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Generalized symmetry integrability test for discrete equations on the square lattice

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Abstract

We present an integrability test for discrete equations on the square lattice, which is based on the existence of a generalized symmetry. We apply this test to a number of equations obtained in different recent papers. As a result we prove the integrability of seven equations which differ essentially from the Q_V equation introduced by Viallet and thus from the Adler–Bobenko–Suris list of equations contained therein.

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1. Introduction

As is well known, the generalized symmetry method allows one to classify integrable equations of a certain class and to test a given equation for integrability [24, 25]. In the case of 1 + 1 partial differential equations, it has been used to develop a computer PC-package DELiA, written in Turbo PASCAL by Bocharov [6]. This program can be used to prove integrability and compute symmetries of given evolutionary partial differential equations. The symmetry approach has also been applied with success to study the integrability of differential difference equations [3, 41]. Here we apply, using the theoretical results contained in [22], the method to partial difference equations defined on a quad-graph. We will consider autonomous equations of the form

$$\mathcal{E}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = 0, \tag{1.1}$$

where n, m are arbitrary integers, i.e. autonomous equations which have no explicit dependence on the point (n, m) of the lattice and consequently are invariant with respect to n and

m translations. Following [22], we use as the integrability criterion the existence of an autonomous five-point generalized symmetry:

$$u_{n,m,t} = \mathcal{G}(u_{n+1,m}, u_{n-1,m}, u_{n,m}, u_{n,m+1}, u_{n,m-1}), \tag{1.2}$$

where t denotes the group parameter. Due to the complexity of the discrete case, we limit ourselves to the simplest nontrivial case given by symmetries of the form (1.2). Moreover we assume that the symmetry is autonomous as so is the equation. As we shall see, autonomous symmetries of the form (1.2) are general enough to cover a wide class of integrable equations. Theoretically, however, an integrable equation may have a symmetry depending on more lattice points and having lattice-dependent coefficients.

We assume that nontrivial nonlinear partial difference equations which have generalized symmetries of the form (1.2) are integrable. However, there might be classes of nonlinear equations contained in (1.1) which are integrable but have no symmetry of the form (1.2).

We define as *trivial* a nonlinear lattice equation (1.1) which is factorizable or reducible by summation to a simpler equation, maybe non-autonomous, which depends on a lower number of lattice points. Example of trivial equations are

$$\mathcal{E} = \omega(u_{n,m}, u_{n,m+1})\omega'(u_{n+1,m}, u_{n+1,m+1}) = 0,$$
$$\mathcal{E} = \omega(u_{n+1,m}, u_{n+1,m+1}) - \omega(u_{n,m}, u_{n,m+1}) = 0,$$

with ω and ω' given functions of their arguments.

Symmetry-related integrability tests can also be obtained considering approaches different from the one presented in this paper.

(1) One can construct integrability tests based on symmetries which are given by lattice-dependent equations, possibly related with master symmetries. An example of such a symmetry is given by lattice-dependent Volterra-type equations studied in [21, 29]. Lattice-dependent discrete equations possessing such symmetries have been recently presented in [38].
(2) An integrability test can also be obtained considering Bäcklund transformations instead of generalized symmetries. These have been used, for example, to study equations of KdV/sine-Gordon type (see [1], for instance).

Alternative methods for testing and classifying difference equations not strictly related to symmetries are:

(1) The 3D-consistency method firstly introduced as a kind of Bianchi identity related to the Bianchi commutativity theorem [31]. As a property of maps it was first proposed in an article by Nijhoff, Ramani, Grammaticos and Ohta [30]. It has been used with success by Adler, Bobenko and Suris to classify some classes of equations on the quad-graph [1, 2].
(2) Grammaticos, Ramani and Papageorgiou [8] proposed in 1991 the singularity confinement criterion. Later, it was shown that the singularity confinement was not sufficient to prove integrability and Viallet and Hietarinta [14] introduced the algebraic entropy, slow growth of complexity, as a way to test both S- and C-integrable equations.
(3) Most recently, the study of the growth of the so-called characteristic algebra has been applied to the case of difference equations and has provided a way to identify and classify integrable equations on the lattice [10, 11]. This method can also be used for the study of linearizable equations [9].
(4) The existence of integrals of (1+1)-dimensional partial difference equations, in both directions, provides a way to test and classify linearizable partial difference equations [4]. The equations which satisfy this test have been called *Darboux integrable equations*.

A recent review of techniques used to show integrability of difference equations can be found in [13].

In section 2, we review the necessary theoretical results contained in [22]. Then we explain how, using those results, we can test any given equation (1.1) for integrability. In section 3, we apply this testing tool to a number of equations which can be found in the recent literature on this subject [2, 12, 15, 18, 20]. In section 4, we collect together and discuss all the equations of the form (1.1) which we have shown to satisfy the generalized symmetry test. In particular, we write down seven integrable equations which are not contained in Q_V [36] and thus are not related to the equations of the Adler–Bobenko–Suris (ABS) list [1].

2. Theoretical results

As equation (1.1) and its symmetry (1.2) are taken to be autonomous, in all generality, applying the translation invariance, we can consider them at the point $n = m = 0$:

$$\mathcal{E}(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = 0, \tag{2.1}$$

$$u_{0,0,t} = g_{0,0} = \mathcal{G}(u_{1,0}, u_{-1,0}, u_{0,0}, u_{0,1}, u_{0,-1}). \tag{2.2}$$

Equation (2.1) must satisfy the following conditions:

$$(\mathcal{E}_{u_{0,0}}, \mathcal{E}_{u_{1,0}}, \mathcal{E}_{u_{0,1}}, \mathcal{E}_{u_{1,1}}) \neq 0, \tag{2.3}$$

where indices in (2.3) denote partial derivatives. These conditions are not sufficient to rule out trivial equations. The equation

$$(u_{0,0} + u_{1,0})(u_{0,1} + u_{1,1} + 1) = 0$$

provides an example of equation which is degenerate but satisfies (2.3).

We require that equation (2.1) be re-writable in the form

$$u_{1,1} = f^{(1,1)}(u_{1,0}, u_{0,0}, u_{0,1}), \tag{2.4}$$

where

$$\left(f^{(1,1)}_{u_{1,0}}, f^{(1,1)}_{u_{0,0}}, f^{(1,1)}_{u_{0,1}}\right) \neq 0 \tag{2.5}$$

and the apex $^{(1,1)}$ indicates that the function f is obtained from (2.1) by explicitating the function u in the point $(1, 1)$. Conditions (2.5) are necessary conditions to prevent triviality of equation (2.1).

It is well known [19, 34, 35] that all the equations (2.1) classified by Adler, Bobenko and Suris have two symmetries of the form

$$u_{0,0,t_1} = \Phi(u_{1,0}, u_{0,0}, u_{-1,0}), \qquad u_{0,0,t_2} = \Psi(u_{0,1}, u_{0,0}, u_{0,-1}), \tag{2.6}$$

where $(\Phi_{u_{1,0}}, \Phi_{u_{-1,0}}, \Psi_{u_{0,1}}, \Psi_{u_{0,-1}}) \neq 0$. Obviously, if (2.6) is valid, we can construct a symmetry $u_{0,0,t} = \Phi + \Psi$ of the form (2.2) with

$$(\mathcal{G}_{u_{1,0}}, \mathcal{G}_{u_{-1,0}}, \mathcal{G}_{u_{0,1}}, \mathcal{G}_{u_{0,-1}}) \neq 0. \tag{2.7}$$

One can also prove in all generality for a symmetry of the form (2.2) (see details and references in [22]) that its rhs must be of the form

$$\mathcal{G} = \Phi(u_{1,0}, u_{0,0}, u_{-1,0}) + \Psi(u_{0,1}, u_{0,0}, u_{0,-1}).$$

So to obtain a coherent result we need to add to the generalized symmetry equation (2.2) conditions (2.7), from now on called *non-degeneracy conditions*.

In [37], the author considered affine linear equations of the form (2.1) and (2.3), where the function $\mathcal{E}$ possesses the Klein symmetry:

$$\begin{aligned}\mathcal{E}(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) &= \pm\mathcal{E}(u_{1,0}, u_{0,0}, u_{1,1}, u_{0,1}),\\ \mathcal{E}(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) &= \pm\mathcal{E}(u_{0,1}, u_{1,1}, u_{0,0}, u_{1,0}).\end{aligned} \tag{2.8}$$

We will call equations possessing symmetry (2.8) *the Klein-type equations*. It has been proved in [37] that Klein-type equations possess two nontrivial generalized symmetries of the form (2.6), i.e. they satisfy our test.

The class of Klein-type equations contains the Q_V equation [36], see the appendix. A generic Q_V is equivalent to Q_4 [1] up to Möbius transformations, see e.g. [37]. The definition of the class of Klein-type equations is constructive and easy to check. Moreover, Klein-type equations are invariant under Möbius (linear-fractional) transformations:

$$u_{n,m} = \frac{\alpha\hat{u}_{n,m} + \beta}{\gamma\hat{u}_{n,m} + \delta}, \tag{2.9}$$

and it is easy to check if an equation cannot be Möbius transformed into the Q_V equation or into an equation of the ABS list.

Equation (2.2) is a generalized symmetry of equation (2.4) if the following compatibility condition is satisfied:

$$\left.\frac{\mathrm{d}(u_{1,1} - f^{(1,1)})}{\mathrm{d}t}\right|_{[u_{1,1}=f^{(1,1)},\ u_{0,0,t}=g_{0,0}]} = 0.$$

Explicitly the compatibility condition reads

$$[g_{1,1} - (g_{1,0}\partial_{u_{1,0}} + g_{0,0}\partial_{u_{0,0}} + g_{0,1}\partial_{u_{0,1}})f^{(1,1)}]|_{u_{1,1}=f^{(1,1)}} = 0, \tag{2.10}$$

where $g_{a,b} = T_1^a T_2^b g_{0,0}$, and T_1, T_2 are the shift operators acting on the first and second indices, respectively, i.e. $T_1 g_{0,0} = g_{1,0}$ and $T_2 g_{0,0} = g_{0,1}$.

