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FAST TRACK COMMUNICATION

Lie point symmetries of differential–difference equations

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Abstract
We present an algorithm for determining the Lie point symmetries of differential equations on fixed non-transforming lattices, i.e. equations involving both continuous and discrete-independent variables. The symmetries of a specific integrable discretization of the Krichever–Novikov equation, the Toda lattice and Toda field theory are presented as examples of the general method.

1. Introduction

Two different but equivalent infinitesimal formalisms exist for calculating Lie point symmetries of differential equations [20]. One is that of 'standard' vector fields

$$\hat{X} = \sum_{i=1}^{p} \xi_i(\vec{x}, \vec{u})\partial_{x_i} + \sum_{\alpha=1}^{q} \phi_\alpha(\vec{x}, \vec{u})\partial_{u_\alpha} \tag{1}$$

acting on the independent variables x_i and the dependent ones u_α in the considered differential equation.

The other is that of the *evolutionary* vector fields

$$\hat{X}^E = \sum_{\alpha=1}^{q} \mathcal{Q}_\alpha(\vec{x}, \vec{u}, \vec{u}_{\vec{x}})\partial_{u_\alpha}, \tag{2}$$

acting only on the dependent variables.

The equivalence of the two formalisms is due to the fact that the total derivatives D_{x_i} are themselves 'generalized' symmetry operators, so for any differential equation

$$\mathcal{E}\big(x_i, u_\alpha, u_{\alpha,x_i}, \ldots\big) = 0 \tag{3}$$

we have

$$\mathrm{pr}\hat{X}^E \mathcal{E}|_{\mathcal{E}=0} = \left(\mathrm{pr}\hat{X} - \sum_{i=1}^{p} \xi_i D_{x_i}\right) \mathcal{E}|_{\mathcal{E}=0} = 0. \tag{4}$$

Here $\mathrm{pr}\hat{X}^E$ and $\mathrm{pr}\hat{X}$ are the appropriate differential prolongations of $\hat{X}^E$ and $\hat{X}$. Relation (4) implies that for point transformations we have

$$Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi_i u_{\alpha,x_i}. \tag{5}$$

For all details we refer to e.g. Olver's textbook [20].

An advantage of the standard formalism is its direct relation to the group transformations obtained by integrating the equations

$$\begin{aligned} &\frac{\mathrm{d}\tilde{x}_i}{\mathrm{d}\lambda} = \xi_i(\tilde{\tilde{x}}, \tilde{\tilde{u}}), \qquad \frac{\mathrm{d}\tilde{u}_\alpha}{\mathrm{d}\lambda} = \phi_\alpha(\tilde{\tilde{x}}, \tilde{\tilde{u}}), \\ &\tilde{x}_i|_{\lambda=0} = x_i, \qquad \tilde{u_\alpha}|_{\lambda=0} = u_\alpha, \qquad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q. \end{aligned} \tag{6}$$

One advantage of the evolutionary formalism is its direct relation with the existence of flows commuting with the studied equation (3)

$$\frac{\mathrm{d}\tilde{u}_\alpha}{\mathrm{d}\lambda} = Q_\alpha, \tag{7}$$

where Q_α is the characteristic of the vector field as in (5).

Another advantage is that the evolutionary formalism can easily be adapted to the case of higher symmetries.

Let us now consider a purely discrete equation, i.e. a difference equation. We restrict to the case of one scalar function defined on a two-dimensional lattice u_{mn}. We shall view u as a continuous variable, introduce two further continuous variables x and t and consider (x, t, u) as being evaluated, or sampled at discrete points on a lattice labeled by the indices m, n. We shall write (x_{mn}, t_{mn}, u_{mn}) for values at these points.

A difference system will consist of the relations

$$\begin{aligned} &\mathcal{E}_a(x_{m+k,n+l}, t_{m+k,n+l}, u_{m+k,n+l}) = 0, \qquad a = 1, \ldots, A, \\ &k_m \leqslant k \leqslant k_M, \qquad l_m \leqslant l \leqslant l_M, \end{aligned} \tag{8}$$

between the variables x, t and u evaluated at a finite number of points on a lattice.

