Physics Letters A 174 (1993) 397–402
North-Holland

PHYSICS LETTERS A

Canonical transformations generated by shifts in nonlinear lattices

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Received 25 November 1992; accepted for publication 22 January 1993
Communicated by A.P. Fordy

The invertible differential substitutions which conserve the standard Poisson brackets and act on Hamiltonians in an appropriate way are considered. These canonical auto-Bäcklund transformations proved to be a very simple and efficient tool in the theory of solitons. In particular, they allow one to prove a general involutivity theorem and to build up simple formulae for soliton-like solutions.

We shall consider local transformations acting, in general, on the vector functions of several independent variables. The property of locality means that the transformation may be written down as

$$\boldsymbol{u}'=\vec{\varphi}[\boldsymbol{u}]=\vec{\varphi}(\boldsymbol{u},\boldsymbol{u}_x,\boldsymbol{u}_y,\ldots,\boldsymbol{u}_{xx},\ldots)\,, \tag{1}$$

where $\vec{\varphi}$ depends on $\boldsymbol{u}$ and its derivatives up to an order m taken at the same point. We call them differential substitutions of order m. It is quite surprising that in contrast to the scalar case there exist invertible differential substitutions of nonzero order. For example the first order substitution

$$u'=a(u)u_x+v\,,\quad v'=u \tag{2}$$

is evidently invertible with $u=v'$, $v=u'-a(v')v'_x$. As usual we shall denote by $\vec{\varphi}_*$ assotiated with (1) the matrix operator acting in the tangent space. Namely, if a function $\boldsymbol{u}$ in (1) depends on an additional parameter τ then this operator arises as the result of differentiation of (1) with respect to the parameter,

$$\begin{aligned}\boldsymbol{u}'_\tau&=\vec{\varphi}_*(\boldsymbol{u}_\tau)\\&=(\vec{\varphi}_{\boldsymbol{u}}+\vec{\varphi}_{\boldsymbol{u}x}\mathrm{D}_x+\vec{\varphi}_{\boldsymbol{u}y}\mathrm{D}_y+\ldots+\vec{\varphi}_{\boldsymbol{u}xx}\mathrm{D}_x^2+\ldots)(\boldsymbol{u}_\tau)\,.\end{aligned} \tag{3}$$

Two properties of the differential substitutions are basic in our discussion,

$$J[\vec{\varphi}[\boldsymbol{u}]]=\vec{\varphi}_*[\boldsymbol{u}]J[\boldsymbol{u}]\vec{\varphi}^{\mathrm{T}}_*[\boldsymbol{u}] \tag{4}$$

$$h[\vec{\varphi}[\boldsymbol{u}]]-h[\boldsymbol{u}]\in\mathrm{Ker}\,\delta/\delta\boldsymbol{u}\,. \tag{5}$$

Here J is a skew symmetric matrix differential operator $J^{\mathrm{T}}=-J$ and by the definition of the formally adjoint operator one has

$$\vec{\varphi}^{\mathrm{T}}_*=\vec{\varphi}^{\mathrm{T}}_{\boldsymbol{u}}-\mathrm{D}_x\circ\vec{\varphi}^{\mathrm{T}}_{\boldsymbol{u}x}-\mathrm{D}_y\circ\vec{\varphi}^{\mathrm{T}}_{\boldsymbol{u}y}+\ldots+\mathrm{D}_x^2\circ\vec{\varphi}^{\mathrm{T}}_{\boldsymbol{u}xx}+\ldots\,. \tag{6}$$

In formula (5) $\mathrm{Ker}\,\delta/\delta\boldsymbol{u}$ denotes the kernel of variational differentiation which is defined for a scalar function $f=f[\boldsymbol{u}]=f(\boldsymbol{u},\boldsymbol{u}_x,\boldsymbol{u}_y,\ldots,\boldsymbol{u}_{xx},\ldots)$ as usual by

$$\frac{\delta f}{\delta\boldsymbol{u}}=f_{\boldsymbol{u}}-\mathrm{D}_x(f_{\boldsymbol{u}x})-\mathrm{D}_y(f_{\boldsymbol{u}y})+\ldots+\mathrm{D}_x^2(f_{\boldsymbol{u}xx})+\ldots\,. \tag{7}$$