To be able to check the compatibility condition between (2.1) and (2.2), we need to define the set of independent variables in terms of which (2.10) can be split into an overdetermined system of independent equations. In this work we choose the functions

$$u_{i,0}, u_{0,j} \tag{2.11}$$

as *independent variables*. Then, using (2.4), all the other functions $u_{i,j}$ can be explicitly written in terms of the independent variables (2.11). So we require that equation (2.10) is satisfied identically for all values of independent variables. Taking into account the form of $\mathcal{G}$, equation (2.10) depends on the variables $u_{1,1}, u_{-1,1}, u_{1,-1}, u_{2,1}, u_{1,2}$, and this is the reason why (2.10) turns out to be a rather complicated functional-difference equation for the functions $f^{(1,1)}$ and $g_{0,0}$ appearing in (2.2) and (2.4).

In [22] we proved the following theorem.

Theorem 1. *If (2.4) possesses a generalized symmetry of the form (2.2), then its solutions must satisfy the following conservation laws:*

$$(T_1 - 1)p_{0,0}^{(k)} = (T_2 - 1)q_{0,0}^{(k)}, \tag{2.12}$$

where

if $\mathcal{G}_{u_{1,0}} \neq 0$, *then*

$$p_{0,0}^{(1)} = \log f_{u_{1,0}}^{(1,1)}, \qquad q_{0,0}^{(1)} = Q^{(1)}(u_{2,0}, u_{1,0}, u_{0,0}); \tag{2.13}$$

if $\mathcal{G}_{u_{-1,0}} \neq 0$, *then*

$$p_{0,0}^{(2)} = \log \frac{f_{u_{0,0}}^{(1,1)}}{f_{u_{0,1}}^{(1,1)}}, \qquad q_{0,0}^{(2)} = Q^{(2)}(u_{2,0}, u_{1,0}, u_{0,0}); \tag{2.14}$$

if $\mathcal{G}_{u_{0,1}} \neq 0$, *then*

$$q_{0,0}^{(3)} = \log f_{u_{0,1}}^{(1,1)}, \qquad p_{0,0}^{(3)} = P^{(3)}(u_{0,2}, u_{0,1}, u_{0,0}); \tag{2.15}$$

if $\mathcal{G}_{u_{0,-1}} \neq 0$, *then*

$$q_{0,0}^{(4)} = \log \frac{f_{u_{0,0}}^{(1,1)}}{f_{u_{1,0}}^{(1,1)}}, \qquad p_{0,0}^{(4)} = P^{(4)}(u_{0,2}, u_{0,1}, u_{0,0}). \tag{2.16}$$

Let us analyze in detail theorem 1 when $k = 1$. In this case we just require $\mathcal{G}_{u_{1,0}} \neq 0$, while the dependence of $\mathcal{G}$ on the other variables is not important. In this case theorem 1 states that there exists a conservation law with a function $p_{0,0}^{(1)}$ completely defined by (2.4) while $q_{0,0}^{(1)}$ is an arbitrary function of $u_{2,0}, u_{1,0}, u_{0,0}$. The other three cases are similar.

Therefore, theorem 1 provides four *integrability conditions* in the form of conservation laws. In the case of a non-degenerate symmetry (2.2), (2.7), all these integrability conditions must be satisfied.

Let us note that from the existence of generalized symmetries, we can easily derive many integrability conditions of this kind. Such conditions have been written down in [26]. However, the other conditions are in general more complicated, and more difficult to use in practice. Work is in progress to check if these further integrability conditions, obtained requiring the existence of a recursion operator for the symmetries, allow us to obtain new integrable equations of this class or if they just correspond to the existence of higher symmetries.

If the integrability conditions given in theorem 1 are satisfied, then there must exist functions $q_{0,0}^{(1)}$, $q_{0,0}^{(2)}$, $p_{0,0}^{(3)}$, $p_{0,0}^{(4)}$ local in their argument. Once we know all such functions, we can check if some autonomous five-point symmetries might exist. In fact in such a case we can construct the four partial derivatives of $\mathcal{G}$ [22]:

$$\mathcal{G}_{u_{1,0}} = \exp\left(-q_{-1,0}^{(1)}\right), \qquad \mathcal{G}_{u_{-1,0}} = \exp\left(q_{-1,0}^{(2)}\right), \tag{2.17}$$

$$\mathcal{G}_{u_{0,1}} = \exp\left(-p_{0,-1}^{(3)}\right), \qquad \mathcal{G}_{u_{0,-1}} = \exp\left(p_{0,-1}^{(4)}\right). \tag{2.18}$$

These partial derivatives must be compatible

$$\mathcal{G}_{u_{1,0},u_{-1,0}} = \mathcal{G}_{u_{-1,0},u_{1,0}}, \qquad \mathcal{G}_{u_{0,1},u_{0,-1}} = \mathcal{G}_{u_{0,-1},u_{0,1}}. \tag{2.19}$$

Equation (2.19) is an additional integrability condition. If this further integrability condition is satisfied we can construct $\mathcal{G}$ in the form

$$\mathcal{G} = \Phi(u_{1,0}, u_{0,0}, u_{-1,0}) + \Psi(u_{0,1}, u_{0,0}, u_{0,-1}) + \nu(u_{0,0}), \tag{2.20}$$

where Φ and Ψ are known functions of their arguments while ν is an unknown arbitrary function which may correspond to a Lie point symmetry of the equation. The function ν can be specified by considering the compatibility condition (2.10), the last and most fundamental integrability condition.

The first problem is to check the integrability conditions given in theorem 1. In the case of differential difference equations we had a similar situation, i.e. the integrability conditions were given by conservation laws depending on arbitrary functions of a limited number of variables. However, such problem was easier to solve as all discrete variables were independent and we could use the variational derivative to check them [22]. Here the calculation of the variational derivative is not sufficient to prove if a given expression is a conservation law.

So, in the following, we present a scheme for solving this problem for any given equation of the form (2.4), i.e. we show how we can solve equations (2.12)–(2.16) to obtain $q_{0,0}^{(1)}$, $q_{0,0}^{(2)}$, $p_{0,0}^{(3)}$, $p_{0,0}^{(4)}$ as local functions of their arguments. We will split the explanation into two steps.

Step 1. First, we consider the integrability conditions (2.12) corresponding to $k = 1, 2$. The unknown functions on the rhs of (2.12) when $k = 1, 2$ contain the dependent variable $u_{2,1}$ which, from (2.4), depends on $u_{2,0}$, $u_{1,0}$, $u_{1,1}$ and thus is not immediately expressed in terms of independent variables, but gives rise to extremely complicated functional expressions of the independent variables. We can avoid this problem by applying the operators T_1^{-1}, T_2^{-1} to (2.12). In this case, we have

$$p_{0,0}^{(k)} - p_{-1,0}^{(k)} = q_{-1,1}^{(k)} - q_{-1,0}^{(k)} = Q^{(k)}(u_{1,1}, u_{0,1}, u_{-1,1}) - Q^{(k)}(u_{1,0}, u_{0,0}, u_{-1,0}), \tag{2.21}$$

$$p_{0,-1}^{(k)} - p_{-1,-1}^{(k)} = q_{-1,0}^{(k)} - q_{-1,-1}^{(k)} = Q^{(k)}(u_{1,0}, u_{0,0}, u_{-1,0}) - Q^{(k)}(u_{1,-1}, u_{0,-1}, u_{-1,-1}). \tag{2.22}$$

Here $p_{i,j}^{(k)}$ are known functions expressed in terms of (2.4). The functions $q_{i,j}^{(k)}$ are unknown, and $\left(q_{-1,1}^{(k)}, q_{-1,-1}^{(k)}\right)$ contain the dependent variables $u_{1,1}$, $u_{-1,1}$, $u_{1,-1}$, $u_{-1,-1}$. Our aim is to derive from (2.21) and (2.22) a set of equations for the unknown function $q_{-1,0}^{(k)}$.

To do so let us extract from (2.1) three further expressions of the form of (2.4) for the dependent variables contained in (2.21) and (2.22):

$$\begin{aligned} &u_{-1,1} = f^{(-1,1)}(u_{-1,0}, u_{0,0}, u_{0,1}), \\ &u_{1,-1} = f^{(1,-1)}(u_{1,0}, u_{0,0}, u_{0,-1}), \qquad u_{-1,-1} = f^{(-1,-1)}(u_{-1,0}, u_{0,0}, u_{0,-1}). \end{aligned} \tag{2.23}$$

All functions $f^{(i,j)}$ have a nontrivial dependence on all their variables, as is the case of $f^{(1,1)}$, and are expressed in terms of independent variables. Let us introduce the two differential operators:

$$A = \partial_{u_{0,0}} - \frac{f_{u_{0,0}}^{(1,1)}}{f_{u_{1,0}}^{(1,1)}} \partial_{u_{1,0}} - \frac{f_{u_{0,0}}^{(-1,1)}}{f_{u_{-1,0}}^{(-1,1)}} \partial_{u_{-1,0}}, \tag{2.24}$$

$$B = \partial_{u_{0,0}} - \frac{f_{u_{0,0}}^{(1,-1)}}{f_{u_{1,0}}^{(1,-1)}} \partial_{u_{1,0}} - \frac{f_{u_{0,0}}^{(-1,-1)}}{f_{u_{-1,0}}^{(-1,-1)}} \partial_{u_{-1,0}}, \tag{2.25}$$

chosen in such a way as to annihilate the functions $q_{-1,1}^{(k)}$ and $q_{-1,-1}^{(k)}$, namely $Aq_{-1,1}^{(k)} = 0$, $Bq_{-1,-1}^{(k)} = 0$. Applying A to (2.21) and B to (2.22), we obtain two equations for the unknown $q_{-1,0}^{(k)}$:

$$Aq_{-1,0}^{(k)} = r^{(k,1)}, \qquad Bq_{-1,0}^{(k)} = r^{(k,2)}, \tag{2.26}$$

where $r^{(k,1)}$, $r^{(k,2)}$ are some explicitly known functions of (2.4). Considering the standard commutator of A and B, $[A, B] = AB - BA$, we can add a further equation

$$[A, B]q_{-1,0}^{(k)} = r^{(k,3)}. \tag{2.27}$$

Equations (2.26) and (2.27) represent a linear partial differential system of three equations for the unknown $q_{-1,0}^{(k)} = Q^{(k)}(u_{1,0}, u_{0,0}, u_{-1,0})$. For the three partial derivatives of $q_{-1,0}^{(k)}$, this is just a linear algebraic system of three equations in three unknowns. In most of the examples considered below, this system is non-degenerate and thus it provides one and only one solution for the three derivatives of $q_{-1,0}^{(k)}$. In these cases, we can find in a unique way the partial derivatives of $q_{0,0}^{(k)}$. Then we can check the consistency of the partial derivatives and, if satisfied, find $q_{0,0}^{(k)}$ up to an arbitrary constant. Finally, we check the integrability condition (2.12) with $k = 1, 2$ in any of the equivalent forms (2.21) or (2.22).