A Lie point symmetry of the system (8) will be generated by a vector field of the form

$$\hat{X}_{mn} = \xi(x_{mn}, t_{mn}, u_{mn})\partial_{x_{mn}} + \tau(x_{mn}, t_{mn}, u_{mn})\partial_{t_{mn}} + \phi(x_{mn}, t_{mn}, u_{mn})\partial_{u_{mn}}. \tag{9}$$

We see that the vector field (9) for difference equations has the same form as (1) for differential ones. Its prolongation is however different, namely

$$\mathrm{pr}\hat{X} = \sum_{k,l} \hat{X}_{m+k,n+l}, \tag{10}$$

where the sum is over all points figuring in the system (8).

In the continuous limit the system (8) goes into a partial differential equation and equation (10) goes into the usual prolongation of a standard vector field (i.e. it also acts on functions of derivatives).

For recent reviews of the theory of continuous symmetries of difference equations see [3, 13, 25].

The purpose of this communication is to consider an intermediate case, that of differential–difference equations. In section 2 we shall take a 'semicontinuous' limit, i.e. leave the variable

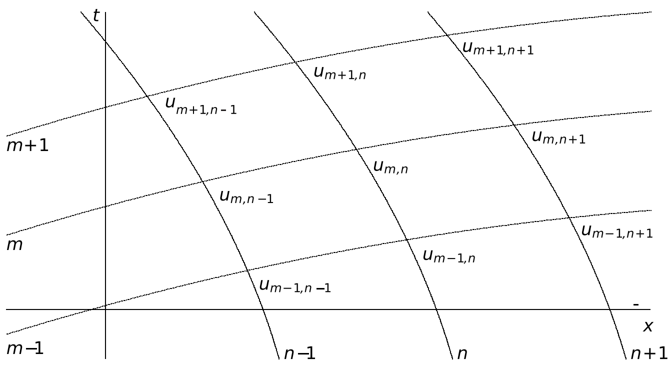

Figure 1. Example of a two-dimensional lattice.

x discrete but let t tend to a continuous variable. This will provide us with both a standard and an evolutionary formalism for calculating point symmetries of differential–difference equations. In section 3 we consider several special cases and prove some theorems that greatly simplify the calculation of symmetries. Section 4 is devoted to examples and section 5 to a summary of the results obtained.

2. Lie point symmetries of difference systems and their semicontinuous limit

2.1. The semicontinuous limit

The starting point to this section is the Lie symmetry formalism for purely difference equations with several (discrete) independent variables [10, 13]. A difference scheme is defined on a *discrete jet space*, a space of independent and dependent variables on a lattice. In this communication we restrict ourselves to the case of two independent variables x, t and one dependent one u defined on a two-dimensional lattice with points labeled by two indices. We shall write

$$\{x_{mn}, t_{mn}, u_{mn} \equiv u(x_{mn}, t_{mn})\} \tag{11}$$

(see figure 1).

The *discrete jet space* will be the set of all variables $\{x_{jk}, t_{jk}, u_{jk}\}$ on the lattice. The dependence of x, t and u on the labels m, n is determined from the difference system

$$\mathcal{E}^a_{mn}(x_{jk}, t_{jk}, u_{jk}) = 0, \qquad a = 1, \ldots, N, \tag{12}$$

and some boundary conditions. In equation (12) N is an integer satisfying $N \geqslant 3$ and (j, k) run over some finite set of values on the lattice while (m, n) is a fixed reference point. Equation (12) thus determines both the difference equation and the lattice.

Lie point symmetries of the system (12) are generated by vector fields of the form

$$\hat{X}^D_{mn} = \xi(x_{mn}, t_{mn}, u_{mn})\partial_{x_{mn}} + \tau(x_{mn}, t_{mn}, u_{mn})\partial_{t_{mn}} + \phi(x_{mn}, t_{mn}, u_{mn})\partial_{u_{mn}} \tag{13}$$

(the superscript D stands for ‘discrete’) satisfying

$$\mathrm{pr}\hat{X}^D \mathcal{E}^a|_{\mathcal{E}^1=\mathcal{E}^2=\cdots=\mathcal{E}^N=0} = 0. \tag{14}$$

In equation (14) $\mathrm{pr}\hat{X}^D$ is the prolongation of the vector field $\hat{X}^D_{mn}$ to the discrete jet space

$$\mathrm{pr}\hat{X}^D = \sum_{j,k} \hat{X}^D_{jk}, \tag{15}$$

where the sum is over all points figuring in the difference system (12).

