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Dilation symmetries and equations on the lattice

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Abstract. We discuss the role of dilation symmetries for differential difference equations depending on nearest-neighbour interactions. In particular, we show that for a simple class of differential difference equations of this kind, symmetries which depend linearly on time are only compatible with linearizable equations.

1. Introduction

The concept of similarity transformations has proved very useful in many scientific and technological applications [1–5] involving differential equations. They are associated with a dilation symmetry of the equation and appear in many differential equations of physical interest.

The situation is different when we consider the case of equations on the lattice. In this case there is an extensive literature concerning symmetries of linear difference equations [6, 7] (and also, by carrying out the continuous limit in one of the discrete variables, differential difference equations) and of integrable nonlinear differential difference equations [8]. In the first instance we can almost always construct a finite-dimensional algebra of symmetries which, in the continuous limit, corresponds to the Lie point symmetry algebra of the corresponding differential equation, while in the second case we are able to construct a denumerable number of generalized symmetries starting from the trivial ones and using the recurrence operator associated with the integrable hierarchy. In all these cases the symmetries are obtained using the properties of the equation under study, be it linearity or integrability. In the case of differential equations the existence of Lie point symmetries allow us to construct group transformations by integration of the infinitesimal generator and to obtain particular solutions by symmetry reduction. This is not the case for differential difference equations as whenever we have a shifted variable in the symmetry we are no longer, in general, able to integrate it to obtain group transformations. We are, in principle, always able to carry out a symmetry reduction. However, this may be very complicated in the case of generalized symmetries for nonlinear differential difference equations.

If we start from a generic equation on the lattice, we can easily obtain intrinsic point symmetries [9–12], i.e. symmetries which depend only on the dependent variables in the generic point n of the lattice. When we try to extend the procedure to the case of neighbouring points we get into trouble as the machinery becomes too complicated to be able to extract a reasonable result in most cases [13]. Moreover, while intrinsic symmetries always form a finite Lie algebra, this is not the case in general for the non-intrinsic ones [8].

In this paper we extend the results presented in [14] by considering symmetries with some explicit t dependence. We do not consider, however, the general problem but we consider just the case of symmetries defined on a finite range in all of its variables, be they continuous or discrete, which in the continuous limit go over to dilation symmetries,

$$u_{n,\tau} = tu_{n,t} + h_n(u_{n+1}, u_n, u_{n-1}) \tag{1}$$

i.e. we consider the simplest extension from the intrinsic symmetries. Here and in the following the function u_n depends on a continuous variable t, on the discrete variable n and on the symmetry variable τ. As in [14], for the sake of simplicity we limit ourselves to equations which depend on nearest-neighbour interactions,

$$u_{n,t} = f_n(u_{n+1}, u_n, u_{n-1}) \tag{2}$$

$$\frac{\partial f_n}{\partial u_{n+1}} \neq 0 \qquad \frac{\partial f_n}{\partial u_{n-1}} \neq 0 \quad \forall\, n \tag{3}$$

as these are the more interesting from the physical point of view. Moreover, in the case of integrable equations belonging to the class (2) and (3), one has constructed symmetries linear in t which, however, turn out to be generalized symmetries [8], more specifically master symmetries.

The main content of this paper is a theorem in which we show that any equation of class (2) and (3) which has a symmetry of the form (1), up to point transformations, can be reduced to a linearizable equation. The presentation of this theorem and its proof is contained in section 2. Section 3 is devoted to some conclusions.

2. Dilation symmetries for differential difference equations

We show here that if an equation of the form (2) and (3) has a symmetry of the form (1) then it is linearizable. Before presenting the corresponding theorem we can show that we can simplify the class of symmetries under study by taking into account equations (2) and (3). Let us consider the commutator

$$u_{n,t\tau} - u_{n,\tau t} = f_{n,\tau} - (tf_n + h_n)_t = \sum_i \frac{\partial f_n}{\partial u_{n+i}}(tf_{n+i} + h_{n+i}) - f_n - tf_{n,t} - h_{n,t}.$$

Then the compatibility condition $u_{n,t\tau} - u_{n,\tau t} = 0$ implies

$$u_{n,ty} - u_{n,yt} = f_{n,y} - h_{n,t} = f_n \tag{4}$$

where the variable y is introduced through the equation $u_{n,y} = h_n$. Condition (4) means that the equation $u_{n,y} = h_n$ is a trivial master symmetry of (2). In our calculation, we will use relation (4) extensively.

