \frac{1}{2}\sum_{k=0}^{N-1}(-T)^k. \end{gathered} \tag{1.13}$$

Here the matrix T corresponds to the cyclic permutation $q_n \to q_{n+1}$, and we use the following notation for column matrices:

$$\mathbf{q} = (q_n, q_{n+1}, \dots, q_{n+N-1})^{\mathsf{T}}, \quad \mathbf{q}^2 = (q_n^2 + \alpha_n, \dots, q_{n+N-1}^2 + \alpha_{n+N-1})^{\mathsf{T}}.$$

We will show below (see Proposition 1.1) that the problem of constructing finite-band potentials for (1.8) reduces to (1.13).

The application of the above general scheme to the canonical spectral problem with

$$U(x, \lambda) = \begin{pmatrix} \lambda & v(x) \\ -u(x) & \lambda \end{pmatrix} \tag{1.14}$$

leads, in the simplest case, to two essentially different chains: to

$$\begin{aligned} -\delta_n q_{nx} &= \delta_n^2 q_{n+1} - \alpha_n q_n + q_n^2 p_n, \\ \delta_n p_{nx} &= \delta_n^2 p_{n-1} - \alpha_n p_n + p_n^2 q_n \end{aligned} \tag{1.15}$$

$(\alpha_n, \delta_n \in \mathbb{C})$, and to the Toda chain [5]

$$q_{nxx} = \exp(q_{n+1} - q_n + \gamma_n) - \exp(q_n - q_{n-1} + \gamma_{n-1}) \tag{1.16}$$

$(\gamma_n \in \mathbb{C})$. In case (1.15), the solution of (1.3) is chosen to be

$$W_n = \begin{pmatrix} \delta_n & -p_n \\ q_n & 2\lambda + \delta_n^{-1}(\alpha_n - p_n q_n) \end{pmatrix} \tag{1.17}$$

$(u_n = q_n,\ v_n = p_{n-1})$. The case of the toda chain (1.16) corresponds to

$$W_n = \begin{pmatrix} 0 & -\exp(-q_n + \gamma_n) \\ \exp q_n & 2\lambda + q_{nx} \end{pmatrix} \tag{1.18}$$

$(u_n = \exp q_n,\ v_n = \exp(\gamma_{n-1} - q_{n-1}))$.

In both of these examples (1.8) and (1.8) and (1.14) the trace of $U(x, \lambda)$ vanishes, so that $d(\det W_n)/dx = 0$ (see (1.3)). Hence not only the trace (1.7) but also the characteristic equation

$$\det(zE - \overset{(N)}{W}(x, \lambda)) = z^2 - z \operatorname{tr} \overset{(N)}{W} + \det \overset{(N)}{W} = 0 \tag{1.19}$$

remains unchanged.

Proposition 1.1. *Let N be odd and let $\mathbf{q}$ be a solution of the dynamical system* (1.13). *Then for every n, the Nth order differential operator*

$$Z_n = \left(\frac{d}{dx} + q_{n+N-1}\right) \cdots \left(\frac{d}{dx} + q_{n+1}\right)\left(\frac{d}{dx} + q_n\right)$$

commutes with $L_n = d^2/dx^2 + u_n$.

Proof. It follows from (1.10) and (1.11) that

$$L_n = (d/dx - q_n)(d/dx + q_n) - \alpha_n$$

and that the differential operators with different subscripts are obtained from one another by conjugation, or more precisely,

$$\begin{aligned} &L_{n+1}(d/dx + q_n) = (d/dx + q_n)(L_n + \alpha_n - \alpha_{n+1}), \\ &L_{n+2}(d/dx + q_{n+1})(d/dx + q_n) \\ &\qquad = (d/dx + q_{n+1})(L_{n+1} + \alpha_{n+1} - \alpha_{n+2})(d/dx + q_n) \\ &\qquad = (d/dx + q_{n+1})(d/dx + q_n)(L_n + \alpha_n - \alpha_{n+2}), \ \dots . \end{aligned}$$

Thus,

$$L_{n+N} Z_n = Z_n(L_n + \alpha_n - \alpha_{n+N}).$$

The vanishing of the commutator now follows from (1.12). ●

The ideas behind the above method for constructing a pair of commuting differential operators by reducing the problem to the system of differential equations (1.13) are similar to the ones in the classical paper [6].

REMARK 1.2. As a complement to Proposition 1.1, one can check that the commuting differential operators are related by an alagebraic equation of the form (1.19), with the replacements $z \to Z_n$, $\lambda \to -L_n$. If the initial data $\mathbf{q}(x)$ at $x = x_0$ for the dynamical system (1.13) are subject to the additional conditions $\operatorname{tr} \overset{(N)}{W}(\lambda) \equiv 0$, where for $N = 2m+1$

$$\operatorname{tr} \overset{(N)}{W}(\lambda) = (-\lambda)^m Q_1(\mathbf{q}) + (-\lambda)^{m-1} Q_2(\mathbf{q}) + \cdots + Q_{m+1}(\mathbf{q}), \tag{1.20}$$

then by (1.7) and (1.9), this algebraic equation (1.19) takes the form of a hyperelliptic curve

$$Z_n^2 - \det \overset{(N)}{W}(-L_n) = Z_n^2 - \prod_{k=n}^{n+N-1} (L_k - \alpha_k) = 0.$$

The Toda chain (1.16) and its generalization (1.15) can also be used to construct a pair of commuting differential operators:

$$L_n = \sigma \frac{d}{dx} - \begin{pmatrix} 0 & u_n \\ v_n & 0 \end{pmatrix}, \qquad Z_n = \overset{(N)}{W}(x, L_n), \qquad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where, for $\overset{(N)}{W}(x, \lambda) = \sum a_k(x)\lambda^k$, we define $\overset{(N)}{W}(x, L_n) = \sum a_k(x) L_n^k$. These operators commute by (1.6); indeed, in case (1.14) we have

$$\sigma(d/dx - U_n(x, \lambda))\Phi = L_n(\Phi) - \lambda\Phi = 0 \Rightarrow Z_n(\Phi) = \overset{(N)}{W}(x, \lambda)\Phi,$$

whence

$$\begin{aligned} L_n Z_n(\Phi) &= (L_n - \lambda) Z_n(\Phi) + Z_n L_n(\Phi) \\ &= \sigma(d/dx - U_n)\overset{(N)}{W}(\Phi) + Z_n L_n(\Phi) \\ &= \sigma(d\overset{(N)}{W}/dx + \overset{(N)}{W} U_n - U_n \overset{(N)}{W})(\Phi) + Z_n L_n(\Phi) = Z_n L_n(\Phi). \end{aligned}$$

Remark 1.2 remains valid in case (1.14), and for a Toda chain, for example, after making the substitutions $z \to Z_n$, $\lambda \to L_n$ we obtain

$$\begin{aligned} Z_n + Z_n^{-1} \exp\left(\sum \gamma_n\right) &= \operatorname{tr} \overset{(N)}{W}(L_n) \\ &= (2L_n)^N + Q_1(\mathbf{q}, \mathbf{q}_x) L_n^{N-1} + \cdots + Q_N(\mathbf{q}, \mathbf{q}_x), \end{aligned} \tag{1.21}$$

where $\mathbf{q} = (q_n, q_{n+1}, \cdots, q_{n+N-1})$, and the functions Q_j are first integrals of the chain (1.16) with the periodicity condition (1.12).

1.2. Liouville's Theorem. The problem of finding explicit formulas for constructing pairs of commuting differential operators is considered in the recent theory of finite-band integration (see [7]). However, we are interested only in the fundamental issues involved, or more precisely, in analyzing the reasons why the explicit formulas for finite-band potentials also yield solutions of nonlinear partial differential equations of the Korteweg-de Vries type. We will show that the answer can be found in the proof of a classical theorem of Liouville

[8] concerning the conditions for a finite-dimensional dynamical system of the general form

$$dq_k/dx = F_k(q_1, \dots, q_N), \qquad k = 1, 2, \dots, N, \tag{1.22}$$

to be solvable by quadrature.

THEOREM 1.3. *Assume we are given commuting vector fields*

$$X_j = \sum_{k=1}^{N} f_{jk}(\mathbf{q})\partial/\partial q_k, \qquad j = 1, \dots, K \le N,$$

satisfying the condition rank $(f_{jk}) = K$, *together with functionally independent first integrals* $Q_j(q_1, \dots, q_N)$, $j = 1, \dots, N-K$, *such that*

$$X_j(Q_i) = 0, \; j = 1, \dots, K, \qquad i = 1, \dots, N-K. \tag{1.23}$$

Then each of the K *dynamical systems*

$$dq_n/dt_j = f_{jn}(q_1, \dots, q_N), \qquad n = 1, \dots, N, \tag{1.24}$$

is solvable by quadrature.