Properties (4) and (5) of transformation (1) in the sequel will provide, respectively, the invariance of the Poisson bracket related to the matrix operator J and the invariance of the Hamiltonian system associated with $h[\boldsymbol{u}]$ (see eqs. (22), (19) below).

At the beginning we consider just how much information can in fact be extracted from the invariance condition (4). We shall now suppose that $\boldsymbol{u}=(u, v)^{\mathrm{T}}$ and we will discuss the two standard symplectic structures related to

$$J_0=\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}, \quad J_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mathrm{D}_x . \qquad (8)$$

One can prove the following

Proposition 1. The invertible differential substitution of order one satisfies the invariance condition (4) with $J=J_1$ iff it can be reduced to

$$u'=a(u)u_x+\alpha v, \quad v'=\alpha^{-1}u+\beta, \quad \alpha, \beta\in\mathbb{C}$$

either by transposition $u\leftrightarrow v$ or by $u'\leftrightarrow v'$ or by both together.

The next example shows what a transformation can do if condition (4) is violated.

Example 1. Let us consider the first order substitution

$$u'=\frac{v_x}{u_x}, \quad v'=-v+uu'=-v+u\frac{v_x}{u_x} .$$

One obtains by differentiation that $v'_x=uu'_x$. Hence

$$u=\frac{v'_x}{u'_x}, \quad v=-v'+u'\frac{v'_x}{u'_x}$$

and there exists the function $h[u]=uv_x/u_x$ which is invariant under this transformation.

One may prove in addition to proposition 1 that in the case $J=J_0$ any first order substitution obeying (4) should be in fact the point transformation

$$u'=U(u, v) , \quad v'=V(u, v) .$$

For this we have the classical formula

$$\vec{\varphi}_* J_0\vec{\varphi}_*^{\mathrm{T}}=J_0 \det\frac{\partial(u', v')}{\partial(u, v)} . \qquad (9)$$

Summing up, one may verify easily, using proposition 1, that any invertible transformation

$$u'=\varphi(u, v, u_x, v_x) , \quad v'=\psi(u, v, u_x, v_x)$$

obeying (4), (8) can be reduced either to the first order substitution (2) or to the preserving area point transformation. It would be very interesting to generalize proposition 1 for the multi-field case. One important generalization of (2) will be given at the end of this paper (see eq. (41)).

The main source of known invertible second order substitutions provides the shift transformation

$$(u, v)=(q_n, q_{n-1})\mapsto(u', v')=(q_{n+1}, q_n) \qquad (10)$$

defined by the Toda chain equations and their generalization. We now give a few examples. Application of (10) to the Toda chain

$$q_{nxx}=\exp(q_{n+1}-q_n)-\exp(q_n-q_{n-1}) \qquad (11)$$

yields the substitution

$$u'=u+\log[u_{xx}+\exp(u-v)] , \quad v'=u .$$

For this substitution

$$\vec{\varphi}_* J_0\vec{\varphi}_*^{\mathrm{T}}=\exp(u-u'+v'-v)J_0 . \qquad (12)$$