The number of variables appearing in h_n can be easily reduced. In fact, differentiating (4) with respect to u_{n-2}, we obtain

$$\frac{\partial f_n}{\partial u_{n-1}} \frac{\partial h_{n-1}}{\partial u_{n-2}} = \frac{\partial h_n}{\partial u_{n-1}} \frac{\partial f_{n-1}}{\partial u_{n-2}}$$

which is equivalent to

$$(D-1)\left(\frac{\partial h_n}{\partial u_{n-1}} \Big/ \frac{\partial f_n}{\partial u_{n-1}}\right) = 0$$

where D is the shift operator. It is not difficult to see that only constants can belong to the kernel of the operator $D-1$. Then

$$\frac{\partial h_n}{\partial u_{n-1}} = c\frac{\partial f_n}{\partial u_{n-1}} \tag{5}$$

where c is a constant. Consequently, h_n can be written as

$$h_n = cf_n + \tilde{h}_n(u_{n+1}, u_n).$$

The addition of cf_n to h_n corresponds to the addition of c to t in the symmetry (1). Then we can set $c=0$ and consider instead of (1), symmetries of the form

$$u_{n,\tau} = tu_{n,t} + h_n(u_{n+1}, u_n). \tag{6}$$

The case when h_n depends only on u_n corresponds to the point symmetry. So we require that

$$\frac{\partial h_n}{\partial u_{n+1}} \neq 0 \quad \text{for some } n. \tag{7}$$

Then we are able to present our main theorem.

Theorem 1. *If a nonlinear equation of the form (2) and (3) has a symmetry of the form (6) and (7), then it is equivalent, up to point transformations*

$$\tilde{t} = \omega t \qquad \tilde{u}_n = \varphi_n(u_n) \qquad \omega \neq 0 \qquad \varphi_n' \neq 0 \quad \forall\, n \tag{8}$$

to an equation of the form

$$\begin{aligned} &u_{n,t} = A_n + B_n \\ &A_n = a_{n+1}\mathrm{e}^{u_{n+1}} - a_n\mathrm{e}^{u_n} - 1 \qquad B_n = a_n\mathrm{e}^{-u_n} - a_{n-1}\mathrm{e}^{-u_{n-1}} - 1 \\ &a_n^2 = n^2 + \alpha n + \beta \neq 0 \qquad \forall\, n \end{aligned} \tag{9}$$

where α and β are arbitrary constants. The symmetry of equation (9) is

$$u_{n,\tau} = tu_{n,t} + A_n \tag{10}$$

and equation (9) is linearizable.

Proof. Let us show that equation (10) is linearizable. In fact, introducing the new dependent variable w_n

$$u_n = \log\frac{w_{n+1}}{w_n} \tag{11}$$

we are led to the linear equation

$$w_{n,t} = a_n w_{n+1} + a_{n-1}w_{n-1} + (c-2n)w_n \tag{12}$$

where $c = c(t)$ is an arbitrary integration function.

In the same way as we derived (5), we easily can obtain the following formula:

$$\frac{\partial h_n}{\partial u_{n+1}} = \varepsilon\frac{\partial f_n}{\partial u_{n+1}}.$$

Due to (7) $\varepsilon \neq 0$. Therefore, using the transformation $\tilde{t} = t/\varepsilon$, we can make $\varepsilon = 1$. Consequently, we can reduce f_n to the form

$$f_n = h_n + g_n \qquad g_n = g_n(u_n, u_{n-1}) \qquad \frac{\partial h_n}{\partial u_{n+1}}\frac{\partial g_n}{\partial u_{n-1}} \neq 0 \quad \forall\, n. \tag{13}$$

Rewriting (4) in terms of h_n, g_n:

$$f_{n,y} - h_{n,t} = (h_n + g_n)_y - \sum_i \frac{\partial h_n}{\partial u_{n+i}}(h_{n+i} + g_{n+i})$$

we obtain

$$\sum_i \left(\frac{\partial g_n}{\partial u_{n+i}} h_{n+i} - \frac{\partial h_n}{\partial u_{n+i}} g_{n+i} \right) = h_n + g_n. \tag{14}$$

Applying the operator $\partial^2/\partial u_{n+1}^2 \partial u_{n-1}^2$, we are led to an equation

$$\frac{\partial^2 g_n}{\partial u_n \partial u_{n-1}} \frac{\partial h_n}{\partial u_{n+1}} = \frac{\partial^2 h_n}{\partial u_{n+1} \partial u_n} \frac{\partial g_n}{\partial u_{n-1}}$$

i.e.