PROOF. The system of equations for determining the K additional first integrals

$$\Phi_j(t_j, q_1, \dots, q_N) = t_j - \varphi_j(q_1, \dots, q_N), \qquad j = 1, \dots, K, \tag{1.25}$$

can be written in the form

$$d/dt_i(\Phi_j) = 0 \Leftrightarrow X_i(\varphi_j) = \delta_{ij}; \qquad i, j = 1, \dots, K. \tag{1.26}$$

For fixed j, the relations (1.26) give us a system of linear algebraic equations for the partial derivatives $\varphi_{jk} = \partial\varphi_j/\partial q_k$ of the function φ_j.

In the case $K = N$, the matrix (φ_{jk}) coincides with the matrix $(f_{jk})^{-1}$ up to transposition. The commutation conditions

$$[X_i, X_j] \equiv \sum [X_i(f_{jk}) - X_j(f_{ik})]\partial/\partial q_k = 0 \tag{1.27}$$

ensure that the differential forms

$$\omega_j = \sum_{k=1}^{N} \varphi_{jk} dq_k$$

are closed, and hence that the functions φ_j are locally unique $(d\varphi_j = \omega_j)$. We observe further that the change of coordinates $p_i = \varphi_i(q_1, \dots, q_N)$, $i = 1, \dots, N$, takes the vector field X_j into $\partial/\partial p_j$, $\forall j$.

In the general case $(K < N)$, the first integrals $Q_1, \dots, Q_{N-K}$ can be used to reduce the order of the dynamical system (1.24) to K. It follows from (1.23) that the reduced system also satisfies the hypotheses of Theorem 1.3. In the light of the above discussion, we can therefore state that for $K < N$, there exists an invertible change of variables

$$\tilde{q}_i = Q_i(q_1, \dots, q_N), \qquad p_j = \varphi_j(q_1, \dots, q_N), \tag{1.28}$$

where $i = 1, \dots, N-K$, $j = 1, \dots, K$, which takes the vector fields $X_1, \dots, X_k$ into $\partial/\partial p_1, \dots, \partial/\partial p_k$, respectively. It is easy to see that the functions φ_j

in (1.28) satisfy (1.26) and that the functions $q_n(t_1, \dots, t_k)$, $n = 1, \dots, N$, found from the relations

$$\begin{aligned} Q_i(q_1, \dots, q_N) &= c_i, \qquad i = 1, \dots, N-K, \\ \varphi_m(q_1, \dots, q_N) &= t_m, \qquad m = 1, \dots, K, \end{aligned} \tag{1.29}$$

give, for fixed t_m, $m \neq j$, a solution of (1.24) depending on the correct number of arbitrary constants. ●

A dynamical system (1.22) with a *complete set of* $N-1$ *independent first integrals* gives the simplest particular case of Theorem 1.3. Here $K = 1$, and the additional first integral (1.25) is readily found from the equation

$$X_1(\varphi_1) = \sum_{k=1}^{N} (F_k \partial\varphi_1/\partial q_k) = 1$$

if, following the proof of the theorem, one introduces new dynamical variables associated with the first integrals. The general solution is then found from (1.29) with $t_1 = x$. In all remaining cases for $K > 1$, the above procedure "Liouville integration") leads by (1.29) to a general solution which is a function not only of x but also of the additional arguments t_j. Typically, $k \simeq N/2$, and in the especially important case of a Hamiltonian system, each first integral can be associated with a corresponding vector field.

The above discussion is well illustrated by the dynamical system (1.13) with $N = 5$. Starting with (1.5) and (1.9), we find by direct computation (cf. (1.20)) that

$$\begin{aligned} \operatorname{tr} \overset{(5)}{W}(\lambda) &= \lambda^2 Q_1 - \lambda Q_2 + Q_3 \\ &= \prod_{k=0}^{4} r_{n+k} + \sum_{k=0}^{4} (\alpha_{n+k} - \lambda) r_{n+k+1} r_{n+k+2} r_{n+k+3} \\ &\quad + \sum_{k=0}^{4} r_{n+k} (\alpha_{n+k+2} - \lambda)(\alpha_{n+k+4} - \lambda), \end{aligned} \tag{1.30}$$

where $r_s = q_s + q_{s+1}$. It is now clear that by virtue of (1.7), this dynamical system has three polynomial first integrals, of which the simplest is

$$Q_1 = \sum_{k=0}^{4} r_{n+k} = 2(q_n + q_{n+1} + \dots + q_{n+4}).$$

These integrals are functionally independent, and it remains to exhibit two vector fields satisfying the conditions (1.23) and (1.27) (one of them always coincides with d/dx). Noting that the dynamical system (1.13) is Hamiltonian, i.e., expressible in the form

$$d\mathbf{q}/dx = J \operatorname{grad} \sum_{k=0}^{N-1} \left(\frac{1}{3} q_{k+n}^3 + \alpha_{k+n} q_{k+n} \right),$$

where the matrix $J = (T+E)^{-1}(T-E)$ is antisymmetric, we can use the vector fields corresponding to the dynamical systems

$$d\mathbf{q}/dt_j = J \operatorname{grad} Q_j \qquad (j = 1, 2, 3). \tag{1.31}$$

It is evident that $\operatorname{grad} Q_1 \in \operatorname{Ker} J$. Straightforward computations show that $d/dt_2 = -d/dx$. The resulting pair of vector fields $X_j = d/dt_j$ $(j = 2, 3)$ and the set (1.30) of first integrals Q_1, Q_2, Q_3 satisfy all the hypotheses of Theorem 1.3. For example, the standard correspondence between Poisson brackets and commutators gives

$$dQ_3/dx = (\operatorname{grad} Q_3)^{\mathrm{T}} J \operatorname{grad} Q_2 = \{Q_3, Q_2\}\Gamma \Rightarrow [X_3, X_2] = 0.$$

The dynamical system (1.13) is thus Liouville-integralbe for $N = 5$, and its general solution depends on the extra argument t_3, in addition to the original independent variable $x = -t_2$. In other words, Liouville integration yields a function $q(t_2, t_3)$ satisfying system (1.31). We suggest that the reader verify that in the special case $\alpha_n = 0$ for all n, the dynamical system (1.31) dual to (1.13), with $j = 3$, reduces to the chain of equations

$$dq_s/d\tau = (q_s + q_{s+1})^{-1} - (q_s + q_{s-1})^{-1}, \tag{1.32}$$

where

$$\tau = c_3 t_3, \qquad c_3 = \prod_{k=0}^{4}(q_{n+k} + q_{n+k+1}) = \text{const},$$

and the periodicity condition $q_{s+5} = q_s$ is imposed. In the general case, a rather lengthy computation shows that (1.31) implies the following relations between the partial derivatives the respect to $x = -t_2$ and t_3 of the general solution of (1.13) with $N = 5$:

$$q_{nt} = q_{nxxx} - 6(q_n^2 + \alpha_n)q_{nx}, \tag{1.33}$$

$$u_{nt} = u_{nxxx} - 6u_n u_{nx}. \tag{1.34}$$

Here (cf. (1.11) $u_n = q_{nx} - q_n^2 - \alpha_n$ and the vector field d/dt is given by the following linear combination of the vector fields (1.31):

$$d/dt = -4d/dt_3 - 2(Q_1^2 + 4\sum_0^4 \alpha_k)d/dx.$$

Summarizing, we may say that the Korteweg-de Vries equation (1.34) arises inevitably in the solution of the dynamical system (1.13). Moreover, it turns out that the dynamical system (1.31) associated with (1.13) with $j = 3$ coincides with the modified Korteweg-de Vries equation (1.33) up to a change of notation. In the next section we will show that these facts, established for the example considered above, are a special case of a very general and simple result concerning pairs of commuting infinite-dimensional vector fields.

1.3. Statement of the problem. The object of study in the sequel will be the following infinite chains of equations generalizing (1.32), (1.16), and (1.15), respectively:

$$q_{nx} = F(q_n, q_{n-1}, q_{n+1}), \tag{1.35}$$

$$q_{nxx} = F(q_{nx}, q_n, q_{n-1}, q_{n+1}), \tag{1.36}$$

$$p_{nx} = F(p_n, q_n, p_{n+1}), \qquad q_{nx} = G(p_n, q_n, q_{n-1}). \tag{1.37}$$

We will also investigate vector chains of the form

$$p_{nx} = F(p_n, q_n, q_{n+1}), \qquad q_{nx} = G(p_n, q_n, q_{n-1}). \tag{1.38}$$

It is easy to check that if $G(a, b, c) = F(b, c, a)$, the vector chain (1.38) reduces to the scalar chain (1.35) if we write $p_n = \tilde{q}_{2n}$, $q_n = q_{2n-1}$. Chains related by point transformations $\tilde{p}_n = \varphi(p_n)$, $\tilde{q}_n = \psi(q_n)$ will be regarded as equivalent, so that the transition from the scalar chain (1.36) to the corresponding vector chain (1.38)

$$p_{nx} = F(p_n, q_n, q_{n+1}), \qquad q_{nx} = F(q_n, p_{n-1}, p_n)$$

extends the class of admissible transformations. It is obvious that the structure of the equations (1.35)–(1.38) is preserved under point transformations; the only thing that changes is the form of the functions appearing on the right-hand sides of the equations.