The non-degeneracy of the system (2.26), (2.27) depends on (2.4) only. So if we have checked the non-degeneracy for $k = 1$, we know that this is also true for $k = 2$ and vice versa. So both functions $q^{(1)}_{0,0}$, $q^{(2)}_{0,0}$ are found in a unique way up to a constant of integration.

If the system (2.26), (2.27) is degenerate, the functions $q^{(1)}_{0,0}$, $q^{(2)}_{0,0}$ are defined up to some arbitrary functions. In this case, the checking of the integrability conditions (2.12) may be more difficult.

In principle the coefficients of the system (2.26), (2.27) may depend, in addition to the natural variables $u_{0,0}, u_{1,0}, u_{-1,0}$, entering in $q^{(k)}_{-1,0}$, on the independent variables $u_{0,1}, u_{0,-1}$. In such a case, we have to require that the solution $q^{(k)}_{-1,0}$ does not depend on them. In this case, we have to split the equations of the system (2.26), (2.27) with respect to the various powers of the independent variables $u_{0,1}, u_{0,-1}$, if (2.4) is rational, and obtain an overdetermined system of equations for $q^{(k)}_{-1,0}$. Moreover, overdetermined systems of equations are usually simpler to solve. There will be some examples of this kind in section 3.

Step 2. Let us consider now conditions (2.12) with $k = 3, 4$. In this case we have a similar situation. By appropriate shifts we rewrite (2.12) in the two equivalent forms:

$$\begin{aligned} p^{(k)}_{1,-1} - p^{(k)}_{0,-1} &= q^{(k)}_{0,0} - q^{(k)}_{0,-1} \\ &= P^{(k)}(u_{1,1}, u_{1,0}, u_{1,-1}) - P^{(k)}(u_{0,1}, u_{0,0}, u_{0,-1}), \end{aligned} \tag{2.28}$$

$$\begin{aligned} p^{(k)}_{0,-1} - p^{(k)}_{-1,-1} &= q^{(k)}_{-1,0} - q^{(k)}_{-1,-1} \\ &= P^{(k)}(u_{0,1}, u_{0,0}, u_{0,-1}) - P^{(k)}(u_{-1,1}, u_{-1,0}, u_{-1,-1}). \end{aligned} \tag{2.29}$$

We can introduce the operators

$$\hat{A} = \partial_{u_{0,0}} - \frac{f^{(1,1)}_{u_{0,0}}}{f^{(1,1)}_{u_{0,1}}}\partial_{u_{0,1}} - \frac{f^{(1,-1)}_{u_{0,0}}}{f^{(1,-1)}_{u_{0,-1}}}\partial_{u_{0,-1}}, \tag{2.30}$$

$$\hat{B} = \partial_{u_{0,0}} - \frac{f^{(-1,1)}_{u_{0,0}}}{f^{(-1,1)}_{u_{0,1}}}\partial_{u_{0,1}} - \frac{f^{(-1,-1)}_{u_{0,0}}}{f^{(-1,-1)}_{u_{0,-1}}}\partial_{u_{0,-1}}, \tag{2.31}$$

such that $\hat{A}p^{(k)}_{1,-1} = 0$ and $\hat{B}p^{(k)}_{-1,-1} = 0$. Then we are led to the system

$$\hat{A}p^{(k)}_{0,-1} = \hat{r}^{(k,1)}, \qquad \hat{B}p^{(k)}_{0,-1} = \hat{r}^{(k,2)}, \qquad [\hat{A}, \hat{B}]p^{(k)}_{0,-1} = \hat{r}^{(k,3)} \tag{2.32}$$

for the function $p^{(k)}_{0,-1}$ depending on $u_{0,1}$, $u_{0,0}$, $u_{0,-1}$, where $\hat{r}^{(k,l)}$ are known functions expressed in terms of $f^{(i,j)}$.

After we have solved (2.12)–(2.16) we can construct a generalized symmetry. When systems (2.26), (2.27) and (2.32) are non-degenerate, we find Φ and Ψ, given by (2.20), up to at most four arbitrary constants. Two of them may be specified by the consistency conditions (2.19), while the remaining constants are specified using the compatibility condition (2.10) together with the function ν. In practice, we always look for symmetries of the form (2.6). Such symmetries are defined uniquely up to multiple factors and the addition of functions of the form $\nu(u_{0,0})$ corresponding to the rhs of point symmetries $u_{0,0,\tau} = \nu(u_{0,0})$. We write down only the generalized symmetry as we are not interested in the point symmetries.

If one of the systems (2.26), (2.27) or (2.32) is degenerate, then Φ and Ψ on the rhs of a generalized symmetry (2.20) may be found up to some arbitrary functions. Those arbitrary functions must be specified using the compatibility condition (2.10). However, in almost all degenerate examples considered below, (2.4) turns out to be trivial and it can be rewritten in one of the following four forms:

$$(T_1 \pm 1)w(u_{0,0}, u_{0,1}) = 0, \qquad (T_2 \pm 1)w(u_{1,0}, u_{0,0}) = 0. \tag{2.33}$$

Equations (2.33) can be integrated once and give equations depending on a reduced number of lattice variables.

In the next section we test equations which depend on arbitrary constants and thus solve some simple classification problems. We look for such particular cases that satisfy our integrability test and are not of Klein type (2.8) or transformable into Klein-type equations by n, m-dependent Möbius transformations.

3. Examples

Here, we apply the test to a number of nonlinear nontrivial partial difference equations introduced by various authors using different approaches to prove their integrability [2, 12, 15–18, 20, 28]. It should be stressed that all equations below are affine linear, i.e. they can be written in the form (2.1) and (2.3), where $\partial^2\mathcal{E}/\partial u_{n,m}^2 = 0$ for all four variables.

Example 1. This will be a simple illustrative example discussed in detail. We consider the equation

$$(u_{1,0}+1)(u_{0,0}-1) = (u_{1,1}-1)(u_{0,1}+1). \tag{3.1}$$

Up to a rotation (change of axes), (3.1) is equivalent to the equations presented in [17] and [28]. In [18, 20, 22], its *L–A* pairs and some conservation laws are presented. Two generalized symmetries of the form (2.6) have been constructed in [22]. So (3.1) satisfies our integrability test, but nevertheless, it is instructive to try out the test with this equation.

The study of this equation splits into two different steps.

Step 1. Let us consider the integrability condition (2.12) with $k = 1$. The corresponding system (2.26), (2.27) reads

$$q_{u_{0,0}} - \frac{u_{1,0}+1}{u_{0,0}-1}q_{u_{1,0}} - \frac{u_{-1,0}-1}{u_{0,0}+1}q_{u_{-1,0}} = \frac{2u_{0,0}}{1-u_{0,0}^2},$$
$$q_{u_{0,0}} - \frac{u_{1,0}-1}{u_{0,0}+1}q_{u_{1,0}} - \frac{u_{-1,0}+1}{u_{0,0}-1}q_{u_{-1,0}} = \frac{2u_{0,0}}{1-u_{0,0}^2},$$
$$(u_{1,0}u_{0,0}+1)q_{u_{1,0}} - (u_{-1,0}u_{0,0}+1)q_{u_{-1,0}} = 0,$$

where $q = q_{-1,0}^{(1)}$ and by the index we denote the argument of the derivative. This system is non-degenerate, and its solution is

$$q_{u_{1,0}} = q_{u_{-1,0}} = 0, \qquad q_{u_{0,0}} = \frac{2u_{0,0}}{1-u_{0,0}^2}.$$

Hence, $q_{0,0}^{(1)} = -\log\left(u_{1,0}^2 - 1\right) + c_1$, where c_1 is an arbitrary constant. $q_{0,0}^{(1)}$ together with $p_{0,0}^{(1)} = \log\frac{u_{0,0}-1}{u_{0,1}+1}$ satisfy relation (2.12) and provide a conservation law for (3.1).

Equation (2.12) with $k = 2$ can be solved in a simpler way. As $p_{0,0}^{(2)} = -p_{0,0}^{(1)} + \log(-1)$, the solution of (2.12) with $k = 2$ is given by $q_{0,0}^{(2)} = -q_{0,0}^{(1)} + c_2$, with c_2 another arbitrary constant. As the corresponding system (2.26), (2.27) is non-degenerate, there is no other solution.

Now we look for Φ, the rhs of the symmetry in (2.6). As follows from (2.17), a candidate for such a symmetry is given by

$$u_{0,0,t_1} = \left(u_{0,0}^2 - 1\right)(\alpha u_{1,0} + \beta u_{-1,0}) + \nu(u_{0,0}).$$

Here α, β are nonzero arbitrary constants, and $\nu(u_{0,0})$ is an arbitrary function of its argument. Rescaling t_1, we can set in all generality $\alpha = 1$. Substituting this into the compatibility condition (2.10) we obtain $\beta = -1$, $\nu(u_{0,0}) \equiv 0$. This symmetry is nothing but the well-known modified Volterra equation [41]

$$u_{0,0,t_1} = \left(u_{0,0}^2 - 1\right)(u_{1,0} - u_{-1,0}). \tag{3.2}$$

Step 2. Let us consider the integrability conditions (2.12) with $k = 3, 4$. In this case the corresponding system (2.32) is degenerate. So we have to modify the procedure presented in the previous section and applied above in the case when $k = 1, 2$. For $k = 3$ this system reads

$$(u_{0,1} + u_{0,0})p_{u_{0,1}} - (u_{0,0} + u_{0,-1})p_{u_{0,-1}} = 2,$$
$$\left(u_{0,0}^2 - 1\right)p_{u_{0,0}} + (u_{0,1}u_{0,0} + 1)p_{u_{0,1}} + (u_{0,0}u_{0,-1} + 1)p_{u_{0,-1}} = 0,$$

where $p = p_{0,-1}^{(3)}$. Its general solution is

$$p = \Omega(\omega) + \log \frac{u_{0,1} + u_{0,0}}{u_{0,0} + u_{0,-1}} \quad \text{with} \quad \omega = \frac{u_{0,0}^2 - 1}{(u_{0,1} + u_{0,0})(u_{0,0} + u_{0,-1})}. \tag{3.3}$$

The integrability condition (2.12) with $k = 3$ is satisfied iff $\Omega(\omega) = \gamma - \log \omega$, where γ is an arbitrary constant. The case $k = 4$ is quite similar, and we easily find the second generalized symmetry of (3.1):

$$u_{0,0,t_2} = \left(u_{0,0}^2 - 1\right)\left(\frac{1}{u_{0,1} + u_{0,0}} - \frac{1}{u_{0,0} + u_{0,-1}}\right). \tag{3.4}$$

Step 2 of this example is not standard. In all the following discrete equations, either the systems (2.26), (2.27) or (2.32) are non-degenerate or the equations themselves are trivial. As a result we have proved the following statement.