In this approach the lattice (x_{mn}, t_{mn}) is in general determined together with u_{mn} from the system (12) and the group transformations generated by the vector field $\hat{X}^D_{mn}$ also transform the lattice. A special case corresponds to an *a priori* determined non-transforming lattice. In that case two of the equations in the system (12) have the form

$$x_{mn} = f(m, n), \qquad t_{mn} = g(m, n), \tag{16}$$

where f and g are given. Such is the case of a uniform orthogonal lattice where (16) takes the form

$$x_{mn} = \sigma_1 n + x_0, \qquad t_{mn} = \sigma_2 m + t_0, \tag{17}$$

and the scale factors (σ_1, σ_2) and the origin (x_0, t_0) are given numbers (e.g. $\sigma_1 = \sigma_2 = 1,\ x_0 = t_0 = 0$).

We are interested in obtaining the form of the vector field in the semicontinuous limit in which t_{mn} becomes a continuous variable t, but x_{mn} remains discrete (and independent of t). Thus x will depend on one discrete label n only and in particular x may be given as $x_n = f(n)$, with $f(n)$ known (e.g. $x_n = hn$ for a uniform lattice, or $x_n = \lambda^n$ for an exponential one).

In this limit the difference system (12) will reduce to a differential–difference equation (DΔE)

$$\mathcal{E}_n\big(t, u_j, \dot{u}_j, \ddot{u}_j, \ldots, u_j^{(K)}\big) = 0, \qquad n - L \leqslant j \leqslant n + M, \tag{18}$$

where dots and (K) denote t-derivatives, and K, L and M are some nonnegative integers. Together with equation (18) we have another equation which determines $x_n = f(n)$. We shall consider the case when $f(n)$ is already given (a known function) so that we can replace the dependence on x_n by a dependence on the integer n (without necessarily assuming that $f(n)$ is linear).

Equation (18) is thus defined on a 'semidiscrete jet space' with local coordinates

$$\big\{t, u_j, \dot{u}_j, \ddot{u}_j, \ldots, u_j^{(K)}\big\}, \tag{19}$$

where j runs over all values on a one-dimensional lattice.

The vector field generating symmetries of equation (18) will have the form

$$\hat{X}^{SD} = \tau_n(t, u_n)\partial_t + \phi_n(t, u_n)\partial_{u_n} \tag{20}$$

$(SD =$ semidiscrete), and its prolongation will be defined on the semidiscrete jet space (19).

Let us consider the simplest nontrivial case, namely that of a difference system (12) involving the three points (m, n), $(m + 1, n)$ and $(m, n + 1)$, i.e. relating the variables x_{jk}, t_{jk} and u_{jk} in these three points:

$$\mathcal{E}_n(x_{mn}, x_{m+1,n}, x_{m,n+1}, t_{mn}, t_{m+1,n}, t_{m,n+1}, u_{mn}, u_{m+1,n}, u_{m,n+1}) = 0. \tag{21}$$

Before taking the limit we change notation and transform to new variables. We choose a reference point (m, n) on the lattice (see figure1) and measure t and x from this point:

$$t_{m+a,n+b} = t + \epsilon_{ab}, \qquad x_{m+a,n+b} = f(n + b) + \theta(\epsilon_{ab}), \tag{22}$$

where $f(n + b)$ is a given function and $\theta(\epsilon_{ab})$ is an analytic function. Instead of $u_{m+j,n+k}$ we introduce a function $v_n(t)$

$$v_{n+b}(t + \epsilon_{ab}) \equiv u_{m+a,n+b}(x_{m+a,n+b}, t_{m+a,n+b}), \tag{23}$$

and assume that the dependence on t is analytical. Thus the dependence on x (which remains discrete in the limit $\epsilon_{ab} \to 0$) is replaced by a dependence on the label $n + b$. For the reference point x_{mn}, t_{mn} we put $\epsilon_{00} = 0$, $\theta(\epsilon_{00}) = 0$.