The nonlinear chains which we consider belong to a large class of *infinite-dimensional systems* of the form

$$dq_n/dx = \mathbf{F}_n, \qquad n = 0, \pm 1, \pm 2, \dots, \tag{1.39}$$

where each of the vector functions $\mathbf{F}_n = (F_n^1, \dots, F_n^d)$ depends on a finite set of *dynamic variables* $\mathbf{q}_k = (q_k^1, \dots, q_n^d)$, the set depending on the number n of the equation in (1.39). For example, setting $q_n^1 = q_n$ and $q_n^2 = p_n = dq_n/dx$, for the generalized Toda chain (1.36) we obtain the infinite-dimensional dynamical system

$$dq_n/dx = p_n, \qquad dp_n/dx = F(p_n, q_n, q_{n-1}, q_{n+1}).$$

Our theory is based on the assumption that there exists an infinite-dimensional dynamical system

$$d\mathbf{q}_n/dt = \mathbf{f}_n, \qquad n = 0, \pm 1, \pm 2, \dots, \tag{1.40}$$

which is *dual* to (1.39) and satisfies the condition that the vector fields corresponding to (1.39) and (1.40) commute:

$$\left[\sum_{-\infty}^{+\infty} \mathbf{F}_n \frac{\partial}{\partial \mathbf{q}_n}, \ \sum_{-\infty}^{+\infty} \mathbf{f}_n \frac{\partial}{\partial \mathbf{q}_n}\right] = 0.$$

These vector fields define the differentiation operators $D = D_x$ and D_t on the set of functions depending on finite sets of the dynamicval variables, and the condition that the vector fields commute is equivalent to the relations

$$D(\mathbf{f}_n) = D_t(\mathbf{F}_n), \qquad n = 0, \pm 1, \pm 2, \dots . \tag{1.41}$$

Our goal is to derive from (1.39) and (1.40) differential relations

$$\Phi(\mathbf{u}, D^j D_t^k(\mathbf{u})) = 0 \tag{1.42}$$

which represent a *closed system of partial differential equations*.([1]) Here $\mathbf{u} = (u^1, \dots, u^m)$ is a distinguished set of components of the dyanmical variables, and the number of "independent" equations in (1.42) is equal to the cardinality $m \geq d$ of this set.

In addition to our basic assumption regarding the existence of an infinite system of equations (1.40) dual to (1.39), we shall also assume that both (1.39)

([1])In what follows, we will indicate some general conditions which ensure that the pair of chains (1.39), (1.40) is equivalent to the associated finite system of partial differential equations (1.42) (see Theorem 2.2).

and (1.40) are *invariant under the shift* $n \to n+1$. As a rule, such infinite-dimensional dynamical systems will simply be called *chains* (cf. (1.35)–(1.38)). In a certain sense, the stringent requirement of shift-invariance replaces the integrability conditions in Theorem 1.3 and enables us, in the cases (1.35)–(1.38) considered below, to effectively find the dual chain (1.40) *and the associated partial differential equation* (1.42) directly in terms of the right-hand side of the original chain (1.39). Here the necessary conditions for the existence of the chain (1.40) dual to (1.39) are closely related to the symetry approach to the problem of determining the integrability of partial differential equations, discussed in the review [9]. To emphasize these connections, we introduce the following.

DEFINITION 1.4. A *local conservation law* for a shift-invariant vector chain (1.39) is a scalar relation of the form

$$D(h) = (T-E)(r), \tag{1.43}$$

where the shift operator T ($T(q_n^i) = q_{n+1}^i$) acts on the set of functions of arbitrary finite sets of the dynamical variables. The function h appearing on the left is called the *density* of the conservation law (1.43). A local conservation law (1.43) is said to be *trivial* if the variational derivatives of its density vanish:

$$\delta h/\delta q_0^i = \sum_{-\infty}^{+\infty} \partial(T^n h)/\partial q_0^i = 0, \qquad i = 1, \dots, d. \tag{1.44}$$

As in the continuous case (see [10]), one can prove that the vanishing of the variational derivatives (1.44) guarantees that h is expressible in the form

$$h = (T-E)\tilde{h} + \text{const},$$

where $\tilde{h}$ is a suitable function of finitely many dynamical variables. It is also evident that

$$(\delta/\delta q_0^i)(T-E) \equiv 0, \qquad i = 1, \dots, d. \tag{1.45}$$

We conclude this section by noting that every shift-invariant infinite-dimensional dynamical system (1.39) admits a finite-dimensional reduction, namely the periodic closure

$$\mathbf{q}_{n+N} = \mathbf{q}_n, \qquad \text{for all} \quad n. \tag{1.46}$$

With the system closed in this way, the shift operator becomes the cyclic permutation

$$T\colon (q_{n+1}, \dots, q_{n+N}) \to (q_{n+2}, \dots, q_{n+N}, q_{n+1}),$$

and the resulting finite-dimensional dynamical system is invariant under cyclic permutations of the variables. The local conservation laws for the original unclosed chain are the inverse images of the first integrals of the reduced system, and for the solutions of the latter we have by (1.43) that

$$\frac{d}{dx}\sum_{k=n}^{n+N-1} T^k(h) = \sum_{k=n}^{n+N-1} T^k\left(\frac{dh}{dx}\right) = \sum_{k=n}^{n+N-1} (T^{k+1} - T^k)(r) = 0.$$

It can be verified that the vector fields (see (1.41)) continue to commute after periodic closure is imposed, because the differentiation operators corresponding to these fields commute with the shift ($TD = DT$, $TD_t = D_tT$).

2. General theory and examples

2.1. Regularity condition. To construct solutions of the infinite-dimensional dynamical system (1.39), we can specify the components of the vector $\mathbf{q}_n$ at one or several neighboring points ($n \simeq n_0$) as functions of x and try to find the solution by passing to the subsequent equations in the system. For instance, for the chain (1.38) we find

$$\begin{aligned} q_{n+1} &= \varphi(p_n, q_n, p_{nx}), \\ p_{n+1} &= \psi(p_n, q_{n+1}, (q_{n+1})_x) = \tilde{\psi}(p_n, q_n, p_{nx}, q_{nx}, p_{nxx}) \end{aligned}$$

with similar formulas for p_{n-1} and q_{n-1}. By specifying the values $p_n = u(x)$, $q_n = v(x)$ of the dynamical variables at the point $n = n_0$, we can find $p_n(x)$, $q_n(x)$ for any $n \in \mathbf{Z}$ by expressing them in terms of $u, v, u_x, v_x, \dots$. In other words, the general solution of the chain (1.38) contains two arbitrary functions.

We define the *rank* of the infinite-dimensional dynamical system (1.39) to be the number of arbitrary functions of x appearing in the general solution. An example of a rank-one system is the chain

$$q_{nx} = F(q_n, q_{n+1}).$$

However, in this case the condition $q_n = u(x)$ for $n = n_0$ determines $q_n(x)$ uniquely only for $n \geq n_0$; for $n < n_0$, additional constants of integration appear in the solution.

For the case of a vector field of the general form

$$\begin{aligned} p_{nx} &= F(p_n, q_n, p_{n+1}, q_{n+1}, p_{n-1}, q_{n-1}), \\ q_{nx} &= G(p_n, q_n, p_{n+1}, q_{n+1}, p_{n-1}, q_{n-1}) \end{aligned} \tag{2.1}$$

the choice of initial data

$$p_n = u(x), \qquad q_n = v(x) \qquad (n = n_0) \tag{2.2}$$

is possible, provided the chain (2.1) is equivalent to the following chain

$$\begin{aligned} F_+(p_n, q_n, p_{nx}, q_{nx}, p_{n+1}, q_{n+1}) &= 0, \\ F_-(p_n, q_n, p_{nx}, q_{nx}, p_{n-1}, q_{n-1}) &= 0. \end{aligned}$$

The functions $p_{n+1}(x)$, $q_{n+1}(x)$ are then determined by the equations

$$\begin{aligned} F_+(p_n, q_n, p_{nx}, q_{nx}, p_{n+1}, q_{n+1}) &= 0, \\ F_-(p_{n+1}, q_{n+1}, (p_{n+1})_x, (q_{n+1})_x, p_n, q_n) &= 0. \end{aligned}$$

Analogous equations hold for the functions $p_{n-1}(x)$ and $q_{n-1}(x)$. Typical of such chains (2.1) are given by (1.37) and (1.38).