Theorem 2. *Equation (3.1) satisfies the generalized symmetry test and possesses the symmetries (3.2) and (3.4).*

Example 2. Let us consider a known equation closely related to (3.1) and studied in [16, 18, 23, 27]

$$u_{1,1}(u_{0,1} + c)(u_{1,0} - 1) = u_{0,0}(u_{1,0} + c)(u_{0,1} - 1), \tag{3.5}$$

where $c \neq -1, 0$. In particular, its L–A pair can be found in [27]. If $c = -1$, it is trivial. If $c = 0$, using the point transformation

$$u_{n,m} = \frac{2}{1 - \hat{u}_{n,m}}, \tag{3.6}$$

we can reduce it to (3.1) for $\hat{u}_{n,m}$. So (3.5) generalizes (3.1). Thus, it will not be surprising that this equation satisfies our test. The calculation is quite similar to the one shown in example 1, step 1. We easily find two generalized symmetries of the form (2.6):

$$\frac{u_{0,0,t_1}}{u_{0,0}(u_{0,0} - 1)} = (T_1 - 1)\frac{1}{u_{0,0}u_{-1,0} + c(u_{0,0} + u_{-1,0} - 1)}, \tag{3.7}$$

$$\frac{u_{0,0,t_2}}{u_{0,0}(u_{0,0}+c)} = (T_2-1)\frac{1}{u_{0,0}u_{0,-1}-(u_{0,0}+u_{0,-1}+c)}. \tag{3.8}$$

As a result we obtain

Theorem 3. *Equation (3.5) satisfies the generalized symmetry test and possesses the symmetries (3.7) and (3.8).*

Example 3. A further example is given by the equation

$$u_{1,1}u_{0,0}(u_{1,0}-1)(u_{0,1}+1)+(u_{1,0}+1)(u_{0,1}-1)=0 \tag{3.9}$$

taken from [18]. It is an equation which possesses five non-autonomous conservation laws of the form

$$(T_1-1)p_{n,m}(u_{n,m},u_{n,m+1}) = (T_2-1)q_{n,m}(u_{n,m},u_{n+1,m}), \tag{3.10}$$

where $p_{n,m}$, $q_{n,m}$ depend explicitly on the discrete variables n, m. In [18], the authors also calculated the algebraic entropy for (3.9) and demonstrated in this way that the equation should be integrable.

This example does not satisfy our test. The system (2.26), (2.27), corresponding to the first of the integrability conditions (2.12), is non-degenerate, and we find from it $q_{0,0}^{(1)}$ in a unique way. However, this function does not satisfy condition (2.12). The same is true for all four integrability conditions. This means that all four assumptions of theorem 1 are not satisfied.

Theorem 4. *Equation (3.9) does not satisfy any of the four integrability conditions (2.12)–(2.16). This equation cannot have an autonomous nontrivial generalized symmetry of the form (2.2).*

Equation (3.9) might have, however, a non-autonomous generalized symmetry. The extension of the method to non-autonomous generalized symmetries for partial difference equations is an open problem which is left for future work.

Example 4. Let us consider the equation

$$(1+u_{0,0}u_{1,0})(\nu u_{1,1}+u_{0,1}) = (1+u_{0,1}u_{1,1})(\nu u_{0,0}+u_{1,0}), \tag{3.11}$$

where the constant ν is such that $\nu^2 \neq 1$. When $\nu = \pm 1$ the equations are trivial, as they are equivalent to (2.33). Equation (3.11) has been obtained in [20] by combining Miura-type transformations relating differential difference equations of the Volterra type. In [32] Miura-type transformations have been found relating this equation to integrable equations of the form (2.1). Equation (3.11) satisfies our test, and we find two generalized symmetries:

$$u_{0,0,t_1} = \frac{\left(u_{0,0}^2-\nu\right)\left(\nu u_{0,0}^2-1\right)}{u_{0,0}}\left(\frac{1}{u_{1,0}u_{0,0}+1}-\frac{1}{u_{0,0}u_{-1,0}+1}\right), \tag{3.12}$$

$$u_{0,0,t_2} = \frac{\left(u_{0,0}^2-\nu\right)\left(\nu u_{0,0}^2-1\right)}{u_{0,0}}\left(\frac{1}{u_{0,1}u_{0,0}-1}-\frac{1}{u_{0,0}u_{0,-1}-1}\right). \tag{3.13}$$

In the particular case $\nu = 0$, (3.11) reduces to

$$u_{1,1}-u_{0,0} = \frac{1}{u_{1,0}}-\frac{1}{u_{0,1}}, \tag{3.14}$$

and (3.12), (3.13) to its generalized symmetries. Equation (3.14), up to point transformations, can be found in [8, 17, 18]. As a result of this example we can state the following theorem.

Theorem 5. *Equation (3.11) satisfies the generalized symmetry test and possesses the symmetries (3.12) and (3.13).*

Example 5. A further interesting example is provided by the equation [12]

$$
\begin{aligned}
2(u_{0,0}+u_{1,1})+u_{1,0}+u_{0,1}+\gamma(4u_{0,0}u_{1,1}+2u_{1,0}u_{0,1}+3(u_{0,0}+u_{1,1})(u_{1,0}+u_{0,1}))\\
+(\xi_2+\xi_4)u_{0,0}u_{1,1}(u_{1,0}+u_{0,1})+(\xi_2-\xi_4)u_{1,0}u_{0,1}(u_{0,0}+u_{1,1})\\
+\zeta u_{0,0}u_{1,1}u_{1,0}u_{0,1}=0,
\end{aligned}
\tag{3.15}
$$

where γ, ξ_2, ξ_4 and ζ are constant coefficients. This equation is obtained as a subclass of the most general multilinear dispersive equation on the square lattice, $\mathcal{Q}_+$, whose linear part is a linear combination with arbitrary coefficients of $u_{0,0}+u_{1,1}$ and $u_{1,0}+u_{0,1}$. Equation (3.15) is contained in the intersection of five of the six classes of equations belonging to $\mathcal{Q}_+$ which are reduced to an integrable nonlinear Schrödinger equation under a multiple scale reduction.

Using the transformation $u_{n,m} = 1/(\hat{u}_{n,m}-\gamma)$ and redefining the original constants entered in (3.15):

$$\alpha=\xi_2+\xi_4-5\gamma^2,\qquad \beta=\xi_2-\xi_4-4\gamma^2,\qquad \delta=\zeta+12\gamma^3-4\gamma\xi_2,$$

we obtain for $\hat{u}_{n,m}$ a simpler equation depending on just three free parameters:

$$(\hat{u}_{0,0}\hat{u}_{1,1}+\alpha)(\hat{u}_{1,0}+\hat{u}_{0,1})+(2\hat{u}_{1,0}\hat{u}_{0,1}+\beta)(\hat{u}_{0,0}+\hat{u}_{1,1})+\delta=0. \tag{3.16}$$

If the three parameters are null, (3.16) is a linear equation in $\tilde{u}_{0,0}=1/\hat{u}_{0,0}$, and thus trivially integrable.

For (3.16) the test is more complicate, as the system (2.26), (2.27) depends on the additional variables $\hat{u}_{0,1}$, $\hat{u}_{0,-1}$. It is written as a polynomial system and setting to zero the coefficients of the different powers of $\hat{u}_{0,1}$ and $\hat{u}_{0,-1}$, we obtain a simpler system of equations for $q^{(k)}_{-1,0}$. The same is also true in the case of system (2.32).

Equation (3.16) is a simple classification problem, as it depends on three arbitrary constants, and we search for all integrable cases, if any, contained in it. By looking at its generalized symmetries we find two integrable non-linearizable cases:

(1) $\alpha=2\beta\neq0$ and $\delta=0$, i.e. $\xi_2=3\xi_4+3\gamma^2$, $\zeta=12\gamma\xi_4$;
(2) $\beta=2\alpha\neq0$ and $\delta=0$, i.e. $\xi_2=6\gamma^2-3\xi_4$, $\zeta=12\gamma(\gamma^2-\xi_4)$.

In case 1, using the transformation $\hat{u}_{n,m}=u_{n,m}(-1)^m\beta^{1/2}$, we obtain (3.11) with $\nu=1/2$. In case 2 we can always choose $\alpha=1$ and the equation reads

$$(u_{0,0}u_{1,1}+1)(u_{1,0}+u_{0,1})+2(u_{1,0}u_{0,1}+1)(u_{0,0}+u_{1,1})=0. \tag{3.17}$$

By applying the procedure presented in the previous section we find the symmetries

$$u_{0,0,t_1}=\left(u_{0,0}^2-1\right)\frac{u_{1,0}-u_{-1,0}}{u_{1,0}u_{-1,0}-1},\qquad u_{0,0,t_2}=\left(u_{0,0}^2-1\right)\frac{u_{0,1}-u_{0,-1}}{u_{0,1}u_{0,-1}-1}. \tag{3.18}$$

This last example (3.17) seems to be a new integrable model. More comments on this will be presented in section 4. This result can be formulated as the following theorem.

Theorem 6. *There are two nontrivial cases when (3.16) satisfies the generalized symmetry test. The first one is given by the relations $\alpha=2\beta\neq0$, $\delta=0$, and the equation is transformed into (3.11). In the second case, an equation can be written as (3.17) which possesses the symmetries (3.18).*

Example 6. This example is also an equation with arbitrary constant coefficients, obtained by Hietarinta and Viallet [15] as an equation with good factorization properties and considered to be an equation worth further study:

$$(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) + (u_{0,0} - u_{1,1})r_4 + (u_{0,1} - u_{1,0})r_3 + r = 0. \tag{3.19}$$

The authors claim that (3.19) is integrable for all values of the coefficients, as it has a quadratic growth of the iterations in the calculation of its algebraic entropy.

Here we see that if $r_4 = r_3 = r = 0$, the equation is trivial. So we consider only those cases when the triple of parameters r_4, r_3 and r is different from zero.