So, in the case of the stencil (m, n), $(m + 1, n)$ and $(m, n + 1)$ we have

$$\begin{aligned}
&t_{mn} \equiv t, \qquad t_{m+1,n} \equiv t + \epsilon_{10}, \qquad t_{m,n+1} \equiv t + \epsilon_{01},\\
&x_{mn} \equiv f(n), \qquad x_{m+1,n} = f(n) + \theta(\epsilon_{10}), \qquad x_{m,n+1} \equiv f(n+1) + \theta(\epsilon_{01}),\\
&u_{mn}(x_{mn}, t_{mn}) = v_n(t), \qquad u_{m+1,n} \equiv v_n(t) + \epsilon_{10} v_{n,t}(t),\\
&u_{m,n+1}(x_{m,n+1}, t_{m,n+1}) = v_{n+1}(t + \epsilon_{01}) = v_{n+1}(t) + \sum_{j=1}^{\infty} \frac{\epsilon_{01}^j}{j!} v_{n+1}^{(j)}(t),
\end{aligned} \tag{24}$$

where $v_{n+1}^{(j)}(t) = \frac{\mathrm{d}^j v_{n+1}(t)}{\mathrm{d}t^j}$ and $v_{n,t}$ is the 'discrete derivative' of $v_n(t)$ given by $v_n(t) = \frac{u_{m+1,n} - u_{m,n}}{\epsilon_{10}}$.

As long as time is discrete the introduction of ϵ_{10} and ϵ_{01} amounts to a change of variables within one stencil. In the continuous time limit we will assume that the quantities ϵ go to zero uniformly in all points (i.e. independently of time). A similar assumption is used when constructing symmetry preserving discretizations of ordinary differential equations [2, 21]. This assumption is used in the Taylor expansion in (24).

Since $v_n(t)$ is by assumption analytical, the Taylor series in (24) are convergent. Using equation (24) we can also express the derivatives $\{\partial_{t_{mn}}, \partial_{t_{m+1,n}}, \partial_{t_{m,n+1}}, \partial_{u_{mn}}, \partial_{u_{m+1,n}}, \partial_{u_{m,n+1}}\}$ in terms of $\{\partial_t, \partial_{\epsilon_{10}}, \partial_{\epsilon_{01}}, \partial_{v_n}, \partial_{v_{n,t}}, \partial_{v_{n+1}}\}$ and thus transform the prolongation of the vector field (15). We obtain

$$\begin{aligned}
\mathrm{pr}\hat{X}^D = {} & \tau_{mn}\partial_t + \phi_{mn}\partial_{v_n} + (\tau_{m+1,n} - \tau_{mn})\partial_{\epsilon_{10}} + (\tau_{m,n+1} - \tau_{mn})\partial_{\epsilon_{01}}\\
& + [(\phi_{m+1,n} - \phi_{mn}) + (\tau_{mn} - \tau_{m+1,n})v_{n,t}]\frac{1}{\epsilon_{10}}\partial_{v_{n,t}}\\
& + \left[\phi_{m,n+1} + (\tau_{mn} - \tau_{m,n+1})\sum_{j=1}^{\infty}\frac{(t_{m,n+1} - t_{mn})^{j-1}}{(j-1)!}v_{n+1}^{(j)}\right]\partial_{v_{n+1}}.
\end{aligned} \tag{25}$$

Further, we put

$$\tau_{mn} \equiv \tau_n(t, v_n), \qquad \phi_{mn} \equiv \phi_n(t, v_n), \tag{26}$$

and expand $\tau_{m+1,n}$ and $\phi_{m+1,n}$ about $\epsilon_{10} = 0$, $\tau_{m,n+1}$ and $\phi_{m,n+1}$ about $\epsilon_{01} = 0$ and then let $\mathrm{pr}\hat{X}^D$ act on functions

$$\mathcal{E}_n = \mathcal{E}_n(t, v_n, v_{n+1}, v_{n,t}), \tag{27}$$

obtained as the limit of equation (21). In the semicontinuous limit we take $\epsilon_{10} \to 0$, $\epsilon_{01} \to 0$ and we obtain