In the case of general position, the vector chain

$$\mathbf{q}_{nx} = \mathbf{F}_n = \mathbf{F}(\mathbf{q}_n, \mathbf{q}_{n+1}, \mathbf{q}_{n-1}), \qquad \mathbf{q} = (q^1, \dots, q^m), \tag{2.3}$$

has rank $2m$, since the nondegeneracy conditions

$$\det \partial \mathbf{F}/\partial \mathbf{q}_{n+1} \neq 0, \qquad \det \partial \mathbf{F}/\partial \mathbf{q}_{n-1} \neq 0 \tag{2.4}$$

ensure the unique solvability of the initial-value problem

$$\mathbf{q}_n = \mathbf{u}(x), \qquad \mathbf{q}_{n-1} = \mathbf{v}(x) \qquad (n = n_0). \tag{2.5}$$

It is clear that the scalar chain (1.35) is of rank 2 in the case of general position. The generalized Toda chain (1.36) also has rank 2, since when the nondegeneracy conditions analogous to (2.4) are satisfied, the initial data (2.5) can be used to find $q_n(x)$ for all n by means of the formulas

$$q_{n+1} = F_+(q_n, q_{nx}, q_{nxx}, q_{n-1}),$$
$$q_{n-2} = F_-(q_n, q_{n-1}, (q_{n-1})_x, (q_{n-1})_{xx}). \tag{2.6}$$

Thus, the general solutioin of the chain (1.36) contains two arbitrary functions.

DEFINITION 2.1. A nonlinear chain will be called *regular* if the initial data for $n \simeq n_0$ uniquely determine the functions $\mathbf{q}_n(x)$, $\forall n \in \mathbf{Z}$.

For a regular chain of rank M, all the components of the solution $\{\mathbf{q}_n(x), n \in \mathbf{Z}\}$ are uniquely expressible in terms of the vector function $\mathbf{u} = (u^1(x), \dots, u^M(x))$ of the initial data for the Cauchy problem in the discrete variable n by means of the formulas

$$\mathbf{q}_n = \mathbf{Q}_n(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots), \tag{2.7}$$

where the number of derivatives on the right is finite for any n but becomes unbounded as $n \to \pm\infty$. The above chains (1.35), (1.36), and (1.38) are examples of regular chains. However, the chain (1.37) is not regular, because additional integration constants appear in the solution of the Cauchy problem with respect to the discrete variable.

THEOREM 2.2. *Assume we are given finite-dimensional dynamical systems* (1.39), (1.40) *satisfying the conditions* (1.41) *for the corresponding vector fields to commute. Assume that these systems are shift-invariant and that the first system* (1.39) *is a regular chain of* rank M. *Then the problem of constructing a general solution of equations* (1.39), (1.40) *is equivalent to solving the Cauchy problem for the evolution partial differential equation*

$$\mathbf{u}_t = \mathbf{G}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots), \tag{2.8}$$

where the right-hand side of the equation for $u^j = q^{i(j)}_{n(j)}$ *is obtained by substituting* (2.7) *into the right-hand side of the corresponding equation in the chain* (1.40).

SKETCH OF PROOF. The equations (2.8) are a consequence of (1.39), (1.40), and therefore the existence of a simultaneous solution $\{q_n(x, t), n \in \mathbf{Z}\}$ of the chains implies the solvability of (2.8).

Now let $\mathbf{u}(x, t)$ be a solution of the Cauchy problem for the evolution equation (2.8) with the initial data

$$u^i(x, t) = \varphi^i(x); \qquad i = 1, \dots, M; \quad t = 0. \tag{2.9}$$

The formulas (2.7) can be used to find the vector functions $\mathbf{q}_n(x, t)$ for all $n \in \mathbf{Z}$. One must check that the functions so constructed satisfy all the equations in the chains (1.39), (1.40). By definition, formulas (2.7) give a solution of (1.39) for any choice of the functions $u^i(x, t)$. Since $u^i = q^{j(i)}_{n(i)}$, formulas (2.8) guarantee that the equations $D_t(q^{j(i)}_{n(i)}) = F^{j(i)}_{n(i)}$ are satisfied. The conditions (1.41) for the vector fields to commute ensure that the remaining equations in the chain (1.40) are satisfied. More precisely, the commutation conditions (1.41) for the vector fields imply that a solution of (2.8) remains a solution under the transformation (2.7). ●

Theorem 2.2 shows that in the infinite-dimensional case, the simultaneous solution of the chains (1.39), (1.40) depends on M arbitrary functions (2.9), where M is the rank of the chain (1.39). We recall that in the finite-dimensional case, the condition that the vector fields commute is necessary and sufficient for the existence of a simultaneous solution of the dynamical systems

$$q_{nx} = F_n, \qquad q_{nt} = f_n \qquad (n = 1, \dots, N),$$

corresponding to the fields, the solution depending on N arbitrary constants.

We illustrate Theorem 2.2 for the Toda chain

$$q_{nxx} = \exp(q_{n+1} - q_n) - \exp(q_n - q_{n-1}). \tag{2.10}$$

The reader can verify by direct calculation that the following chains are dual to (2.10):

$$q_{nt} = q_{nx}^2 + \exp(q_{n+1} - q_n) + \exp(q_n - q_{n-1}), \tag{2.11}$$

$$\begin{aligned} q_{nt} = q_{nx}^3 &+ (2q_{nx} + q_{n+1,x}) \exp(q_{n+1} - q_n) \\ &+ (2q_{nx} + q_{n-1,x}) \exp(q_n - q_{n-1}). \end{aligned} \tag{2.12}$$

As already noted, chains of the form (1.36) are regular, and setting

$$u_n = \exp q_n, \qquad v_n = \exp(-q_{n-1}), \tag{2.13}$$

(cf. (2.5)), we can express the rigt-hand sides of the dual chains in terms of derivatives of the functions (2.13). For (2.11), simple computations lead to the system of equations

$$u_{nt} = u_{nxx} + 2u_n^2 v_n, \qquad -v_{nt} = v_{nxx} + 2v_n^2 u_n. \tag{2.14}$$

Thus, the equation of the form (2.8) associated with (2.10), (2.11) coincides with the generalized Schrödinger equation (2.14) with a cubic nonlinearity. Similar calculations for (2.10) and (2.12) lead to the generalized modified Korteweg-de Vries equation

$$u_{nt} = u_{nxxx} + 6u_n v_n u_{nx}, \qquad v_{nt} = v_{nxxx} + 6u_n v_n v_{nx}. \tag{2.15}$$

Theorem 2.2 asserts that any solution $u_m(x, t)$, $v_m(x, t)$ of the associated equation (2.14) or (2.15) can, in conjunction with formulas of the type (2.13), be used to construct a simultaneous solution $\{q_n(x, t), n \in \mathbf{Z}\}$ of the chains (2.10), (2.11) or (2.10), (2.12).

The Toda chain (2.10) is often expressed in terms of the variables $u_n = \exp(q_{n+1} - q_n)$, $v_n = q_{nx}$:

$$u_{nx} = u_n(v_{n+1} - v_n), \qquad v_{nx} = u_n - u_{n-1}. \tag{2.16}$$

This chain exemplifies a regular chain of the form (1.38). The following systems of equations (cf. [11], [12]) are associated with (2.16):

$$u_{nt} = u_{nxx} + 2(u_n v_n), \qquad v_{nt} = -v_{nxx} + (v_n^2 + 2u_n)_x, \tag{2.17}$$

$$\begin{aligned} u_{nt} &= u_{nxxx} + 3(v_n u_{nx} + u_n^2 + u_n v_n^2)_x, \\ v_{nt} &= v_{nxxx} + (-3v_n v_{nx} + v_n^3 + 6u_n v_n)_x. \end{aligned} \tag{2.18}$$

It is evident that for a regular chain (1.39), the form of the dual chain (1.40) can be recovered uniquely from the associated partial differential equation (2.8). This general fact is illustrated by the pairs of equations (2.11), (2.14), and (2.12), (2.15). The reader can use (2.17) and (2.18) to reconstruct the chains

(1.40) and verify that the commutation conditions (1.41) hold for the vector fields.

Clearly, the form of the associated partial differential equation (2.8) is independent of the choice of the point $n \in \mathbf{Z}$ at which the initial data for the original chain (1.39) are specified. In the regular case, formulas (2.7) can be used to find an explicit expression for the action of the shift operator $n \to n+1$ on the solutions of (2.8). As noted at the end of the proof of Theorem 2.2, this operator preserves the solution set of the associated equation. Under the hypotheses of Theorem 2.2, we thus have the next result.