Thus the invariance condition (4) is satisfied up to the point transformations. By comparison of (9) with (12) one can choose the new coordinates

$$u=\exp(q_n) , \quad v=\exp(-q_{n-1})$$
$$(u'=\exp(q_{n+1}), v'=\exp(-q_n))$$

and find the canonical transformation

$$u'=u_{xx}-u^{-1}u_x^2+u^2v , \quad v'=u^{-1} , \qquad (13)$$

which is generated by a shift in the Toda chain. It is useful to note that the above discussed substitution (2) is also generated by the shift (10) in the chain

$$a(q_n)q_{nx}=q_{n+1}-q_{n-1} . \qquad (14)$$

A two-dimensional generalization of (13)

$$u'=u_{xy}-u^{-1}u_xu_y+u^2v , \quad v'=u^{-1} \qquad (15)$$

is connected with the shift in the well-known two-dimensional Toda chain

$$q_{nxy}=\exp(q_{n+1}-q_n)-\exp(q_n-q_{n-1}) .$$

This gives an example of the invertible differential substitution in the case of two independent variables. The reader can readily verify that this shift transformation (15) is also canonical, i.e. the invariance condition (4) holds for (15).

The properties of superpositions of transformations (1) have great importance in the symmetry approach. Here we give a useful formula for the power of the substitution $\boldsymbol{u}'=\vec{\varphi}(\boldsymbol{u})$ defined by (15). Let us choose (u_0, v_0), $u_0=u(x, y)$, $v_0=0$ as the initial state and call u_n, v_n, $n=1, 2, \ldots$, the result of the iterations of transformation (15). Then

$$(u_n, v_n)=\vec{\varphi}^n(u, 0)\,,$$
$$u_n=w_n/w_{n-1}\,,\quad v_n=v_{n-1}^{-1}\,, \tag{16}$$

where $u_0=w_0=u(x, y)$, $w_1=uu_{xy}-u_xu_y$ and for general $n>0$

$$w_n=\det(\partial_x^i\partial_y^j u)\,,\quad i,j=0, 1, \ldots, n\,. \tag{17}$$

This formula provides the solution of the recursion problem

$$\partial_x\partial_y(\phi_n)=\exp(\phi_{n-1}-2\phi_n+\phi_{n-1})\,,$$
$$\phi_n=0\,,\quad n<0\,,\quad \phi_0=u(x,y)\,. \tag{18}$$

Namely one obtains from (18) that $\exp(\phi_1)=w_1$ and by induction that $\exp(\phi_n)=w_n$, $n=1, 2, \ldots$ (see ref. [1]).

Now we proceed to the discussion of the second condition (5) which has been formulated at the beginning of this paper. Firstly, we note that one may construct a formally Hamiltonian evolution equation

$$\boldsymbol{u}_t=J\frac{\delta h}{\delta \boldsymbol{u}} \tag{19}$$

by using any scalar function $h=[\boldsymbol{u}]=h(\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_{xx}, \ldots)$ and that the deformation of (19) under transformation (1) is defined in virtue of (3), (7) by the following formula,

$$\boldsymbol{u}_t'=\vec{\varphi}_*J\vec{\varphi}_*^{\mathrm{T}}\frac{\delta h'}{\delta \boldsymbol{u}'}=J'\frac{\delta h'}{\delta \boldsymbol{u}'}\,. \tag{20}$$

Here the new Hamiltonian density $h'(u')=h(u)$ is well defined if the transformation may be inverted. The foregoing formula proves

Proposition 2. Let J be skew symmetric (more refined properties of J related to the Jacobi identity are here irrelevant) and $\vec{\varphi}$ be the differential substitution obeying (4). Then the mapping $\boldsymbol{u}'=\vec{\varphi}[\boldsymbol{u}]$ transforms the solution of (19) into a solution of the same equation if the Hamiltonian satisfies condition (5).

We are now going to prove our main result. Let us denote

$$\mathrm{Ker}\,\vec{\varphi}=\{f\,|\,f[\vec{\varphi}(u)]-f[u]=0\}\,,$$
$$\widetilde{\mathrm{Ker}}\,\vec{\varphi}=\{f\,|\,f[\vec{\varphi}(u)]-f[u]\in\mathrm{Ker}\,\delta/\delta u\}\,. \tag{21}$$

It follows immediately from (19) that the Hamiltonian function h is the conserved density, i.e. $h_t\in\mathrm{Ker}\,\delta/\delta\boldsymbol{u}$. We now indicate the conditions which insure that any element of $\widetilde{\mathrm{Ker}}\,\vec{\varphi}$ is a conserved density for the Hamiltonian equation (19).