Let $r_4 + r_3 = \nu = 0$ in (3.19). We apply an n, m-dependent point transformation $u_{n,m} = \hat{u}_{n,m} + (n+m)r_4$ and obtain for $\hat{u}_{n,m}$ the equation

$$(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) + r - r_4^2 = 0$$

of Klein type, more precisely a particular case of the Q_V equation. However, it is obviously trivial whenever $r = r_4^2$. If $r \neq r_4^2$, we can rewrite it as

$$(T_1 + 1)\left[\log(u_{0,0} - u_{0,1}) - \tfrac{1}{2}\log\left(r_4^2 - r\right)\right] = 0,$$

i.e. the equation is trivial in this case too.

The other possible case is when $r_4 + r_3 = \nu \neq 0$. By the transformation $u_{n,m} = \nu\hat{u}_{n,m}$ we obtain the following two-parameter equation:

$$(u_{0,0} - u_{0,1} + a)(u_{1,0} - u_{1,1} + a) + u_{0,1} - u_{1,0} + b = 0, \tag{3.20}$$

where

$$r_4 = a\nu, \qquad r_3 = (1-a)\nu, \qquad r = (b + a^2)\nu^2.$$

This equation has two generalized symmetries which we can construct using our procedure

$$u_{0,0,t_1} = (u_{1,0} - u_{0,0} - a - b)(u_{0,0} - u_{-1,0} - a - b), \tag{3.21}$$

$$u_{0,0,t_2} = \frac{(1 - y_{0,0})(1 - y_{0,-1})}{y_{0,0} + y_{0,-1}} + 1, \qquad y_{n,m} = 2(u_{n,m+1} - u_{n,m}) - 2a + 1, \tag{3.22}$$

showing its integrability. This result can be formulated as

Theorem 7. *In the case $r_4 + r_3 = 0$, (3.19) is equivalent to a trivial equation. In the case $r_4 + r_3 \neq 0$, it can be rewritten in the form (3.20). Equation (3.20) satisfies the generalized symmetry test and possesses the symmetries (3.21) and (3.22).*

Example 7. The next example is also taken from [15]:

$$u_{0,0}u_{0,1}c_5 + u_{1,0}u_{1,1}c_6 + u_{0,0}u_{1,0}c_1 + u_{0,1}u_{1,1}c_3 + (u_{0,0}u_{1,1} + u_{1,0}u_{0,1})c_2 = 0. \tag{3.23}$$

This equation is proven to be integrable for all values of constants c_i by checking its algebraic entropy. Also in this case we have a kind of classification problem once we exclude, up to some simple transformations, all Klein type and trivial subequations.

Let us observe at first that if $c_5 = c_6$ and $c_1 = c_3$, (3.23) is of Klein type, and if moreover $c_1 = c_5 = 0$, it is trivial. We can construct some point transformations which leave (3.23) invariant, but transform the coefficients among themselves. By the transformation

$$u_{n,m} = \hat{u}_{m,n}, \tag{3.24}$$

$c_5 \leftrightarrow c_1$, $c_6 \leftrightarrow c_3$, and by the transformation

$$u_{n,m} = 1/\hat{u}_{n,m}, \tag{3.25}$$

$c_5 \leftrightarrow c_6$, $c_1 \leftrightarrow c_3$. In both cases c_2 remains unchanged. Moreover, the n, m dependent transformation

$$u_{n,m} = \hat{u}_{n,m}\kappa_1^n\kappa_2^m, \qquad \kappa_i \neq 0,\ i = 1, 2, \tag{3.26}$$

leaves the equation invariant with the following transformation of the coefficients:

$$\hat{c}_5 = c_5/\kappa_1, \qquad \hat{c}_6 = c_6\kappa_1, \qquad \hat{c}_1 = c_1/\kappa_2, \qquad \hat{c}_3 = c_3\kappa_2, \qquad \hat{c}_2 = c_2.$$

So if at least one of the coefficients c_i $(i \neq 2)$ is different from zero, using the transformations (3.24) and (3.25), we can make $c_5 \neq 0$. Let us assume that also $c_6 \neq 0$. If either c_1 or c_3 is equal to zero then, using the transformations (3.24) and (3.25), we can make $c_6 = 0$. If both c_1 and c_3 are either zero or different from zero, using the transformation (3.26), we can make $c_1 = c_3$ and $c_5 = c_6$, i.e. we obtain a Klein-type equation. So the only possible remaining case is when $c_5 \neq 0$, $c_6 = 0$ and without loss of generality we can set

$$c_5 = 1, \qquad c_6 = 0. \tag{3.27}$$

The non-degeneracy conditions (2.3) give two restrictions $c_2 \neq 0$ or $c_2 = 0$, $c_1c_3 \neq 0$. In these two cases, the equation can be nontrivially rewritten in the form of (2.4). If $c_2 \neq 0$ and $c_1 = c_3 = 0$, (3.23) is trivial, as it is equivalent to

$$(T_2 + 1)\left(c_2\frac{u_{1,0}}{u_{0,0}} + \frac{1}{2}\right) = 0.$$

So at the end we get two admissible cases:

$$c_2 = 0: \qquad c_1c_3 \neq 0, \tag{3.28}$$

$$c_2 \neq 0: \qquad c_1 \text{ or } c_3 \neq 0. \tag{3.29}$$

Any equation (3.23), (3.27) satisfying conditions (3.28), (3.29) possesses two generalized symmetries. The first symmetry depends on the number $c_1c_3 - c_2^2$. If

$$c_1c_3 - c_2^2 \neq 0, \tag{3.30}$$

the condition (3.28) is satisfied automatically. The symmetry reads

$$u_{0,0,t_1} = (u_{1,0} - cu_{0,0})\left(\frac{u_{0,0}}{u_{-1,0}} - c\right), \qquad c = \frac{c_2}{c_1c_3 - c_2^2}. \tag{3.31}$$

In the case when $c_1c_3 = c_2^2$, as $c_2 \neq 0$ due to condition (3.28), condition (3.29) is satisfied automatically. In this case

$$c_1c_3 = c_2^2 \neq 0 \tag{3.32}$$

and the symmetry reads

$$u_{0,0,t_1} = u_{1,0} + \frac{u_{0,0}^2}{u_{-1,0}}. \tag{3.33}$$

The form of the second symmetry depends on the number c_1c_3. If

$$c_1c_3 \neq 0, \tag{3.34}$$

then both non-degeneracy conditions are satisfied, and we have the symmetry

$$u_{0,0,t_2} = \frac{c_2c_3c_1\left(u_{0,1}u_{0,-1} + u_{0,0}^2\right) + \frac{1}{2}\left(c_2^2 + c_3c_1\right)u_{0,0}(u_{0,1}c_3 + u_{0,-1}c_1)}{u_{0,1}c_3 - u_{0,-1}c_1}. \tag{3.35}$$

If $c_1c_3 = 0$, then we cannot have $c_2 = 0$ due to condition (3.28). So $c_2 \neq 0$, and we use condition (3.29). We have here two cases for which both non-degeneracy conditions are satisfied. First of them is

$$c_3 = 0, \qquad c_1c_2 \neq 0, \tag{3.36}$$

and the corresponding symmetry has the form

$$u_{0,0,t_2} = \left(u_{0,1} + u_{0,0}\frac{c_1}{c_2}\right)\left(\frac{u_{0,0}}{u_{0,-1}} + \frac{c_1}{c_2}\right). \tag{3.37}$$

The second case is

$$c_1 = 0, \qquad c_2c_3 \neq 0, \tag{3.38}$$

and the symmetry reads

$$u_{0,0,t_2} = \left(\frac{u_{0,0}}{u_{0,1}} + \frac{c_3}{c_2}\right)\left(u_{0,-1} + u_{0,0}\frac{c_3}{c_2}\right). \tag{3.39}$$

Theorem 8. *For (3.23) we have the following.*

(1) If it is not equivalent to a Klein-type equation, then it can be rewritten in the form (3.27) using transformations (3.24) and (3.25).
(2) Nontrivial equations (3.23) and (3.27) must satisfy conditions (3.28) and (3.29).
(3) Equations (3.23) and (3.27) with the restrictions (3.28) and (3.29) satisfy the generalized symmetry test for any values of c_1, c_2, c_3.
(4) The first symmetry of this equation is of the form (3.31) in case (3.30) and of the form (3.33) in case (3.32);
(5) The second symmetry is of the form (3.35) in case (3.34), of the form (3.37) in case (3.36) and of the form (3.39) in case (3.38).

The resulting equations (3.23) and (3.27) satisfying conditions (3.28) and (3.29) will be written down in a simpler explicit form in section 4. One of these equations, in a slightly different form, can be found in [33], where its L–A pair is given. There it is stated that hierarchies of generalized symmetries and conservation laws exist.

Example 8. The last example is taken from an article by Adler, Bobenko and Suris [2], where an extended definition of 3D-consistency is discussed and the so-called deformations of H equations are presented. As an example, let us consider here one of them, namely,

$$(u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) = (\alpha - \beta)(1 - \epsilon u_{1,0}u_{0,1}), \tag{3.40}$$

where $\alpha \neq \beta$ and ϵ are constants. Equation (3.40) is a generalization of the well-known discrete potential KdV or H_1 equation which is reobtained when $\epsilon = 0$.

Let us use the integrability condition (2.12) with $k = 1$ and obtain the system (2.26), (2.27). The first equation of this system depends on the additional variable $u_{0,1}$. We rewrite the equation in polynomial form and obtain a fourth degree polynomial in $u_{0,1}$. The coefficients of this polynomial provide us with five more equations for $q^{(1)}_{-1,0}$. Using these equations, we easily obtain as an integrability condition that $\epsilon = 0$. The other integrability conditions are similar, and none of them is satisfied if $\epsilon \neq 0$.