COROLLARY 2.3. *The evolution equation* (2.8) *associated with the regular chain* (1.39) *admits an invertible differential substitution*

$$\mathbf{U} = \mathbf{U}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots), \tag{2.19}$$

which takes solutions of (2.8) *again into solutions.*

In the examples considered above, the form of the differential substitution (2.19) is easily found from (2.10) and (2.16):

$$(2.10) \Leftrightarrow u_{n+1} = u_n[(\ln u_n)_{xx} + u_n v_n], \quad v_{n+1} = 1/u_n;$$
$$(2.16) \Leftrightarrow u_{n+1} = u_n[(\ln u_n)_{xx} + v_{nx}], \quad v_{n+1} = v_n + (\ln u_n)_x.$$

The periodic closure condition $u_{n+N} = u_n$, $v_{n+N} = v_n$ leads to periodic solutions of the chains (2.10) and (2.16) and to finite-band solutions of equations (2.14), (2.15) and (2.17), (2.18), respectively (see §1).

2.2. Examples of quasi-regular chains. The general scheme for reducing the chains (1.39), (1.40) to partial differential equations can be generalized to the nonregular case. As an example, we consider the linear rank-one chain

$$\partial_x(q_n) = q_{n+1}. \tag{2.20}$$

Any chain of the form

$$\partial_t(q_n) = q_{n+m} \tag{2.21}$$

is clearly dual to (2.20). For $m = 2$ we find, as in the regular case, that a simultaneous solution of (2.20), (2.21) satisfies the evolution equation $u_t = u_{xx}$ $(u = q_n)$. The case $m = -1$ leads to the hyperbolic equation

$$u_{xt} = u. \tag{2.22}$$

We note that the substitution

$$p_n = q_{n+1}/q_n \tag{2.23}$$

takes the chain (2.20) into the discrete analog of the Burgers equation

$$p_{nx} = p_n(p_{n+1} - p_n). \tag{2.24}$$

Chains consistent with (2.24) are obtained from (2.21) by the substitution (2.23) and for $m = 2$ lead to the Burgers equation

$$v_t = v_{xx} + 2vv_x \qquad (v = p_n) \tag{2.25}$$

and for $m = -1$ to the hyperbolic equation

$$w_{xt} + (e^w)_t - (e^{-w})_x = 0 \qquad (p_n = e^w). \tag{2.26}$$

Using (2.20), we see that (2.23) coincides with the substitution $v = u_x/u$, which linearizes the Burgers equation (2.25). A similar substitution reduces (2.26) to (2.22).

As we noted in 2.1, §2, unlike chains of the form (1.38), chains of the type (1.37) are not regular, and in this case two different choices of initial data are possible:

$$p_n = u, \qquad q_n = v; \tag{2.7}$$

$$p_n = u, \qquad q_{n-1} = v. \tag{2.28}$$

For the chains (1.37) we are unable to express all the dynamical variables in terms of the initial data u, v. Nevertheless, the examples given below will show that in general, the dual chains can be used to derive a closed system of partial differential equations for the functions (2.27), (2.28) (we call such chains *quasiregular*). As in the regular case, the resulting system is a consequence of the corresponding pair of chains (1.39), (1.40). It is clear from (1.37) that solutions of the associated partial differential equation corresponding to the choice of initial data (2.28) can be obtained from the solution of the equation corresponding to the choice (2.27) by formulas of the type

$$\tilde{u} = u, \qquad \tilde{v} = V(u, v, v_x). \tag{2.29}$$

As an example, we consider a local version of the chain (1.15):

$$p_{nx} = p_{n+1} + p_n^2 q_n, \qquad -q_{nx} = q_{n-1} + q_n^2 p_n. \tag{2.30}$$

This chain is an infinite Hamiltonian system of the form (see (1.44))

$$p_{nx} = r_n \delta h_n / \delta q_n, \qquad q_{nx} = -r_n \delta h_n / \delta p_n, \tag{2.31}$$

where $r_n = 1$, $h_n = p_{n+1} q_n + p_n^2 q_n^2 / 2$. Using the Hamiltonian property, we can construct a dual chain by using a local conservation law (see Definition 1.4) with density

$$h_n = p_{n+2} q_n + p_{n+1}^2 q_{n+1} q_n + p_{n+1} p_n q_n^2 + \tfrac{1}{3} p_n^3 q_n^3.$$

As in the case of the Toda chain, the choice of initial data (2.28) leads to the system (2.14), while the choice (2.27) gives another familiar integrable system ([12], [13])

$$\begin{aligned} u_t &= u_{xx} - 2(u^2 v_x + u^3 v^2), \\ -v_t &= v_{xx} + 2(v^2 u_x - v^3 u^2). \end{aligned} \tag{2.32}$$

The solutions of system (2.14) are obtained from those of (2.32) by the formula $\tilde{u} = u$, $\tilde{v} = -(uv + v_x^2)$ (cf. (2.29)).

The requirement that a dual chain should exist severely restricts the form of the right-hand side of the chain (1.37). By analyzing chains of the form (2.31), (1.37) we have succeeded in finding two more interesting examples of such chains (see [14]). In the first case,

$$h_n = (p_{n+1} - p_n) q_n, \qquad r_n = \alpha p_n q_n + \beta (p_n + q_n) + \gamma, \tag{2.33}$$

and in the second,

$$\begin{aligned} h_n &= \tfrac{1}{2} \ln r_n - \ln(p_{n+1} - q_n), \\ r_n &= r(p_n, q_n) = \alpha p_n^2 q_n^2 + \beta p_n q_n (p_n + q_n) \\ &\quad + \gamma p_n q_n + \delta (p_n + q_n)^2 + \varepsilon (p_n + q_n) + \mu. \end{aligned} \tag{2.34}$$

The case (2.31), (2.33) leads to known generalizations of the systems (2.14), (2.32) (see [9], [12]). The chain (2.31), (2.34) merits a more detailed discussion.

We recall the Landau-Lifshits equation

$$\mathbf{S}_t = \mathbf{S} \times S_{xx} + \mathbf{S} \times j\mathbf{S}, \qquad \mathbf{S} = (S_1, S_2, S_3), \quad S_1^2 + S_2^2 + S_3^2 = 1, \tag{2.35}$$

which was solved in [15] and [16]. Under the stereographic projection

$$u = \frac{S_1 + iS_2}{1 + S_3}, \qquad v = \frac{S_1 - iS_2}{1 + S_3}$$

it becomes the system of equations

$$\begin{aligned} iu_t + u_{xx} - 2v\frac{u_x^2 + R(u)}{uv + 1} + \tfrac{1}{2}R'(u) = 0, \\ -iv_t + v_{xx} - 2u\frac{v_x^2 + R(v)}{uv + 1} + \tfrac{1}{2}R'(v) = 0, \end{aligned} \tag{2.36}$$

where

$$R(u) = \alpha u^4 + \beta u^3 + \gamma u^2 - \beta u + \alpha. \tag{2.37}$$

If $J = \operatorname{diag}(J_1, J_2, J_3)$ we have

$$\alpha = \tfrac{1}{4}(J_2, J_1), \qquad \gamma = \tfrac{1}{2}(J_1 + J_2) - J_3, \qquad \beta = 0. \tag{2.38}$$

The problem of finding solvable generalizations of the classical Landau-Lifshits model (2.35) was studied in [9], [12], and [17]. The table at the end of this paper lists four such generalizations of (2.35), referred to as II(b), IV(b), VI(b), and VI(c). It is easily seen that II(b), with $P(u) \equiv R(u)$, goes into (2.36) after the change of variables

$$\tilde{u} = u, \qquad \hat{v} = -1/v, \qquad \hat{t} = -it,$$

i.e., it coincides in essence with the original model (2.35).

As will be shown below, the Hamiltonian chain (2.13), (2.34) and a formula of the type (2.29) can be used to answer the question concerning the existence of higher-order conservation laws for the system of equations VI(b).([2]) The function

$$\begin{aligned} h_n = & -\frac{r_n}{(p_{n+1} - q_n)(p_n - q_{n-1})} + \frac{1}{2}\frac{\partial r_n/\partial p_n}{p_{n+1} - q_n} \\ & + \frac{1}{2}\frac{\partial r_n/\partial q_n}{p_n - q_{n-1}} - \frac{1}{4}\frac{\partial^2 r_n}{\partial p_n \partial q_n}, \end{aligned} \tag{2.39}$$

where r_n is given by (2.34), is the density of a local conservation law defining the desired chain. The alternative choice (2.38) of initial data leads to a system II(b) with

$$P(u) = \frac{1}{2} r r_{vv} - \frac{1}{4} r_v^2, \tag{2.40}$$

where $r = r(u, v)$ is the polynomial in (2.34). In other words, the choice (2.28) leads to the Landau-Lifshits equation. In the same notation, the case (2.37) leads precisely to a system VI(b). In the corresponding substitution (2.29)

$$v = u - r(u, v)[v_x + \tfrac{1}{2} r_u(u, v)]^{-1}. \tag{2.41}$$

([2])This question was left open in the papers [9], [12], [17] cited above.