Theorem 1. Let the conditions of proposition 2 hold and $\mathrm{Ker}\,\vec{\varphi}\subset\mathrm{Ker}\,\delta/\delta\boldsymbol{u}$. Then $g\in\widetilde{\mathrm{Ker}}\,\vec{\varphi}\Rightarrow g_t\in\mathrm{Ker}\,\delta/\delta\boldsymbol{u}$.

Proof. Let us call

$$f=\{g, h\}=(\delta g/\delta\boldsymbol{u})^{\mathrm{T}}J(\boldsymbol{u})\delta h/\delta\boldsymbol{u}\,. \tag{22}$$

It follows from (19) and (22) that $g_t-f\in\mathrm{Ker}\,\delta/\delta\boldsymbol{u}$. Now we have to verify that $f\in\mathrm{Ker}\,\vec{\varphi}$. In virtue of (4) the bracket (22) is invariant under the substitution

$$\left(\frac{\delta g[\vec{\varphi}(\boldsymbol{u})]}{\delta\boldsymbol{u}}\right)^{\mathrm{T}}J(\boldsymbol{u})\,\frac{\delta h[\vec{\varphi}(\boldsymbol{u})]}{\delta\boldsymbol{u}}$$
$$=\left(\frac{\delta g[\vec{\varphi}(\boldsymbol{u}')]}{\delta\boldsymbol{u}'}\right)^{\mathrm{T}}J(\boldsymbol{u}')\,\frac{\delta h[\vec{\varphi}(\boldsymbol{u}')]}{\delta\boldsymbol{u}'}\,,\quad \boldsymbol{u}'=\vec{\varphi}[\boldsymbol{u}]\,.$$

Since g, $h\in\widetilde{\mathrm{Ker}}\,\vec{\varphi}$ one may replace $g[\vec{\varphi}(\boldsymbol{u})]$ and $h[\vec{\varphi}(\boldsymbol{u})]$ by $g[\boldsymbol{u}]$ and $h[\boldsymbol{u}]$ in the left-hand side of the above relation. This yields $f[\boldsymbol{u}]=f[\boldsymbol{u}']$. Thus $f\in\mathrm{Ker}\,\vec{\varphi}$ and the proof is completed.

The result just proved means that

$$\{g, h\}\in\mathrm{Ker}\,\delta/\delta\boldsymbol{u}\quad\forall g, h\in\widetilde{\mathrm{Ker}}\,\vec{\varphi}\,. \tag{23}$$

If J defines the symplectic structure and the Jacobi identity holds then (23) implies commutativity of the corresponding vector fields $J\delta g/\delta\boldsymbol{u}$, $J\delta h/\delta\boldsymbol{u}$ (see ref. [2]). Thus the following statement is valid.

Corollary. Let in addition to the conditions of theorem 1 the Jacobi identity hold. Then any evolution differentiations of type (19), (5) commutate.

Now we consider simple examples which should elucidate our discussion of discrete symmetries. Though this discrete symmetry approach may be accounted as self-contained it appears to be more convenient to incorporate this approach in the well-developed theory of solitons.