$$\lim_{(\epsilon_{10},\epsilon_{01})\to(0,0)} \mathrm{pr}\hat{X}^D = \mathrm{pr}\hat{X}^{SD} = \tau_n\partial_t + \phi_n\partial_{v_n} + \phi_n^{[t]}\partial_{v_{n,t}} + \phi_n^{[n+1]}\partial_{v_{n+1}^{(1)}}, \tag{28}$$

$$\phi_n^{[t]} = D_t\phi_n - (D_t\tau_n)\dot{v}_n, \tag{29}$$

$$\phi_n^{[n+1]} = \phi_{n+1} + (\tau_n - \tau_{n+1})v_{n+1}^{(1)}. \tag{30}$$

The form (29) of the coefficient $\phi_n^{[t]}$ is the 'obvious' generalization of the first prolongation for ordinary differential equations. The presence of the second term in (30) is less obvious and follows from the above analysis of the semicontinuous limit. We see that the prolongation of the vector field $\hat{X}^{SD}$ to derivatives is the same as for differential equations. The prolongation to other points x_n on the lattice does, however, not consist of merely shifting n in ϕ_n.

We mention that the additional term in $\phi_n^{[n+1]}$ was missed in [11].

If we start from the set of all nine points on the stencil of figure 1 and take the semicontinuous limit in the same way, we arrive at a more general DΔE, namely

$$\mathcal{E}_n(t, u_n, u_{n+1}, u_{n-1}, \dot{u}_n, \dot{u}_{n+1}, \dot{u}_{n-1}, \ddot{u}_n, \ddot{u}_{n+1}, \ddot{u}_{n-1}) = 0 \tag{31}$$

(with the change of notation to u_n). We also obtain the prolongation $\mathrm{pr}\hat{X}^{SD}$ for such an equation (see below).

2.2. *The evolutionary formalism and commuting flows for differential–difference equations*

An alternative method of calculating symmetries of DΔE on a fixed lattice is to construct commuting flows in two variables. Let us again consider equation (27), this time solved for the first derivative, and change the notation from $\dot{v}_n$ to $u_{n,t}$, which now denotes an ordinary time derivative:

$$\dot{u}_n \equiv u_{n,t} = \mathcal{F}_n(t, u_n, u_{n+1}). \tag{32}$$

We introduce an additional variable λ, the group parameter and consider the flow on $u_n(t, \lambda)$ in this variable:

$$u_{n,\lambda} = \mathcal{Q}_n(t, u_n, \dot{u}_n). \tag{33}$$

Let us now require that the flows (32) and (33) be compatible, i.e. commute. Thus we impose

$$u_{n,t\lambda} = u_{n,\lambda t}. \tag{34}$$

We replace $u_{n,\lambda}$ using equation (33), $\dot{u}_n$ and $\ddot{u}_n$ using (32) and its differential consequences and obtain

$$\mathcal{Q}_{n,t} + \mathcal{Q}_{n,u_n}\mathcal{F}_n + \mathcal{Q}_{n,\dot{u}_n}\left(\mathcal{F}_{n,t} + \mathcal{F}_{n,u_n}\mathcal{F}_n + \mathcal{F}_{n,u_{n+1}}\mathcal{F}_{n+1}\right) = \mathcal{F}_{n,u_n}\mathcal{Q}_n + \mathcal{F}_{n,u_{n+1}}\mathcal{Q}_{n+1}. \tag{35}$$

This derivation of (35) is completely equivalent to the following procedure. We first introduce an evolutionary vector field

$$\hat{X}_E = \mathcal{Q}_n(t, u_n, \dot{u}_n, \ldots)\partial_{u_n}, \tag{36}$$

and its prolongation

$$\mathrm{pr}\hat{X}_E = \mathcal{Q}_n\partial_{u_n} + \mathcal{Q}_{n+1}\partial_{u_{n+1}} + (D_t\mathcal{Q}_n)\partial_{\dot{u}_n} + \cdots. \tag{37}$$

We then apply this prolonged field to equation (32), require

$$\mathrm{pr}\hat{X}_E[\dot{u}_n - \mathcal{F}_n(t, u_n, u_{n+1})]|_{(\dot{u}_n=\mathcal{F}_n, \ddot{u}_n=D_t\mathcal{F}_n)} = 0, \tag{38}$$

and reobtain equation (35).