The existence of the differential substitution (2.29), (2.41) demonstrates, in particular, that the system of equations VI(b) possesses higher-order conservation laws.

In closing this section, we consider Hamiltonian chains of the form (2.31) with $r_n = p_n q_n - 1$ and $h_n = p_{n+1} q_n$, $h_n = p_{n-1} q_n$ (cf. (2.33)):

$$p_{nx} = (p_n q_n - 1) p_{n+1}, \qquad -q_{nx} = (p_n q_n - 1) q_{n-1}, \tag{2.42}$$

$$p_{nt} = (p_n q_n - 1) p_{n-1}, \qquad -q_{nt} = (p_n q_n - 1) q_{n+1}. \tag{2.43}$$

Althouth we are unable to construct an evolution system of partial differential equations associated with these chains, by introducing the initial data (2.27) we can express the mixed derivatives in terms of the data (cf. (2.22)). As a result, we get the system of hyperbolic equations

$$u_{xt} = (uv - 1)^{-1} v u_x u_t - (uv - 1) u,$$
$$v_{xt} = (uv - 1)^{-1} u v_x v_t - (uv - 1) v,$$

which is a consequence of the chains (2.42), (2.43). The reduction $u = v = \sin(w/2)$ of this known system ([18], [19]) is expressible in the form

$$w_{xt} = \sin w.$$

2.3. Complex structure. Reductions. Returning to the regular chains (1.35), (1.36), and (1.38), we now indicate some conditions under which the associated partial differential equation (2.8) is expressible as a system of equations

$$u_t = u_{xx} + f(u, v, u_x, v_x), \qquad -v_t = v_{xx} + g(u, v, u_x, v_x), \tag{2.44}$$

generalizing (2.14), (2.17), (2.32), (2.36). We first consider the scalar chains (1.35).

THEOREM 2.4. *Assume we are given scalar chains*

$$q_{nx} = F(q_n, q_{n+1}, q_{n-1}),$$
$$q_{nt} = G(q_n, q_{n+1}, q_{n-1}, q_{n+2}, q_{n-2}), \tag{2.45}$$

whose corresponding vector fields satisfy the commutation condition (1.41). *Assume that these chains are invariant under one of the following involutions:*

$$m = -n, \qquad \tilde{x} = -x, \qquad \tilde{t} = -t, \qquad \tilde{q}_m = \varepsilon q_n \qquad (\varepsilon = \pm 1) \tag{2.46}$$

and that the nondegeneracy conditions are satisfied:

$$\partial F_n / \partial q_{n+1} \neq 0 \quad (i = \pm 1), \qquad \partial G_n / \partial q_{n+j} \neq 0 \quad (j = \pm 2). \tag{2.47}$$

Then for each n, *the functions*

$$u(x, t) = q_n(x, t), \qquad v(x, t) = \varepsilon q_{n-1}(x, t) \tag{2.48}$$

satisfy, for a suitable choice of scale ($\tilde{t} = \lambda t$), *a system of equations* (2.44) *invariant under the involution*

$$\tilde{x} = -x, \qquad \tilde{t} = -t, \qquad \tilde{u} = v, \qquad \tilde{v} = -u. \tag{2.49}$$

PROOF. The consistency condition (1.41) for the chains (2.45) reads

$$(\partial F_n / \partial q_{n+1}) G_{n+1} + (\partial F_n / \partial q_n) G_n + (\partial F_n / \partial q_{n-1}) G_{n-1} = \sum_k (\partial G_n / \partial q_{n+k}) F_{n+k}. \tag{2.50}$$

Differentiation with respect to q_{n+3} leads to the relation

$$(\partial F_n/\partial q_{n+1})(\partial G_{n+1}/\partial q_{n+3}) = (\partial G_n/\partial q_{n+2})(\partial F_{n+2}/\partial q_{n+3}). \tag{2.51}$$

Using the nondegeneracy condition (2.47), we find from (2.51) that

$$(\partial G_n/\partial q_{n+2}) = \alpha(\partial F_n/\partial q_{n+1})(\partial F_{n+1}/\partial q_{n+2}), \tag{2.52}$$

where $\alpha = \text{const}$. Comparing this formula with

$$q_{nxx} = (\partial F_n/\partial q_{n+1})F_{n+1} + (\partial F_n/\partial q_n)F_n + (\partial F_n/\partial q_{n-1})F_{n-1}$$

we get that

$$u_t - \alpha u_{xx} = \tilde{f}(q_{n+1}, q_n, q_{n-1}, q_{n-2}) = f(u, v, u_x, v_x).$$

The equality

$$-v_t - \beta v_{xx} = g(u, v, u_x, v_x)$$

is established similarly. The invariance of the chains (2.45) under the involution (2.46) implies that the resulting system of equations for u, v is invariant under the change of variables (2.49). It follows, in particular, that $\alpha = \beta$. ●

The assertion of Theorem 2.4 remains valid for the chains (1.36) and (1.38). For the generalized Toda chains, the chains dual to (1.36) are chosen in the form

$$q_{nt} = G_n = G(q_{n+1}, q_{n-1}, q_n, q_{nx}), \qquad \partial G_n/\partial q_{n+j} \neq 0 \qquad (j = \pm 1). \tag{2.53}$$

Here we assume that the generalized Toda chain satisfies the nondegeneracy conditions, and that the system of equations (1.36), (2.53) is invariant under one of the involutions (2.46). For example, for a Toda chain the system of equations (2.10), (2.11) admits the involution in (2.46) with $\varepsilon = -1$.

In the case of chains (1.38) satisfying the regularity conditions

$$\partial F_n/\partial q_n, \ \partial F_n/\partial q_{n+1}, \ \partial G_n/\partial p_n, \ \partial G_n/\partial p_{n-1} \neq 0,$$

the dual chain is of the form

$$\begin{aligned} p_{nt} &= \varphi(p_n, q_n, p_{n+1}, q_{n+1}, p_{n-1}, q_{n-1}), \\ q_{nt} &= \psi(p_n, q_n, p_{n+1}, q_{n+1}, p_{n-1}, q_{n-1}) \end{aligned} \tag{2.54}$$

and satisfies the conditions

$$\det[\partial(\varphi_n, \psi_n)/\partial(p_{n+k}, q_{n+k})] \neq 0, \qquad k = \pm 1.$$

The involutions (2.46) are replaced by the involution

$$m = -n, \quad \tilde{x} = -x, \quad \tilde{t} = -t, \quad \tilde{q}_m = \varepsilon p_n, \quad \tilde{p}_m = \varepsilon q_n, \quad \varepsilon = \pm 1, \tag{2.55}$$

while formulas (2.48) for the transition to (2.44) are replaced by the formulas

$$u(x, t) = p_n(x, t), \qquad v(x, t) = \varepsilon q_n(x, t).$$

The system of equations

$$\begin{aligned} iu_t + u_{xx} &= f(u, v, iu_x, iv_x), \\ -iv_t + v_{xx} &= g(u, v, iu_x, iv_x) \end{aligned} \tag{2.56}$$

admits a complex reduction ($v = \bar{u}$) if the functions f, g satisfy the conditions

$$g(a, b, c, d) = f(b, a, -d, -c), \tag{2.57}$$

$$\overline{f(a, b, c, d)} = f(\bar{a}, \bar{b}, \bar{c}, \bar{d}). \tag{2.58}$$

To specify the complex structure for the system (2.44) and the associated chains (2.45), one must introduce new independent variables $\tilde{x} = iv$, $\tilde{t} =$ it and require the chains (2.45) to be invariant under the involution

$$\tilde{x} = -x, \qquad \tilde{t} = -t, \qquad \tilde{q}_n = \bar{q}_n \tag{2.59}$$

in addition to the involution (2.46). The involutions (2.46) and (2.59) then ensure that conditions (2.57) and (2.58) hold, respectively. For chains of the form (1.38), the additional involution needed to define the complex structure is given by the formulas

$$\tilde{x} = -x, \qquad \tilde{t} = -t, \qquad \tilde{p}_n = \bar{p}_n, \qquad \tilde{q}_n = \bar{q}_n. \tag{2.60}$$

For chains with a complex structure which are invariant under the pair of involutions (2.46), (2.59) or (2.55), (2.60), the corresponding system (2.56) admits a complex reduction. For the chains (1.35), (1.36) the condition

$$\varepsilon q_{-n-1} = \bar{q}_n, \qquad \forall n \in \mathbb{Z}, \tag{2.61}$$

and for chains (1.38) the condition

$$\varepsilon p_{-n} = \bar{q}_n, \qquad \forall n \in \mathbb{Z}, \tag{2.62}$$

ensure that the function $u = q_0(x, t)$ (or $u = p_0(x, t)$ for (1.38)) satisfies the reduced equation

$$iu_t + u_{xx} = f(u, \bar{u}, iu_x, i\bar{u}_x). \tag{2.63}$$

Indeed, for (1.35) and (1.36), e.g., the pair of functions $u = q_0$, $v = \varepsilon q_{-1}$ satisfies the system of equations (2.56), and by (2.61) we have $v = \bar{u}$. Since these chains are invariant under the involutions defining the complex structure, it suffices to impose the conditions (2.61), (2.62) on the initial data. It is also clear that these conditions are compatible with periodic closure of the chains. For example, to construct finite-band solutions of the nonlinear Schrödinger equation

$$iu_t + u_{xx} = 2|u|^2 u$$

the condition $-q_{-n-1} = \bar{q}_n$ must be imposed on the initial data for the chains (2.10), (2.11) with $x = -i\tilde{x}$, $t = -i\tilde{t}$.