It can be verified directly that substitution (13) is the discrete symmetry of the nonlinear Schrödinger coupled system

$$\mathrm{i}u_t = u_{xx} + 2u^2 v\,, \quad -\mathrm{i}v_t = v_{xx} + 2v^2 u\,. \tag{24}$$

Namely (u', v') satisfies (24) as well as (u, v). Moreover, it can be proved that the following relation holds (cf. (5)),

$$h_j[u', v'] - h_j[u, v] = \mathrm{D}_x Q_j[u, v]\,, \quad j = 0, 1, 2, \ldots\,. \tag{25}$$

Here $h_0 = uv$, $h_1 = vu_x$, $h_2 = u^2v^2 - u_x v_x$, ... are the densities of the conservation laws for (24) and Q_j are the densities of the conservation laws for the Toda chain (11). The shifted variables $q_{n\pm1}$, $q_{n\pm2}$, ... are expressed in terms of Q_j by derivatives of $u = \exp(q_n)$, $v = \exp(-q_{n-1})$. For example

$$Q_0 = q_{nx} = (\ln u)_x\,,$$

$$Q_1 = \tfrac{1}{2}q_{nx}^2 + \exp(q_n - q_{n-1}) + q_{nxx} = (\ln u)_{xx} + \tfrac{1}{2}(\ln u)_x^2 + uv\,,$$

$$Q_2 = (\ln u)_x (\ln u)_{xx} + \tfrac{1}{3}(\ln u)_x^3 + 2vu_x\,.$$

We may now apply the above discussed general theory to substitution (13). The densities h_j of the conservation laws of (24) satisfy (5) by (25). Therefore substitution (13) is a discrete symmetry not only for (24) but for any equation

$$\mathrm{D}_{t_j}\boldsymbol{u} = J\frac{\delta h_j(\boldsymbol{u})}{\delta \boldsymbol{u}}\,, \quad j = 0, 1, 2, \ldots\,, \tag{26}$$

where the matrix $J = J_0$ (proposition 2). Furthermore, the commutativity of the differentiations (26) (i.e. $\mathrm{D}_{t_j}\mathrm{D}_{t_k} = \mathrm{D}_{t_k}\mathrm{D}_{t_j}$) is an immediate consequence of (25) (corollary to theorem 1). One needs only to verify that

$$f[\vec{\varphi}(\boldsymbol{u})] = f[\boldsymbol{u}] \Rightarrow f = \text{const}\,. \tag{27}$$

That is an easy problem for substitution (13).

Next we give a quite elementary exposition of the Hirota type formula for the N-soliton solution of (24). One can choose as starting point any solution (u_0, v_0) of (24) and obtain the new one by (13). We set $v_0 = 0$. Then the function u_0 satisfies the linear equation $\mathrm{i}u_{0t} = u_{0xx}$ and we have by (16), (17) the following explicit formulae for the iterations of (13),

$$u_n = w_n / w_{n-1}\,, \quad v_n = u_{n-1}^{-1}\,,$$

$$w_n = \det[\partial_x^{i+j} u_0(x, t)]\,, \quad i, j = 0, \ldots, n\,. \tag{28}$$

The formula of the N-soliton solution corresponds to the following special case,

$$u_0(x, t) = \sum_1^{2N} c_k \exp(\lambda_k x - \mathrm{i}\lambda_k^2 t)\,,$$

$$c_j c_{j'}^* = \prod_{k \neq j} (\lambda_j - \lambda_k)^{-2}\,. \tag{29}$$

Here the set λ_j, $j = 1, \ldots, 2N$, is supposed to be symmetric under reflection: $\lambda_j \mapsto \lambda_{j'} = -\lambda_j^*$ relative to the axis $\mathrm{Re}\,\lambda = 0$. In the case (29) choosing $n = N$ in (28) one obtains that $v_n = u_n^*$ and therefore $\mathrm{i}u_{nt} = u_{nxx} + 2|u_n|^2 u_n$. A slight modification in (29) leads to N-soliton solutions for other equations (26). It is interesting to notice that the choice of the function u_0 as the sum of m different exponentials results after m iterations in the reversed initial state ($u_m = 0$, $v_m \neq 0$). It has been shown in ref. [3] that the periodicity condition $u_0 = u_n$, $v_0 = v_n$ leads to finite-gap solutions of (24) (see also ref. [4]).