Theorem 9. *Equation (3.40) with $\epsilon \neq 0$ satisfies none of the four integrability conditions (2.12)–(2.16). This equation cannot have an autonomous nontrivial generalized symmetry of the form (2.2).*

The result is not surprising, as (3.40) is 3D-consistent on the so-called black-white lattice. This means that to check 3D-consistency we have to use (3.40) together with another equation, i.e. (3.40) is conditionally 3D-consistent. Generalized symmetries might exist in a similar indirect sense when we consider the complete 3D-consistent system. The following n, m-dependent equation

$$\begin{aligned}(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - (\alpha - \beta)\\ + \epsilon(\alpha - \beta)\left(\frac{1 + (-1)^{n+m}}{2} u_{n+1,m}u_{n,m+1} + \frac{1 - (-1)^{n+m}}{2} u_{n,m}u_{n+1,m+1}\right) = 0\end{aligned} \tag{3.41}$$

is obtained in [38] instead of (3.40). Equation (3.40) is obtained when $n+m$ is even, while if $n+m$ is odd we have a different equation. An n, m-dependent L–A pair and n, m-dependent generalized symmetries have been constructed in [38] for (3.41). Such n, m-dependent generalized symmetries could be possibly constructed, starting from its L–A pair.

4. General picture

We have applied our test to a number of discrete equations on the square lattice and have constructed generalized symmetries for some of them. Such equations have automatically a few simple conservation laws. Here we collect together all these equations satisfying the test in order to discuss and compare them.

Discrete-differential equations of the Volterra type

$$u_{k,t} = \phi(u_{k+1}, u_k, u_{k-1}) \tag{4.1}$$

play an important role in this discussion. The main representative of this class is the well-known Volterra equation. A complete list of integrable equations of the Volterra type has been obtained using the generalized symmetry method in [39], see the review [41] for details. As (4.1) are autonomous, they will be written down below at $k = 0$.

The most interesting example of an equation of the class (4.1), apart from the Volterra equation, is the equation

$$\dot{u}_0 = \frac{r}{u_1 - u_{-1}} - \frac{1}{2}\frac{\partial r}{\partial u_1}, \tag{4.2}$$

where $r = r(u_1, u_0)$ is an arbitrary bi-quadratic and symmetric polynomial in its two arguments with six constant coefficients. This is an integrable discretization found by Yamilov in [39], from now on abbreviated as the YdKN equation, of the well-known Krichever–Novikov equation. Two different representations of (4.2) can be found in [39] and [41]. The generic YdKN equation (i.e. its main, non-degenerate component) can be obtained as the continuous limit of the Q_4 equation [5]. This limit preserves the 3D-consistency condition and thus the YdKN equation is a symmetry of the Q_4.[3]

It has been observed in [19] that generalized symmetries (2.6) of any equation of the ABS list are of the form (4.2), i.e. they are subcases of the YdKN equation. In [19], it has been explained that equations of the ABS list can be interpreted as auto-Bäcklund transformations of their symmetries, i.e. of YdKN-type equations. Moreover particular cases of the generalized symmetries of the YdKN equation are generalized symmetries of the equations of the ABS list.

All generalized symmetries mentioned in the previous section can be identified, up to point transformations $u_{n,m} = \omega(\hat{u}_{n,m})$, with an equation of the complete list of integrable

[3] Communication of the referee.

equations of the Volterra type (4.1), presented in [41]. Following [19], we are going to use here this relation between Volterra-type equations and discrete equations on the square lattice. Let us present here some of the reasons why it is convenient to use this connection.

(1) One can always interpret, as has been done in [19], discrete equations on the square satisfying our test as Bäcklund transformations for their symmetries. Bäcklund transformations of integrable equations are integrable equations too, as they are characterized by a Lax pair as the nonlinear equations themselves.
(2) If we know the hierarchy of generalized symmetries for a Volterra-type equation, we automatically obtain the hierarchy of generalized symmetries for its Bäcklund transformation.
(3) Classification of integrable Volterra-type equations up to Miura-type transformations can be found in [41]. This suggests the Miura-type transformations relating different discrete equations on the square lattice.
(4) Is is not so easy to check whether two discrete equations are different up to Möbius transformations. A relatively easy way to do it is by comparing their symmetries[4], i.e. equations (4.1), as generalized symmetries for these equations have been constructed in a unique way up to point symmetries. We can even do that for n, m-dependent Möbius transformations.

In [37], it is shown that generalized symmetries (2.6) for Klein-type equations are always given by particular cases of the YdKN equation. The two symmetries may be different, but both are particular cases of the YdKN equation. As a result we have the following picture:

$$\text{YdKN equation:} \quad \text{Klein-type equations} \;\equiv\; Q_V \text{ equation} \;\supset\; \text{ABS list,}$$

i.e. Klein-type equations are auto-Bäcklund transformations for particular cases of the YdKN equation. The Klein-type equations are essentially equivalent to the Q_V equation, see the appendix, which includes in its turn the ABS list [37]. This picture is true only in the autonomous case. In general, the ABS equations may be lattice dependent and are not included in Q_V. Moreover, generically Q_V is just equivalent to Q_4 up to a Möbius transformation[5]. Q_4, due to its special parametrization of the coefficients, possesses a Lax pair characterized by copies of the Lax operator while this is not the case for Q_V. The ABS list includes a number of well-known nonlinear partial difference equations, see a review in [1]. A hierarchy of generalized symmetries for the YdKN equation has been constructed, using a master symmetry [3], see also a detailed discussion in [19]. In this way we obtain generalized symmetries for all Klein-type equations. An alternative way for constructing symmetries for Klein-type equations can be found in [37]. Many subcases of the YdKN equation can be transformed, using Miura-type transformations, into the Volterra or Toda lattice equations [41].

Let us write down in table 1 the nontrivial non-Klein-type equations satisfying the test, together with their generalized symmetries. We present those equations in a simpler or slightly different form convenient for this section.

Equation (T1) of table 1 is nothing but (3.5) of example 2 with its symmetries. If $c = -1$, it is degenerate. If $c = 0$, the transformation (3.6) gives (3.1) with both its symmetries, as is explained in example 2.

Equation (T2) of table 1 is obtained from (3.11) of example 4. If in (3.11) we set $\nu = 0$, then we have (3.14). By the lattice-dependent point transformation $u_{n,m} = \mathrm{i}\frac{\hat{u}_{n,m}-1}{\hat{u}_{n,m}+1}(-1)^m$ we transform (3.11) into (T2) by defining $\kappa = 2\frac{1-\nu}{1+\nu}$. In order to obtain its symmetries, we rescale t_1 and t_2. The case when $\kappa = 0$ is trivial is in the sense of (2.33). If $\kappa = -2$, the transformation

[4] An alternative way to compare discrete equations is by the use of the invariants of Möbius transformations [2].

[5] Communication of the referee.

Table 1. The nontrivial non-Klein-type partial difference equations analyzed in section 3 and their generalized symmetries written in the form (2.6). In the last column we give the number of an example from section 3 where the corresponding equation is considered.

Equation no	Difference equations	$\Phi(u_{1,0},u_{0,0},u_{-1,0})$ $\Psi(u_{0,1},u_{0,0},u_{0,-1})$	Example No.
T1	$u_{1,1}(u_{0,1}+c)(u_{1,0}-1)=$ $u_{0,0}(u_{1,0}+c)(u_{0,1}-1)$	$\frac{u_{0,0}(u_{0,0}-1)}{u_{1,0}u_{0,0}+c(u_{1,0}+u_{0,0}-1)}-\frac{u_{0,0}(u_{0,0}-1)}{u_{0,0}u_{-1,0}+c(u_{0,0}+u_{-1,0}-1)}$	2
	$c\neq-1$	$\frac{u_{0,0}(u_{0,0}+c)}{u_{0,1}u_{0,0}-(u_{0,1}+u_{0,0}+c)}-\frac{u_{0,0}(u_{0,0}+c)}{u_{0,0}u_{0,-1}-(u_{0,0}+u_{0,-1}+c)}$	
T2	$(u_{0,0}u_{1,1}-1)(u_{1,0}+u_{0,1}-\kappa)=$ $(1-u_{1,0}u_{0,1})(u_{0,0}+u_{1,1}+\kappa)$	$(u_{0,0}^4+(2-\kappa^2)u_{0,0}^2+1)\left(\frac{1}{u_{1,0}+u_{0,0}}-\frac{1}{u_{0,0}+u_{-1,0}}\right)$	4
	$\kappa\neq0$	$(u_{0,0}^4+(2-\kappa^2)u_{0,0}^2+1)\left(\frac{1}{u_{0,1}+u_{0,0}}-\frac{1}{u_{0,0}+u_{0,-1}}\right)$	
T3	$3(u_{0,0}u_{1,1}u_{1,0}u_{0,1}-1)+$	$2u_{0,0}\frac{u_{1,0}-u_{-1,0}}{u_{1,0}+u_{-1,0}}$	5
	$u_{0,0}u_{1,1}-u_{1,0}u_{0,1}=0$	$2u_{0,0}\frac{u_{0,1}-u_{0,-1}}{u_{0,1}+u_{0,-1}}$	
T4	$(u_{0,0}-u_{0,1}+1/2)(u_{1,0}-$	$(u_{1,0}-u_{0,0}-1/2)(u_{0,0}-u_{-1,0}-1/2)$	6
	$u_{1,1}+1/2)+u_{0,1}-u_{1,0}=0$	$\frac{(u_{0,1}-u_{0,0})(u_{0,0}-u_{0,-1})+1/4}{u_{0,1}-u_{0,-1}}$	
T5	$u_{0,0}(u_{1,0}+u_{0,1}+$	$(u_{1,0}+u_{0,0})\left(\frac{u_{0,0}}{u_{-1,0}}+1\right)$	7
	$u_{1,1})+u_{1,0}u_{0,1}=0$	$(u_{0,1}+u_{0,0})\left(\frac{u_{0,0}}{u_{0,-1}}+1\right)$	
T6	$(u_{1,1}-u_{1,0})(u_{0,1}-u_{0,0})+$	$u_{1,0}+\frac{u_{0,0}^2}{u_{-1,0}}$	7
	$u_{0,1}u_{0,0}=0$	$\frac{(u_{0,1}-u_{0,0})(u_{0,0}-u_{0,-1})}{u_{0,1}-u_{0,-1}}$	
T7	$u_{0,1}u_{1,1}+u_{0,0}(u_{1,0}+u_{0,1})+$ $(u_{0,0}u_{1,1}+u_{1,0}u_{0,1})c_2=0$	$\left(u_{1,0}+u_{0,0}\frac{c_2}{c_2^2-1}\right)\left(\frac{u_{0,0}}{u_{-1,0}}+\frac{c_2}{c_2^2-1}\right)$	7
	$c_2^2\neq1$	$\frac{2c_2(u_{0,1}u_{0,-1}+u_{0,0}^2)+(c_2^2+1)u_{0,0}(u_{0,1}+u_{0,-1})}{u_{0,1}-u_{0,-1}}$	
T1*	$(u_{1,0}+1)(u_{0,0}-1)=$	$\left(u_{0,0}^2-1\right)(u_{1,0}-u_{-1,0})$	1
	$(u_{1,1}-1)(u_{0,1}+1)$	$(u_{0,0}^2-1)\left(\frac{1}{u_{0,1}+u_{0,0}}-\frac{1}{u_{0,0}+u_{0,-1}}\right)$	2
T2*	$u_{1,1}-u_{0,0}=\frac{1}{u_{1,0}}-\frac{1}{u_{0,1}}$	$u_{0,0}\left(\frac{1}{u_{1,0}u_{0,0}+1}-\frac{1}{u_{0,0}u_{-1,0}+1}\right)$	4
		$u_{0,0}\left(\frac{1}{u_{0,1}u_{0,0}-1}-\frac{1}{u_{0,0}u_{0,-1}-1}\right)$	

is undefined. However, the symmetries are compatible with the equation for any value of the constant κ.