Let us now specialize to the case of point symmetries. The quantity $\mathcal{Q}_n$ in (33) and (36) is the *characteristic* of the vector field $\hat{X}_E$. For point symmetries it has the form

$$\mathcal{Q}_n(t, u_n, \dot{u}_n, \ldots) = \phi_n(t, u_n) - \tau_n(t, u_n)\dot{u}_n. \tag{39}$$

The total derivative D_t is itself a (generalized) symmetry of the DΔE (18) and in particular (32). This provides us with a relation between ordinary and evolutionary vector fields and their prolongations, namely

$$\mathrm{pr}\hat{X} = \mathrm{pr}\hat{X}_E + \tau_n(t, u_n)D_t. \tag{40}$$

Substituting (39) and (37) into (40) we reobtain equations (28)–(30).

We see that the ‘obvious’ prolongation (37) of the evolutionary vector field (36) provides, via equation (40), the correct prolongation (28) of the ordinary vector field (20).

2.3. General algorithm for calculating Lie point symmetries of a differential–difference equation

Let us consider a DΔE involving $L+M+1$ points and t-derivatives up to order K as in equation (18). The Lie point symmetries of equation (18) can be obtained using the evolutionary formalism by imposing

$$\mathrm{pr}\hat{X}_E\mathcal{E}_n|_{\mathcal{E}_n=0,D_t^k\mathcal{E}_n=0}=0, \qquad k=1,\ldots,K. \tag{41}$$

Thus the expression $\mathrm{pr}\hat{X}_E\mathcal{E}_n$ is annihilated on the solution set of equation (18) and of all differential consequences of the equation.

The vector field $\hat{X}_E$ has the form (36) with $\mathcal{Q}_n$ as in (39). The prolongation of $\hat{X}_E$ is

$$\mathrm{pr}\hat{X}_E=\sum_j\mathcal{Q}_j\partial_{u_j}+\sum_{k=1}^{K}\sum_{j^{\text{‘}}}\left(D_t^k\mathcal{Q}_j\right)\partial_{u_j^{(k)}}, \tag{42}$$

where the j summation is over all points figuring in equation (18) and $u_j^{(k)}$ denotes the kth t-derivative of u_j.

The standard vector field $\hat{X}$ generating Lie point symmetries and its prolongation are given by formula (40). Explicitly the prolongation formula is

$$\mathrm{pr}\hat{X}=\phi_n\partial_{u_n}+\tau_n\partial_t+\sum_{j\neq n}\phi_j\partial_{u_j}+\sum_j\sum_{k=1}^{K}\phi_j^{[k]}\partial_{u_j^{(k)}}+\sum_j\sum_{k=1}^{K}(\tau_n-\tau_j)\left(D_t^{k+1}u_j\right)\partial_{u_j^{(k)}}, \tag{43}$$

$$\phi_j^{[k]}=D_t\phi_j^{[k-1]}-(D_t\tau_j)u_j^{(k)}, \qquad D_t^ku_j\equiv u_j^{(k)}. \tag{44}$$

Note that $\phi_j^{[k]}$ is the same as for a differential equation [20] but the last term in (43) has no analog in the continuous case. The coefficients ϕ_n and τ_n in the vector field $\hat{X}$ itself are *a priori* functions of n, t and u_n (see equation (20)). In the following section we will examine some cases when $\tau_n(t,u_n)$ simplifies.

Equation (43) is also obtained as the semicontinuous limit of the discrete prolongation (15).

3. Theorems simplifying the calculation of symmetries of DΔE

3.1. General comments

Lie point symmetries of DΔE of the form (18) are generated by vector fields of the form (20). We shall now investigate three important cases when the coefficient $\tau_n(t,u_n)$ actually depends on t alone.