The class of PDEs with discrete symmetries is reasonably large. In particular, in ref. [3] one can find an exhaustive, in a certain sense, list of the systems

$$u_t = u_{xx} + f(u, v, u_x, v_x)\,,$$

$$-v_t = v_{xx} + g(u, v, u_x, v_x)\,, \tag{30}$$

which are invariant under second order substitutions $\boldsymbol{u}' = \vec{\varphi}(\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_{xx})$. Most of these systems can be put in the Hamiltonian form (19) with J either as in proposition 1 or

$$J = \exp[\theta(u, v)] J_0\,, \quad \theta_{uv}\exp(\theta) = 2\,. \tag{31}$$

In the case $J = J_1$,

$$h = u_x v + \alpha u^2 v^2 + \gamma(u^2 v + v^2 u) + \beta(u^2 + v^2)\,,$$

where α, β, $\gamma \in \mathbb{C}$. The corresponding discrete symmetry is defined by the shift in the chain (14) with $a^{-1}(q)=\alpha q^2+\gamma q+\beta$. In the case (31) the Hamiltonian has in general the form $h=\exp(-\theta)\times(u_xv_x+au_x+bv_x+c)$.

The Landau-Lifshitz model corresponds to a special choice of the solution of the Liouville equation in (31): $\exp(\theta)=(u-v)^2$. By that

$$\exp(\theta)h=u_xv_x+2\alpha u^2v^2+\beta(u^2v+uv^2)$$
$$+\gamma uv+\delta(u+v)+2\epsilon\,.$$

The discrete symmetry is generated by the shift (10) in the chain

$$q_{nxx}+\tfrac{1}{2}P'(q_n)+[P(q_n)+q_{nx}^2]$$
$$\times[(q_{n+1}-q_n)^{-1}-(q_n-q_{n-1})^{-1}]=0\,, \qquad (32)$$

where

$$P(q)=\alpha q^4+\beta q^3+\gamma q^2+\delta q+\epsilon\,,$$
$$P'(q)=4\alpha q^3+3\beta q^2+2\gamma q+\delta\,.$$

A family of generalizations of (24) is generated by a solution of the Liouville equation of the following form,

$$\theta(u,v)=\theta(u-v)\,,$$
$$\theta'^2\exp(\theta)=4+\delta\exp(\theta)\,,\quad \delta\neq 0\,.$$

In this case the substitution is defined by the shift (10) in the chain

$$q_{nxx}=P(q_{nx})\frac{\partial}{\partial q_n}[\theta(q_n,q_{n-1})+\theta(q_{n+1},q_n)]\,,$$
$$P(q)=-\tfrac{1}{2}q^2+\alpha q+\beta\,. \qquad (33)$$

It can be proved that property (27) still holds for substitutions generated by shifts in (32), (33). Therefore theorem 1 may be used for the above discussed generalizations of the nonlinear Schrödinger system. Formula (33) suggests the invariant form of the generalized Toda chains related to the symplectic structure (31). For example, consider the chain (32) with $P=0$. One can start by any solution of the Liouville equation $\theta_{uv}\exp(\theta)=2$ and check out the invariance of the chain

$$2\frac{q_{nxx}}{q_{nx}^2}+\frac{\partial}{\partial q_n}[\theta(q_n,q_{n-1})+\theta(q_{n+1},q_n)]=0\,,$$
$$\theta_{uv}\exp(\theta)=2$$

under the point transformations $q_n'=Q(q_n)$.