$$\frac{\partial}{\partial u_n} \log \frac{\partial h_n}{\partial u_{n+1}} = \frac{\partial}{\partial u_n} \log \frac{\partial g_n}{\partial u_{n-1}} = p_n$$

where p_n is a function which can depend only on u_n. Integrating and exponentiating, we obtain

$$\frac{\partial h_n}{\partial u_{n+1}} = A'_{n+1} C_n \qquad \frac{\partial g_n}{\partial u_{n-1}} = C_n B'_{n-1}$$

where A_n, B_n and C_n are some functions depending only on u_n. Consequently, we have

$$h_n = A_{n+1} C_n + R_n \qquad g_n = C_n B_{n-1} + S_n \tag{15}$$

where also R_n and S_n are functions depending only on u_n and, according to (13),

$$A'_n B'_n C_n \neq 0 \qquad \forall\, n. \tag{16}$$

Taking into account the point transformations (8), we can set

$$C_n = 1 \qquad \forall\, n. \tag{17}$$

Let us now find the functions A_n, B_n, R_n and S_n. To do so we can rewrite equation (14) in the following way:

$$\begin{aligned} A_n B'_{n-1} - R'_n B_{n-1} + R_{n-1} B'_{n-1} - B_n A'_{n+1} + S'_n A_{n+1} - S_{n+1} A'_{n+1} + R_n S'_n - S_n R'_n \\ = A_{n+1} + B_{n-1} + R_n + S_n. \end{aligned} \tag{18}$$

Differentiating equation (18) with respect to u_{n-1} and then dividing by B'_{n-1}, we obtain

$$A_n (\log B'_{n-1})' - R'_n + R'_{n-1} + R_{n-1} (\log B'_{n-1})' = 1 \tag{19}$$

and by differentiating it again with respect to u_n we have

$$A'_n (\log B'_{n-1})' = R''_n. \tag{20}$$

From equation (20) we see that $(\log B'_n)'' = 0$ for any n, i.e. B_n has to satisfy the following differential equation:

$$B''_n = \beta_n B'_n \tag{21}$$

where β_n is an n-dependent constant. Then from equation (20) we obtain

$$R'_n = \beta_{n-1} A_n + \rho_n \tag{22}$$

where ρ_n is another n-dependent constant and thus the relation (19) can be rewritten as

$$\beta_n R_n + \beta_{n-1} A_n = \rho_{n+1} - \rho_n + 1. \tag{23}$$

In a similar way (by differentiating (18) with respect to u_{n+1}), we are led to three more differential equations:

$$A_n'' = \alpha_n A_n' \tag{24}$$

$$S_n' = \alpha_{n+1} B_n + \sigma_n \tag{25}$$

$$\alpha_n S_n + \alpha_{n+1} B_n = \sigma_{n-1} - \sigma_n - 1 \tag{26}$$

where α_n and σ_n are n-dependent integration constants. We can solve the ordinary differential equations (21) and (24) and obtain

$$A_n' = a_n \mathrm{e}^{\alpha_n u_n} \qquad B_n' = b_n \mathrm{e}^{\beta_n u_n} \qquad a_n b_n \neq 0 \quad \forall\, n.$$

To integrate them once more, it is convenient to represent α_n, β_n in the following way:

$$\alpha_n = \mu_n \alpha_n^* \qquad \alpha_n^* \neq 0 \qquad \mu_n^2 = \mu_n \quad \forall\, n \tag{27}$$

$$\beta_n = \nu_n \beta_n^* \qquad \beta_n^* \neq 0 \qquad \nu_n^2 = \nu_n \quad \forall\, n. \tag{28}$$

So we obtain

$$A_n = \mu_n \frac{a_n}{\alpha_n^*} \mathrm{e}^{\alpha_n^* u_n} + (1 - \mu_n) a_n u_n + \tilde{a}_n \tag{29}$$

$$B_n = \nu_n \frac{b_n}{\beta_n^*} \mathrm{e}^{\beta_n^* u_n} + (1 - \nu_n) b_n u_n + \tilde{b}_n \tag{30}$$

and, taking into account (22) and (25),

$$R_n = \beta_{n-1} \left(\mu_n \frac{a_n}{(\alpha_n^*)^2} \mathrm{e}^{\alpha_n^* u_n} + (1 - \mu_n) \frac{a_n}{2} u_n^2 \right) + \tilde{\rho}_n u_n + \hat{\rho}_n \tag{31}$$

$$S_n = \alpha_{n+1} \left(\nu_n \frac{b_n}{(\beta_n^*)^2} \mathrm{e}^{\beta_n^* u_n} + (1 - \nu_n) \frac{b_n}{2} u_n^2 \right) + \tilde{\sigma}_n u_n + \hat{\sigma}_n \tag{32}$$

where

$$\tilde{\rho}_n = \beta_{n-1} \tilde{a}_n + \rho_n \qquad \tilde{\sigma}_n = \alpha_{n+1} \tilde{b}_n + \sigma_n.$$