The three classes of DΔE are

$$\dot{u}_n=f_n(t,u_{n-1},u_n,u_{n+1}), \tag{45}$$

$$\ddot{u}_n=f_n(t,\dot{u}_n,u_{n-1},u_n,u_{n+1}), \tag{46}$$

$$u_{n,xy}=f_n(x,y,u_{n,x},u_{n,y},u_{n-1},u_n,u_{n+1}). \tag{47}$$

Equation (45) contains integrable Volterra, modified Volterra and discrete Burgers-type equations [26]. A list of integrable Toda-type equations of the form (46) can be found in [27]. Class (47) involves two continuous variables and contains the two-dimensional Toda model [5, 16]. A list of integrable cases exists [22] and Lie point symmetries of this class have been studied.

For equations (45) and (46) Lie point symmetries correspond to commuting flows of the form (33) with $\mathcal{Q}_n$ given by equation (39) while for equation (47) the form is

$$u_{n,\lambda} = \psi_n(x, y, u_n, u_{n,x}, u_{n,y}), \tag{48a}$$

$$\psi_n(x, y, u_n, u_{n,x}, u_{n,y}) = \phi_n(x, y, u_n) - \xi_n(x, y, u_n)u_{n,x} - \eta_n(x, y, u_n)u_{n,y}. \tag{48b}$$

For all equations ((45)–(47)), we assume everywhere below that at least one of the following two conditions is satisfied:

$$\frac{\partial f_n}{\partial u_{n+1}} \neq 0, \quad \text{for all } n, \qquad \text{or} \qquad \frac{\partial f_n}{\partial u_{n-1}} \neq 0, \quad \text{for all } n. \tag{49}$$

3.2. Volterra-type equations and their generalizations

Let us consider equation (45).

Theorem 3.1. *If (45) satisfies at least one of the conditions (49) and (33) represents a point symmetry of (45), then we have*

$$\tau_n(t, u_n) = \tau(t). \tag{50}$$

Proof. The compatibility condition (34) of equations (45) and (33), (39) implies

$$\sum_{l=-1}^{1} f_{n,u_{n+l}}[\phi_{n+l} - \tau_{n+l} f_{n+l}] + \left(\tau_{n,t} + \tau_{n,u_n} f_n\right) f_n$$
$$+ \tau_n \left[f_{n,t} + \sum_{l=-1}^{1} f_{n,u_{n+l}} f_{n+l} \right] - \phi_{n,t} - \phi_{n,u_n} f_n = 0, \tag{51}$$

where the indices t, u_{n+l} denote partial derivatives. Taking the derivative of equation (51) with respect to u_{n+2} and separately with respect to u_{n-2}, we obtain two relations:

$$f_{n+1,u_{n+2}} f_{n,u_{n+1}}(\tau_n - \tau_{n+1}) = 0, \qquad f_{n-1,u_{n-2}} f_{n,u_{n-1}}(\tau_n - \tau_{n-1}) = 0. \tag{52}$$

In view of conditions (49), equations (52) imply

$$\tau_{n+1}(t, u_{n+1}) = \tau_n(t, u_n) \qquad \text{or} \qquad \tau_{n-1}(t, u_{n-1}) = \tau_n(t, u_n). \tag{53}$$

Each of these conditions must be satisfied for any n and they are equivalent. Since $u_0, u_1, u_{-1}, \ldots$ are independent, we find that $\tau_n(t, u_n)$ depends on t alone and this proves theorem 3.1. □

A somewhat weaker theorem can be proved for a more general differential–difference equation, namely

$$\dot{u}_n = f_n(t, u_{n+k}, u_{n+k+1}, \ldots, u_{n+m}), \qquad k \leqslant m. \tag{54}$$

Theorem 3.2. *Let equations (33) and (39) represent a symmetry of equation (54). If the function f_n in (54) satisfies*

$$m > 0, \qquad \frac{\partial f_n}{\partial u_{n+m}} \neq 0 \quad \textit{for all } n, \tag{55}$$

then the function $\tau_n(t, u_n)$ is such that

$$\tau_n(t, u_n) = \tau_n(t), \qquad \tau_{n+m}(t) = \tau_n(t). \tag{56}$$

If the function f_n satisfies

$$k < 0, \qquad \frac{\partial f_n}{\partial u_{n+k}} \neq 0 \quad \textit{for all } n, \tag{57}$$

then we have

$$\tau_n(t, u_n) = \tau_n(t), \qquad \tau_{n+k}(t) = \tau_n(t). \tag{58}$$

Proof. The compatibility condition for equations (33), (39) and (54) will be the same as for equation (51) but all sums will be from $l = k$ to $l = m$. If equation (55) is satisfied we can differentiate equation (51) with respect to u_{n+2m} and obtain $\tau_n(t, u_n) = \tau_{n+m}(t, u_{n+m})$ which implies (56). If (57) is satisfied we differentiate (51) with respect to u_{n+2k} and obtain $\tau_n(t, u_n) = \tau_{n+k}(t, u_{n+k})$ which implies (58). □