Next, we would like to account briefly (cf. ref. [5]) the matrix generalization of substitutions (2), (15). The following formula generalizes the Toda shift (13),

$$u'=u_{xx}-u_xu^{-1}u_x+uvu\,,\quad v'=u^{-1}\,. \qquad (34)$$

In this formula u, v are noncommutative variables which we suppose to be $N\times N$ nondegenerate matrices. One may verify directly that the invariance condition (4) holds with the matrix J which is a $2N\times 2N$ analog of J_0:

$$J=\begin{pmatrix}0 & -\mathrm{I}\\ \mathrm{I} & 0\end{pmatrix}. \qquad (35)$$

Though

$$u'v'-uv=\mathrm{D}_x(u_xu^{-1})$$

the trace operation is needed for the generalization of (25). In particular for $h=\mathrm{Tr}[u_xv_x-(uv)^2]$ one obtains that

$$h'-h=\mathrm{D}_x\,\mathrm{Tr}[u'v_x-u_xv-\tfrac{1}{3}(u_xu^{-1})^3]\,.$$

The matrix analog of (24) related to this Hamiltonian density h has the following form,

$$\mathrm{i}u_t=u_{xx}+2uvu\,,\quad -\mathrm{i}v_t=v_{xx}+2vuv\,. \qquad (36)$$

Naturally, this system may be put in the Hamiltonian form (19)

$$\begin{bmatrix}u\\ v\end{bmatrix}_t=J\begin{bmatrix}\delta h/\delta u\\ \delta h/\delta v\end{bmatrix}. \qquad (37)$$

Let us notice that the above "matrix variational derivatives" are well defined only for functionals which may be written down as a trace of some $N\times N$ matrix. Namely for $f[u]=\mathrm{Tr}\,F[u]$ the matrix $\delta f/\delta u$ was defined from the equation $\mathrm{d}f[u]/\mathrm{d}\tau=\mathrm{Tr}(u_\tau\delta f/\delta u)$. By these conditions the analog of the bracket (22) takes the usual form

$$\{f,g\}=\mathrm{Tr}\left(\left[\frac{\delta f}{\delta u},\frac{\delta f}{\delta v}\right]J\begin{bmatrix}\delta g/\delta u\\ \delta g/\delta v\end{bmatrix}\right). \qquad (38)$$

One may easily check that the second order substitution (34) corresponds to a shift in the matrix Toda chain

$$(w_{nx}w_n^{-1})_x=w_{n+1}w_n^{-1}-w_nw_{n-1}^{-1}\,. \qquad (39)$$

Another matrix chain

$$w_{nx}=w_n(w_{n+1}-w_{n-1})w_n \tag{40}$$

generates the first order substitution

$$u'=u^{-1}u_xu^{-1}+v, \quad v'=u. \tag{41}$$

This substitution (cf. proposition 1) satisfies the invariance condition (4) with

$$J=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}D_x \tag{42}$$

and defines the variational discrete symmetry for the matrix evolution system of equations

$$u_t=u_{xx}+2(uvu)_x, \quad v_t=-v_{xx}+2(vuv)_x, \tag{43}$$

which corresponds to $h=\mathrm{Tr}[u_xv+(uv)^2]$ in (37).

A large family of 2+1 integrable Hamiltonian systems can be introduced and investigated by means of the two-dimensional generalization of the above discussed canonical differential substitutions. For example, along the same lines as in the 1+1 case (see ref. [3]), the two-dimensional generalization (15) of the Toda shift (13) results in the Davey–Stewartson coupled system of equations (see ref. [6])

$$\mathrm{i}u_t=u_{xx}+u_{yy}+u(D_x^{-1}D_y+D_y^{-1}D_x)(uv),$$

$$-\mathrm{i}v_t=v_{xx}+v_{yy}+v(D_x^{-1}D_y+D_y^{-1}D_x)(uv). \tag{44}$$

This system is not unique because the variety of symmetries of the two-dimensional Toda chain generates also more elementary but not so symmetrical systems (cf. ref. [7]). It can be verified straightforwardly that the Davey–Stewartson system (44) is invariant under substitution (15) but in applications of formulae (16), (17) one should overcome the ambiguity related to the equality $D_y^{-1}D_xv_{xy}=v_{xx}$ on which the invariance is based. The four-dimensional self-duality system can also be included in the theory. One can find an explicit form of the differential substitutions for these equations in ref. [8].

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