For scalar reductions the matter is more delicate. For example, the Korteweg-de Vries equation can be derived as a scalar reduction of the modified Toda chain (2.16). This chain admits the reduction $u_n = u_{-n-1}$, $v_n = -v_{-n}$. Since $v_0 = 0$, the function u_0 satisfies (see (2.18)) the equation

$$u_{0t} = u_{0xxx} + 6u_0 u_{0x}.$$

The relationship between the chain (2.16) and the Korteweg-de Vries equation was discussed in [2]. The example of the Toda chain (see (2.15)) shows that a regular chain of rank 2 can be used to construct solutions of the modified Korteweg-de Vries equation. As in the case of a complex reduction, the pair of chains (1.39), (1.40) must admit another involution in addition to (2.46). For the chains (1.35), (1.36) this involution is given by the formulas

$$\tilde{x} = -x, \qquad \tilde{t} = -t, \qquad \tilde{q}_n = \delta q_n, \qquad \delta = \pm 1. \tag{2.64}$$

Imposing the conditions

$$\varepsilon\delta q_{n-1} = q_n \tag{2.65}$$

on the initial data, we obtain a solution $u = q_0(x, t)$ of the reduced system (2.8). For instance, the Toda chain (2.10) admits both of the additional involutions (2.59) and (2.64) with $\delta = 1$. However, of the dual chains (2.11) and (2.12), only the latter is invariant under the involution (2.64) with $\delta = 1$. The condition (2.65) ($\varepsilon\delta = -1$) on the initial data for the chains (2.10), (2.12) leads to solutions of the modified Korteweg-de Vries equation

$$u_t = u_{xxx} + 6\kappa u^2 u_x, \tag{2.66}$$

where $\kappa = 1$. The pair of chains (2.10), (2.11) does not admit this reduction.

Equation (2.66) can be derived in the same way, as a reduction of the familiar Volterra chain [20],

$$q_{nx} = q_n(q_{n+1} - q_{n-1}). \tag{2.67}$$

The standard choice of variables $u = q_n$, $v = q_{n-1}$ for the chains (1.35) leads to the system of equations

$$u_t = [u_{xx} + 3(u+v)u_x + u^3 + 6u^2 v + 3uv^2]_x,$$
$$v_t = [v_{xx} - 3(u+v)v_x + v^3 + 6v^2 u + 3vu^2]_x,$$

whose scalar reduction ($v = -u$) is equation (2.66) with $\kappa = -1$.

A more complicated example of a scalar reduction is provided by the chain IV(a) in the table at the end of this article. For $Q(q_n) = \beta q_n^3 + \delta q_n$ this chain admits the involutions (2.46) and (2.64) with $\varepsilon = 1$ and $\delta = -1$, respectively. One can check that with an appropriate choice of dual chain, the scalar reduction (2.65) yields the familiar equation

$$u_t = u_{xxx} - 3(u_x u_{xx})/u + 3u_x^3/2u^2 - \tfrac{3}{16}(\beta u + \delta/u)^2 u_x$$

of the Korteweg-de Vries type. This equation is more often expressed in the gauge $u = \exp v$ [21]–[23].

The quasiregular chains considered in §2, 2.2, also admit complex and scalar reductions. For example, the pair of chains (2.42), (2.43) admits the scalar reduction $p_n = (-1)^n q_{-n}$, which takes the equation $\sin(w/2) = p_0(x, t)$ into the sine-Gordon equation. We note that periodic closure is possible here only for even periods. As in the case of a Toda chain, the reductions of the chain (2.30) lead to a nonlinear Schrödinger equation and modified Korteweg-de Vries equation.

In order to get a complex structure for the chain (2.31), (2.34) consistent with the complex structure for the Landau-Lifshits model (2.36), one must impose the constraint $\alpha - \mu = \beta + \varepsilon = 0$ on the coefficients of the polynomial r_n in (2.34). These chains then admit the additional complex reduction $\bar{p}_n = -(q_{-n-1})^{-1}$ ($\tilde{x} = ix$, $\tilde{t} = it$).

3. Conclusion

At the end of this paper we present a table in which we attempt to summarize the results of the classification for systems of partial differential equations of the form (2.44) which are integrable, and for the regular chains (1.35), (1.36), and (1.38). For a long time these two classification problems were regarded as independent (see, e.g., the review [9]). The idea of comparing the two lists arose quite recently, and the comparison was initially carried out along the lines discussed in the proof of Theorem 2.4. In this approach, one starts with the complete lists of integrable chians of the form (1.35), (1.36) obtained in the papers [24]–[26].

Corollary 2.3 of Theorem 2.2 in §2, 2.1, suggests another approach. For a given system of equations (2.44), one can directly find an invertible differential substitution (an explicit Bäcklund transformation) admitted by the system. It is not hard to see that all three cases (1.35), (1.36), and (1.38) correspond to differential substitutions of the form

$$u_{n+1} = U(u_n, v_n, v_{nx}), \qquad v_{n+1} = V(v_n, u_{n+1}, (u_{n+1})_x). \tag{3.1}$$

If the system of equations is associated with the chains (1.35), (1.36), then formulas (3.1) can be simplified substantially (see the examples in 2.1 at the end of §2). There are no difficulties in recovering the chain from the explicit Bäcklund transformation (3.1).

The table lists the key systems of equations from [9], [12], along with their corresponding regular chains.([3]) To the Landau-Lifshits equation (2.36), (2.37) corresponds to the chain II(a) with $P(u) = \alpha u^4 + \gamma u^2 + \alpha$, and the original complex structure is recovered by means of the reduction $\bar{q}_n = -1/q_{-n-1}$ (cf. the end of 2.3 in §2). The most complicated systems of equations occur for the scalar chains IV(a), V(a), and VI(a) in [24]. The system (2.18), whose reduction yields a nonlinear Schrödinger equation, corresponds to system I(b) with $P(a) = 1$, $y = \exp(u+v)$.

The theory of transformations developed for systems of the form (2.44) (see [9], [12]) carries over in a natural way to chains of equations [14]. For example, any system of the form

$$\begin{aligned} u_t &= u_{xx} + Au_x^2 + Bu_xv_x + Cu_x + Dv_x + E, \\ -v_t &= v_{xx} + Av_x^2 + Bu_xv_x - Cv_x - Du_x + E, \end{aligned} \tag{3.2}$$

where $A, B, \dots, E$ are functions of the variable $u+v$, reduces to system I(b) (see the table) after a substitution of the form

$$\tilde{u} + \tilde{v} = a(u+v), \qquad \tilde{u}_x = a'(u+v)u_x + b(u+v), \tag{3.3}$$

where $a'u_x + b$ $(a' \neq 0)$ is the density of the conservation law for system I(b). The variables $\tilde{u}, \tilde{v}$ in (3.3) pertain to the system (3.2). The functions a', b in (3.3) can be expressed linearly in terms of the function $y(u+v)$ appearing on the right-hand side of the system of equations I(b). The chains corresponding to (3.2) are obtained from I(a) by the change of variables

$$\begin{aligned} \tilde{p}_n + \tilde{q}_n &= c_1\Theta(q_n - q_{n-1}) + c_2(q_n - q_{n-1}), \\ \tilde{q}_n - \tilde{q}_{n+1} &= c_1[\Theta(q_n - q_{n-1}) + R(q_{nx})] + c_2(q_n - q_{n-1}) + c_3S(q_{nx}), \end{aligned} \tag{3.4}$$

where the functions Θ, R, S are specified by the relations $\Theta'(z) = y(z)$, $R'(z) = z/P(z)$, $S'(z) = 1/P(z)$, and c_1, c_2, and c_3 are constants. These chains are of the type (1.38), and one can pass from them to the systems (3.2) by choosing variables in accordance with the formulas $u = p_n$, $v = q_n$. Like I(b), the system of equations V(b) can be generalized by making changes of variables of the form (3.3). The chains (1.38) associated with these generalizations are obtained from V(a) by transformations analogous to (3.4).

([3])Systems of the Boussinesq type (see [9], [12]) are excluded from consideration.