Equation (T3) of table 1 is obtained from (3.17) of example 5 by applying the transformation $u_{n,m}=\frac{\hat{u}_{n,m}-1}{\hat{u}_{n,m}+1}$.

Equation (T4) of table 1 is derived from (3.20) of example 6, using the lattice-dependent point transformation

$$u_{n,m}=\hat{u}_{n,m}+(b+a-1/2)n+(a-1/2)m,$$

which allows $a=1/2$, $b=0$.

Equations (T5)–(T7) of table 1, together with the example

$$u_{0,1}(u_{0,0}+u_{1,1}+u_{1,0})+u_{0,0}u_{1,1}=0, \tag{4.3}$$

$$u_{0,0,t_1}=(u_{1,0}+u_{0,0})\left(\frac{u_{0,0}}{u_{-1,0}}+1\right),\qquad u_{0,0,t_2}=\left(\frac{u_{0,0}}{u_{0,1}}+1\right)(u_{0,-1}+u_{0,0}), \tag{4.4}$$

are obtained from example 7, i.e. (3.23) and (3.27) satisfying conditions (3.28) and (3.29). We consider all possible cases and remove some constants by using transformations of the form (3.26). Equation (4.3) together with its symmetries (4.4) is transformed into (T5) of table 1 by the transformation $u_{n,m} = \hat{u}_{n,-m}$ which is not standard for this paper. For this reason, (4.3) is not included in table 1.

Equations (T1*) and (T2*) of table 1 are particular cases of (T1) and (T2), respectively, as shown in examples 2 and 4. However, these equations are interesting and well known as themselves and are included in table 1 to provide a more complete picture.

Comparing the generalized symmetries, we can easily show that the main seven equations of the table are different. More precisely, we have the following statement.

Theorem 10. *Up to (n, m)-dependent point transformations $u_{n,m} = \omega_{n,m}(\hat{u}_{n,m})$, (T1)–(T7) of table 1 are different from Klein-type equations and from each other.*

The proof is more or less obvious. We will give below some special comments only in the case of (T1) and (T3) of table 1.

The second symmetries of (T4), (T6) and (T7) of table 1 are particular cases of the YdKN equation. In these cases, we have no problem finding further generalized symmetries. This result shows that we can also obtain for the YdKN equation auto-Bäcklund transformations which are not equations of the Klein type.

From the point of view of its generalized symmetries, (T3) of table 1 is close to a Klein-type equation. Indeed, using the lattice-dependent point transformation $u_{n,m} = \hat{u}_{n,m} i^{n+m}$, we obtain from its symmetries the equations

$$u_{0,0,t_1} = 2u_{0,0}\frac{u_{1,0} + u_{-1,0}}{u_{1,0} - u_{-1,0}}, \qquad u_{0,0,t_2} = 2u_{0,0}\frac{u_{0,1} + u_{0,-1}}{u_{0,1} - u_{0,-1}},$$

which are both of the YdKN type as in the case of Klein-type equations. By such a transformation this equation becomes, however, explicitly lattice dependent

$$3(u_{n,m}u_{n+1,m+1}u_{n+1,m}u_{n,m+1} - 1) = (-1)^{n+m}(u_{n,m}u_{n+1,m+1} - u_{n+1,m}u_{n,m+1}).$$

So this equation is not a Klein-type equation, but it provides n, m-dependent Bäcklund transformation for a YdKN-type equation. Generalized symmetries for this equation can be obtained, starting from the YdKN equation, but those symmetries may be explicitly n, m-dependent due to the involved transformation.

Let us consider the following integrable Volterra-type equations (4.1):

$$\dot{u}_0 = (\alpha u_0^2 + \beta u_0 + \gamma)(u_1 - u_{-1}), \tag{4.5}$$

$$\dot{u}_0 = \left(\alpha u_0^4 + \beta u_0^2 + \gamma\right)\left(\frac{1}{u_1 + u_0} - \frac{1}{u_0 + u_{-1}}\right), \tag{4.6}$$

with α, β and γ constant coefficients. Up to linear point transformations, (4.5) contains two nonlinear equations: the Volterra equation if $\alpha = \gamma = 0$, $\beta = 1$ and the modified Volterra equation if $\alpha = 1$, $\beta = 0$. Equation (4.6) can be called a twice modified Volterra equation, as there is a Miura-type transformation from (4.6) into the modified Volterra equation [40]. Generalized symmetries of equations (4.5) can be constructed in many different ways, see e.g. [41]. Generalized symmetries for (4.6) can be obtained, using a master symmetry found in [7]. The additional equation (T1*) of table 1 provides us examples of symmetries of both types (4.5) and (4.6).

The three-point symmetries of (T2) of table 1 have the form (4.6). Also the symmetries of (T1) of table 1 (in the generic case $c \neq 0$) can be rewritten in the form (4.6), using Möbius transformations (2.9). However, we cannot rewrite them both as (4.6), using the same Möbius transformation. In particular (T1) cannot be written in the symmetric form as in the case of (T2). So (T1) and (T2) of table 1 are different. Their generalized symmetries can be taken from (4.6).

All the other generalized symmetries are related to the following integrable Volterra-type equations

$$\dot{u}_0 = (u_1 - u_0 + \delta)(u_0 - u_{-1} + \delta), \tag{4.7}$$

$$\dot{u}_0 = (\mathrm{e}^{u_1 - u_0} + \delta)(\mathrm{e}^{u_0 - u_{-1}} + \delta), \tag{4.8}$$

$$\dot{u}_0 = \mathrm{e}^{u_1 - u_0} + \mathrm{e}^{u_0 - u_{-1}}, \tag{4.9}$$

where δ is constant. In fact, the first symmetry of (T4) of table 1 is of the form of (4.7). The first symmetry of (T6) of table 1 is obtained from (4.9) by point transformation $\hat{u}_k = \mathrm{e}^{u_k}$. The other symmetries are obtained from (4.8): both symmetries of (T5) and the first symmetry of (T7) of table 1 are obtained, using the transformation $\hat{u}_k = \mathrm{e}^{u_k}$.

Equations (4.7)–(4.9) are slight modifications of (4.5). Indeed, using the transformations

$$\hat{u}_0 = u_1 - u_0 + \delta, \tag{4.10}$$

$$\hat{u}_0 = \mathrm{e}^{u_1 - u_0} + \delta, \tag{4.11}$$

$$\hat{u}_0 = \mathrm{e}^{u_1 - u_0}, \tag{4.12}$$

respectively, we transform (4.7)–(4.9) into equations of the form (4.5). As the transformations (4.10)–(4.12) are very simple, we can use these transformations, together with the symmetries of (4.5), to construct generalized symmetries for (4.7)–(4.9).

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Appendix. Klein symmetries and Q_V

Let us consider the most general multilinear equation (2.1)

$$\begin{aligned} &u_{0,0}u_{1,0}u_{0,1}u_{1,1}k_1 + u_{1,0}u_{0,1}u_{1,1}k_2 + u_{0,0}u_{0,1}u_{1,1}k_3 + u_{0,0}u_{1,0}u_{1,1}k_4 + u_{0,0}u_{1,0}u_{0,1}k_5 \\ &\quad + u_{0,0}u_{1,0}k_6 + u_{0,0}u_{0,1}k_7 + u_{0,0}u_{1,1}k_8 + u_{1,0}u_{0,1}k_9 + u_{1,0}u_{1,1}k_{10} + u_{0,1}u_{1,1}k_{11} \\ &\quad + u_{0,0}k_{12} + u_{1,0}k_{13} + u_{0,1}k_{14} + u_{1,1}k_{15} + k_{16} = 0 \end{aligned} \tag{A.1}$$

Imposing the discrete symmetries

$$\mathcal{E}(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = \mathcal{E}(u_{1,0}, u_{0,0}, u_{1,1}, u_{0,1}) = \mathcal{E}(u_{0,1}, u_{1,1}, u_{0,0}, u_{1,0}), \tag{A.2}$$

we obtain the Q_V equation

$$\begin{aligned} &u_{0,0}u_{1,0}u_{0,1}u_{1,1}k_1 + (u_{1,0}u_{0,1}u_{1,1} + u_{0,0}u_{0,1}u_{1,1} + u_{0,0}u_{1,0}u_{1,1} + u_{0,0}u_{1,0}u_{0,1})k_2 \\ &\quad + (u_{0,0}u_{1,0} + u_{0,1}u_{1,1})k_6 + (u_{0,0}u_{0,1} + u_{1,0}u_{1,1})k_7 + (u_{0,0}u_{1,1} + u_{1,0}u_{0,1})k_8 \\ &\quad + (u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1})k_{12} + k_{16} = 0. \end{aligned} \tag{A.3}$$

By direct calculation, we can check that the Q_V equation is invariant under an n, m-independent Möbius transformation.

A Klein-type equation satisfies the following discrete symmetries:

$$\mathcal{E}(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = \pi_1 \mathcal{E}(u_{1,0}, u_{0,0}, u_{1,1}, u_{0,1}) = \pi_2 \mathcal{E}(u_{0,1}, u_{1,1}, u_{0,0}, u_{1,0}), \tag{A.4}$$

where $\pi_1 = \pm 1$, $\pi_2 = \pm 1$. In addition to the equation Q_V, we have three other possible cases.