Form equations (23) and (26), collecting coefficients at exponents and of powers of u_n, we obtain the following constraints:

$$\beta_{n-1}(\alpha_n + \beta_n) = 0 \qquad\qquad \alpha_{n+1}(\alpha_n + \beta_n) = 0 \tag{33}$$

$$\beta_n \tilde{\rho}_n + a_n (1 - \mu_n) \beta_{n-1} = 0 \qquad\qquad \alpha_n \tilde{\sigma}_n + b_n (1 - \nu_n) \alpha_{n+1} = 0 \tag{34}$$

$$\beta_n \hat{\rho}_n + \tilde{\rho}_n - \tilde{\rho}_{n+1} + \beta_n \tilde{a}_{n+1} - 1 = 0 \tag{35}$$

$$\alpha_n \hat{\sigma}_n + \tilde{\sigma}_n - \tilde{\sigma}_{n-1} + \alpha_n \tilde{b}_{n-1} + 1 = 0. \tag{36}$$

Using equations (33)–(36) we can simplify the formulae for A_n, B_n, R_n and S_n. From equations (33) and (34), taking into account equations (27) and (28), we obtain

$$\mu_n = \nu_n = \lambda \tag{37}$$

for all n, where λ is a constant. As $\lambda^2 = \lambda$ (see equations (27) and (28)), there are only two possibilities. If $\lambda = 0$, we have a linear equation (see equations (29)–(32)). Then $\lambda = 1$. The constraints (33)–(36) imply

$$\beta_n = -\alpha_n \neq 0 \qquad \tilde{\rho}_n = \tilde{\sigma}_n = 0$$

for any n, and

$$\hat{\rho}_n = -\tilde{a}_{n+1} - 1/\alpha_n \qquad \hat{\sigma}_n = -\tilde{b}_{n-1} - 1/\alpha_n.$$

So we have obtained an equation (2) and its symmetry (1) with the following definitions for f_n and h_n:

$$f_n = \frac{a_{n+1}}{\alpha_{n+1}} e^{\alpha_{n+1}u_{n+1}} - \alpha_{n-1}\frac{a_n}{\alpha_n^2} e^{\alpha_n u_n} + \alpha_{n+1}\frac{b_n}{\alpha_n^2} e^{-\alpha_n u_n} - \frac{b_{n-1}}{\alpha_{n-1}} e^{-\alpha_{n-1}u_{n-1}} - \frac{2}{\alpha_n}$$

$$h_n = \frac{a_{n+1}}{\alpha_{n+1}} e^{\alpha_{n+1}u_{n+1}} - \alpha_{n-1}\frac{a_n}{\alpha_n^2} e^{\alpha_n u_n} - \frac{1}{\alpha_n}.$$

By performing the following point transformation $\tilde{u}_n = \alpha_n u_n$ and redefining a_n and b_n, we will have

$$f_n = a_{n+1} e^{u_{n+1}} - a_n e^{u_n} + b_n e^{-u_n} - b_{n-1} e^{-u_{n-1}} - 2$$

$$h_n = a_{n+1} e^{u_{n+1}} - a_n e^{u_n} - 1.$$

A point transformation of the form $\tilde{u}_n = u_n + c_n$ allows one to make $b_n = a_n$. Inserting the results obtained so far into (4), we can check that it is satisfied iff

$$a_{n+1}^2 - 2a_n^2 + a_{n-1}^2 = 2.$$

This last equation can be easily solved, and its general solution completes the proof of the theorem. □

3. Conclusion

This paper has been a further step in the characterization of symmetries for nonlinear differential difference equations. Here we have shown that among equations (2) and (3) only equation (9) has a symmetry of the form (1). Moreover, equation (9) is linearizable. Thus it has been proved that when we consider symmetries of the form (1) then the equation under study cannot be integrable (see also [8]). However, not all linearizable equations of this class have this kind of symmetry. In fact, we can construct an equation

$$u_{n,t} = (u_{n+1} - u_n)^{1/2}(u_n - u_{n-1})^{1/2} \tag{38}$$

which belongs to the class (2) and (3) and which is linearizable via the transformation

$$v_n = (u_{n+1} - u_n)^{1/2} \tag{39}$$

to the equation

$$v_{n,t} = \tfrac{1}{2}(v_{n+1} - v_{n-1}). \tag{40}$$

Clearly, equation (38) is not transformable via a point transformation (8) to equation (9).

The symmetries (1) considered in this paper depend essentially on shifted variables as otherwise this would not be a dilation symmetry in the continuous limit. In such a case we are not able to integrate it and obtain a group transformation. Work is in progress for constructing symmetries depending on shifted variables but which can be integrated to provide a group of transformations.

The form of the symmetry (1) can be extended, by still keeping its compatibility with a dilation symmetry. Work on this is in progress.

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