Summarizing these results, we note that the chains listed in the table not only determine explicit Bäcklund transformations for the corresponding partial differential equations, but also contain complete information concerning the symmetries and conservation laws for these equations [27] (cf. 1.2 in §1).([4]) By Theorem 2.2, formulas (2.7) can be used to convert any dual chain into an evolution equation embodying a symmetry of the particular equation (2.44) considered.

Similarly, if $h_n = h(\mathbf{q}_{n+m}, \mathbf{q}_{n+m-1}, \dots)$ is the density of a local conservation law for a dual pair of chains (1.39), (1.40), then by Definition 1.4 there exist functions, ρ, σ such that

$$\partial_x h_n = \rho_{n+1} - \rho_n, \qquad \partial_t h_n = \sigma_{n+1} - \sigma_n. \tag{3.5}$$

Writing $c_n = \rho_{nt} - \sigma_{nx}$, we see that $c_{n+1} = c_n$, i.e., $c_n = c = \text{const}$. It thus follows from (3.5) that

$$\rho_{nt} = (\sigma_n + cx)_x. \tag{3.6}$$

It can be shown (see [27]) that $c = 0$ if the hypotheses of Theorem 2.4 hold and $\varepsilon = 1$ in (2.46). Passing to "new dynamical variables" in relation (3.6), for $c = 0$ we get the local conservation law

$$\partial_t \rho(\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \dots) = \partial_x \sigma(\mathbf{u}, \mathbf{u}_1, \dots) \tag{3.7}$$

for the evolution equation (2.44).

As an illustration of the foregoing analysis, we consider in more detail the chain (2.67) and the system of equations III(b) corresponding to it, with $P(z) = Q(z) = z$. An obvious consequence of (2.67) is the relations

$$\begin{aligned} q_{nx} &= a_{n+1} - a_n, \qquad a_n = q_n q_{n-1}, \\ (\ln q_n)_x &= b_{n+1} - b_n = c_{n+1} + c_n, \\ b_n &= q_n + q_{n-1}, \qquad c_n = q_n - q_{n-1}. \end{aligned} \tag{3.8}$$

Switching to the variables (2.48), we find from (3.5) the densities for the conservations laws (3.7) of a corresponding system of type III(b): $a = uv$, $b = u + v$. It is clear that the derivation of (3.7) also generalizes to "nonstandard" conservation laws of the form

$$\partial_x h_n = \rho_{n+1} + \rho_n, \qquad \partial_t h_n = \sigma_{n+1} + \sigma_n. \tag{3.9}$$

In particular, the missing conservation law with the density $\rho = u - v$ can be obtained from $c = q_0 - q_{-1}$ (see (3.8)).

In going from (2.67) to III(a), the nonstandard conservation law (3.9) takes on the standard form (3.5). The change of variables is given by $\tilde{p}_n = q_{2n}$, $\tilde{q}_n = q_{2n-1}$, and the density $\tilde{h}_n$ for the conservation law of the corresponding chain of type III(a) is constructed from the density h_n for the chain (2.67) by $\tilde{h}_n = h_{n+1} - h_n$.

([4])This permits us, in particular, to state that all the systems (2.44) in the reviews [9], [12], including IV(b) and VI(c), possess higher-order conservation laws (cf. the remark in 2.2, §2).

Table

I

(a) $$q_{nxx} = P(q_{nx})[y(q_{n+1} - q_n) - y(q_n - q_{n-1})]$$

(b) $$\begin{cases} u_t = u_{xx} + 2P(u_x)y(u+v) + \gamma u_x^2 \\ -v_t = v_{xx} + 2P(-v_x)y(u+v) + \gamma v_x^2 \end{cases}$$

$$P(a) = \varepsilon a^2 + \alpha a + \beta, \quad y' = \varepsilon y^2 + \gamma y + \delta$$

II

(a) $$-q_{nxx} = \tfrac{1}{2}P'(q_n) + [P(q_n) + q_{nx}^2][(q_{n+1} - q_n)^{-1} - (q_n - q_{n-1})^{-1}]$$

(b) $$\begin{cases} u_t = u_{xx} - 2[u_x^2 + P(u)]/(u-v) + \tfrac{1}{2}P'(u) \\ -v_t = v_{xx} - 2[v_x^2 + P(v)]/(v-u) + \tfrac{1}{2}P'(v) \end{cases} \qquad (P^{(5)} = 0)$$

III

(a) $$p_{nx} = P(p_n)(q_{n+1} - q_n), \qquad q_{nx} = Q(q_n)(p_n - p_{n-1})$$

(b) $$u_t = u_{xx} + [2P(u)v + \gamma u^2]_x, \quad -v_t = v_{xx} - [2Q(v)u + \alpha v^2]_x$$

(c) $$p_{nx} = \varphi(q_{n+1} - q_n), \qquad q_{nx} = \psi(p_n - p_{n-1})$$

(d) $$u_t = u_{xx} + 2P(u_x)v_x + \gamma u_x^2, \quad -v_t = v_{xx} - 2Q(v_x)u_x - \alpha v_x^2$$

$$P(a) = \varepsilon a^2 + \alpha a + \beta, \qquad Q(a) = \varepsilon a^2 + \gamma a + \delta$$

$$\varphi' = P(\varphi), \qquad \psi' = Q(\psi)$$

IV

(a) $$q_{nx} = Q(q_n)[(q_{n+1} - q_n)^{-1} + (q_n - q_{n-1})^{-1}]$$

(b) $$\begin{cases} u_t = u_{xx} - 2u_x^2(u-v)^{-1} + 2[r(u,v)u_x - Q(u)v_x](u-v)^{-2} \\ -v_t = v_{xx} - 2v_x^2(v-u)^{-1} - 2[r(v,u)v_x - Q(v)u_x](v-u)^{-2} \end{cases}$$

$$Q(u) = \alpha u^4 + \beta u^3 + \gamma u^2 + \delta u + \varepsilon$$

$$r(u,v) = 2\alpha u^2 v^2 + \beta uv(u+v) + 2\gamma uv + \delta(u+v) + 2\varepsilon$$

V

$$\text{(a)}\qquad q_{nx} = -2[y(q_{n+1}+q_n) - y(q_n+q_{n-1})]^{-1}$$
$$y' = P(y) = \alpha y^4 + \beta y^3 + \gamma y^2 + \delta y + \varepsilon$$
$$\text{(b)}\quad \begin{cases} u_t = u_{xx} + P(y)u_x^2 v_x + P'(y)u_x^2 - \frac{2}{3}[P''(y) - 2\gamma]u_x + \frac{1}{3}P'''(y) \\ -v_t = v_{xx} - P(y)v_x^2 u_x + P'(y)v_x^2 + \frac{2}{3}[P''(y) - 2\gamma]v_x + \frac{1}{3}P'''(y) \\ y = y(u+v) \end{cases}$$

VI

$$\text{(a)}\qquad q_{nx} = (q_{n+1} - q_{n-1})^{-1}[R_n + \nu(r_n r_{n+1})^{1/2}], \quad \nu = 0, 1$$
$$R_n = R(q_{n+1}, q_n, q_{n-1}), \qquad r_n = r(q_n, q_{n-1})$$
$$R(a, b, c) = (\alpha b^2 + 2\beta b + \gamma)ac + (\beta b^2 + \mu b + \delta)(a + c)$$
$$+\gamma b^2 + 2\delta b + \varepsilon, \qquad r(a, b) = R(a, b, c)$$
$$\text{(b)}\quad \begin{cases} u_t = u_{xx} - [u_x^2 + P(u)][(\ln r)_u + 2v_x r^{-1}] + \frac{1}{2}P'(u) \\ -v_t = v_{xx} - [v_x^2 + P(v)][(\ln r)_v - 2u_x r^{-1}] + \frac{1}{2}P'(v) \end{cases} \qquad (\nu = 0)$$
$$\text{(c)}\quad \begin{cases} u_t = u_{xx} - [u_x^2 + P(u)]v_x r^{-1} - (\ln r)_u^2 u_x - \frac{r}{2}(\ln r)_{uv} u_x \\ -v_t = v_{xx} + [v_x^2 + P(v)]u_x r^{-1} - (\ln r)_v v_x^2 + \frac{r}{2}(\ln r)_{uv} v_x \end{cases}$$
$$4P(u) = 2rr_{vv} - r_v^2, \qquad r = r(u, v) \qquad (\nu = 1)$$

REMARK. In this table we employ the standard transition formulas (see 2.3, §2) for the chains (1.35), (1.36), and (1.38). For the chain I(a), the transition is given by the formula $u = q_n$, $v = -q_{n-1}$; for the chains III(a) and (c), it is given by the formula $u = p_n$, $v = q_n$. The formula $u = q_m$, $v = q_{n-1}$ is used in the remaining cases.

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