If $\pi_1 = 1$ and $\pi_2 = -1$, we obtain

$$(u_{1,0}u_{0,1}u_{1,1} + u_{0,0}u_{0,1}u_{1,1} - u_{0,0}u_{1,0}u_{1,1} - u_{0,0}u_{1,0}u_{0,1})k_2 \\ +(u_{0,0}u_{1,0} - u_{0,1}u_{1,1})k_6 + (u_{0,0} + u_{1,0} - u_{0,1} - u_{1,1})k_{12} = 0. \tag{A.5}$$

The case when $\pi_1 = -1$ and $\pi_2 = 1$ is equivalent to the previous one up to transformation $u_{n,m} = \hat{u}_{m,n}$.

If $\pi_1 = -1$ and $\pi_2 = -1$, we obtain

$$(u_{1,0}u_{0,1}u_{1,1} - u_{0,0}u_{0,1}u_{1,1} - u_{0,0}u_{1,0}u_{1,1} + u_{0,0}u_{1,0}u_{0,1})k_2 \\ +(u_{0,0}u_{1,1} - u_{1,0}u_{0,1})k_8 + (u_{0,0} - u_{1,0} - u_{0,1} + u_{1,1})k_{12} = 0. \tag{A.6}$$

Equations (A.5) and (A.6) are invariant under Möbius transformations, thus showing that this property is valid for all Klein-type equations.

Let us consider (A.5). If $k_2 = k_{12} = 0$, the equation is degenerate:

$$(T_2 - 1)(u_{0,0}u_{1,0}k_6) = 0.$$

If either k_2 or k_{12} is not zero, we can make $k_2 \neq 0$ by using the transformation $u_{n,m} = 1/\hat{u}_{n,m}$. Then, using $u_{n,m} = \hat{u}_{n,m} + k_6/(2k_2)$, we make $k_6 = 0$. Equation (A.5) with $k_6 = 0$ is reduced by the lattice-dependent transformation $u_{n,m} = \hat{u}_{n,m}(-1)^m$ to the Q_V type equation

$$(u_{1,0}u_{0,1}u_{1,1} + u_{0,0}u_{0,1}u_{1,1} + u_{0,0}u_{1,0}u_{1,1} + u_{0,0}u_{1,0}u_{0,1})k_2 \\ +(u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1})k_{12} = 0. \tag{A.7}$$

Consider (A.6). If $k_2 = k_{12} = 0$, the equation is degenerate:

$$(T_2 - 1)(k_8 u_{1,0}/u_{0,0}) = 0.$$

If at least one of k_2, k_{12} is not zero, we can make $k_2 \neq 0$, using transformation $u_{n,m} = 1/\hat{u}_{n,m}$. Then, using $u_{n,m} = \hat{u}_{n,m} + k_8/(2k_2)$, we make $k_8 = 0$. Equation (A.6) with $k_8 = 0$ is reduced by the lattice-dependent transformation $u_{n,m} = \hat{u}_{n,m}(-1)^{n+m}$ to (A.7).

So the Klein-type equations are effectively equivalent to Q_V.

References

[1] Adler V E, Bobenko A I and Suris Yu B 2003 Classification of integrable equations on quad-graphs. The consistency approach *Commun. Math. Phys.* **233** 513–43

[2] Adler V E, Bobenko A I and Suris Yu B 2009 Discrete nonlinear hyperbolic equations. Classification of integrable cases *Funk. Anal. Prilozh.* **43** 3–23

Adler V E, Bobenko A I and Suris Yu B 2009 *Funct. Anal. Appl.* **43** 3–17 (Engl. Transl.)

[3] Adler V E, Shabat A B and Yamilov R I 2000 Symmetry approach to the integrability problem *Teor. Mat. Fiz.* **125** 355–424

Adler V E, Shabat A B and Yamilov R I 2000 *Theor. Math. Phys.* **125** 1603–61 (Engl. Transl.)

[4] Adler V E and Startsev S Ya 1999 On discrete analogues of the Liouville equation *Teor. Mat. Fiz.* **121** 271–84

Adler V E and Startsev S Ya 1999 *Theor. Math. Phys.* **121** 1484–95 (Engl. Transl.)

[5] Adler V E and Suris Yu B 2004 Q4: integrable master equation related to an elliptic curve *Int. Math. Res. Not.* **47** 2523–53

[6] Bocharov A V 1989 DELiA: a system for exact analysis of differential equations using S. Lie approach *DELiA 1.5.1 User Guide* (Beaver Soft Programming Team, Joint Venture OWIMEX Program Systems Institute USSR) (Pereslavl-Zalessky, USSR: Academy of Sciences)

[7] Cherdantsev I Yu and Yamilov R I 1995 Master symmetries for differential-difference equations of the Volterra type *Physica* D **87** 140–4

[8] Grammaticos B, Ramani A and Papageorgiou V 1991 Do integrable mappings have the Painleve property? *Phys. Rev. Lett.* **67** 1825–8

[9] Habibullin I T 2005 Characteristic algebras of fully discrete hyperbolic type equations *SIGMA* **1** 023 (arXiv:nlin/0506027)

[10] Habibullin I T and Gudkova E V 2010 An algebraic method of classification of S-integrable discrete models arXiv:1006.3423v2

[11] Habibullin I T and Gudkova E V 2010 Classification of integrable discrete Klein–Gordon models arXiv:1011.3364v1

[12] Hernandez Heredero R, Levi D and Scimiterna C A discrete integrability test based on multiple scales analysis (in preparation)

[13] Hietarinta J 2011 Definitions and predictions of integrability for difference equations *Symmetries and Integrability of Difference Equations (LMS* vol 381*)* ed D Levi, P Olver, Z Thomova and P Winternitz (Cambridge: Cambridge University Press) pp 83–114

[14] Hietarinta J and Viallet C 2000 Discrete Painlevé I and singularity confinement in projective space *Chaos Solitons Fractals* **11** 29–32

[15] Hietarinta J and Viallet C 2007 Searching for integrable lattice maps using factorization *J. Phys. A: Math. Theor.* **40** 12629–43

[16] Hirota R and Tsujimoto S 1994 Conserved quantities of discrete Lotka–Volterra equation *RIMS Kokyuroku* **868** 31–8 (in Japanese)

[17] Hirota R and Tsujimoto S 1995 Conserved quantities of a class of nonlinear difference–difference equations *J. Phys. Soc. Japan* **64** 3125–7

[18] Hydon P E and Viallet C M 2010 Asymmetric integrable quad-graph equations *Appl. Anal.* **89** 493–506

[19] Levi D, Petrera M, Scimiterna C and Yamilov R 2008 On Miura transformations and Volterra-type equations associated with the Adler–Bobenko–Suris equations *SIGMA* **4** 077 (arXiv:0802.1850)

[20] Levi D and Yamilov R I 2009 On a nonlinear integrable difference equation on the square *Ufa Math. J.* **1** 101–5

[21] Levi D and Yamilov R 1997 Conditions for the existence of higher symmetries of evolutionary equations on the lattice *J. Math. Phys.* **38** 6648–74

[22] Levi D and Yamilov R I 2009 The generalized symmetry method for discrete equations *J. Phys. A: Math. Theor.* **42** 454012

[23] Maruno K, Kajiwara K, Nakao S and Oikawa M 1997 Bilinearization of discrete soliton equations and singularity confinement *Phys. Lett.* A **229** 173–82

[24] Mikhailov A V, Shabat A B and Sokolov V V 1991 The symmetry approach to classification of integrable equations *What Is Integrability?* ed V E Zakharov (Berlin: Springer)

[25] Mikhailov A V, Shabat A B and Yamilov R I 1887 The symmetry approach to the classification of nonlinear equations. Complete lists of integrable systems *Usp. Mat. Nauk* **42** 3–53
Mikhailov A V, Shabat A B and Yamilov R I 1987 *Russ. Math. Surv.* **42** 1–63 (Engl. Transl.)

[26] Mikhailov A V, Wang J P and Xenitidis P 2010 Recursion operators, conservation laws and integrability conditions for difference equations arXiv:1004.5346

[27] Nijhoff F, Atkinson J and Hietarinta J 2009 Soliton solutions for ABS lattice equations: I. Cauchy matrix approach *J. Phys. A: Math. Theor.* **42** 404005

[28] Nijhoff F W and Capel H 1995 The discrete Korteweg–de Vries equation *Acta Appl. Math.* **39** 133–58

[29] Nijhoff F, Hone A and Joshi N 2000 On a Schwarzian PDE associated with the KdV hierarchy *Phys. Lett* A **267** 147–56

[30] Nijhoff F W, Ramani A, Grammaticos B and Ohta Y 2001 On discrete Painlevé equations associated with the lattice KdV systems and the Painlevé VI equation *Stud. Appl. Math.* **106** 261–314

[31] Nimmo J J C and Schief W K 1997 Superposition principles associated with the Moutard transformation: an integrable discretization of a (2+1)-dimensional sine-Gordon system *Proc. R. Soc.* A **453** 255–79

[32] Ramani A, Grammaticos B, Satsuma J and Willox R 2009 On two (not so) new integrable partial difference equations *J. Phys. A: Math. Theor.* **42** 282002

[33] Rasin A G 2010 Infinitely many symmetries and conservation laws for quad-graph equations via the Gardner method arXiv:1001.0724

[34] Rasin O G and Hydon P E 2007 Symmetries of integrable difference equations on the quad-graph *Studies Appl. Math.* **119** 253–69

[35] Tongas A, Tsoubelis D and Xenitidis P 2007 Affine linear and D_4 symmetric lattice equations: symmetry analysis and reductions *J. Phys. A: Math. Theor.* **40** 13353–84

[36] Viallet C M 2009 Integrable lattice maps: Q_V, a rational version of Q_4 *Glasgow Math. J.* **51** 157–63

[37] Xenitidis P 2009 Integrability and symmetries of difference equations: the Adler–Bobenko–Suris case *Proc. 4th Workshop 'Group Analysis of Differential Equations and Integrable Systems'* pp 226–42 (arXiv:0902.3954)

[38] Xenitidis P D and Papageorgiou V G 2009 Symmetries and integrability of discrete equations defined on a black–white lattice *J. Phys. A: Math. Theor.* **42** 454025

[39] Yamilov R I 1983 Classification of discrete evolution equations *Usp. Mat. Nauk* **38** 155–6 (in Russian)

[40] Yamilov R I 1994 Construction scheme for discrete Miura transformations *J. Phys. A: Math. Gen.* **27** 6839–51

[41] Yamilov R 2006 Symmetries as integrability criteria for differential difference equations *J. Phys. A: Math. Gen.* **39** R541–623