This result is valid, in particular, for Burgers-type equations for which $k = 0$, $m > 0$ or $k < 0$, $m = 0$. For all equations in class (54), under the assumptions of this theorem, the function τ_n is independent of u_n and is periodic in n. In particular, if $k = -2$, $m = 2$ it is two-periodic and we can write

$$\tau_n(t) = \frac{1 + (-1)^n}{2}\tau_0(t) + \frac{1 - (-1)^n}{2}\tau_1(t). \tag{59}$$

3.3. Toda-type equations

The compatibility condition for equations (46) and (33), (39) is $u_{n,tt\lambda} = u_{n,\lambda tt}$ and implies

$$\begin{aligned} f_{n,\dot{u}_n}\big[\phi_{n,t} + \big(\phi_{n,u_n} - \tau_{n,t}\big)\dot{u}_n - \tau_{n,u_n}(\dot{u}_n)^2\big] + \sum_{k=-1}^{1} f_{n,u_{n+k}}[\phi_{n+k} + (\tau_n - \tau_{n+k})\dot{u}_{n+k}] \\ - \phi_{n,tt} + \big(- 2\phi_{n,tu_n} + \tau_{n,tt}\big)\dot{u}_n + \big(-\phi_{n,u_nu_n} + 2\tau_{n,tu_n}\big)(\dot{u}_n)^2 \\ + \tau_{n,u_nu_n}(\dot{u}_n)^3 + \tau_n f_{n,t} + \big(2\tau_{n,t} - \phi_{n,u_n} + 3\tau_{n,u_n}\dot{u}_n\big) f_n = 0. \end{aligned} \tag{60}$$

We use equation (60) to prove the following theorem.

Theorem 3.3. *Let equations (33) and (39) represent a point symmetry of equation (46) and let the function f_n in equation (46) satisfy at least one of conditions (49) for all n. Then the function $\tau_n(t, u_n)$ in equation (39) satisfies equation (50), i.e. $\tau_n(t, u_n)$ depends on t alone.*

Proof. None of the functions f_n, ϕ_n, τ_n figuring in equation (60) depend on $\dot{u}_{n+1}$ or $\dot{u}_{n-1}$. These two expressions do however figure explicitly in (60). Their coefficients must hence vanish and we obtain

$$\begin{aligned} f_{n,u_{n+1}}(\tau_n - \tau_{n+1}) = 0, \\ f_{n,u_{n-1}}(\tau_n - \tau_{n-1}) = 0. \end{aligned} \tag{61}$$

In view of conditions (49), we can use one of equations (61), and both of them provide the same:

$$\tau_n(t, u_n) = \tau_{n+1}(t, u_{n+1}) \tag{62}$$

for any n. Hence we again obtain result (50), as stated in theorem 3.3. □

3.4. Toda field theory-type equations

Let us now consider equation (47) and assume that it has a Lie point symmetry represented by (48).

Theorem 3.4. *Let (48) represent a Lie point symmetry of the field equation (47) and let the function* $f_n(x, y, u_{n,x}, u_{n,y}, u_{n-1}, u_n, u_{n+1})$ *satisfy at least one of the conditions*

$$\frac{\partial f_n}{\partial u_{n-1}} \neq 0, \qquad \textit{or} \qquad \frac{\partial f_n}{\partial u_{n+1}} \neq 0. \tag{63}$$

The functions ξ_n *and* η_n *in symmetry (48) then are given by*

$$\xi_n(x, y, u_n) = \xi(x, y), \qquad \eta_n(x, y, u_n) = \eta(x, y). \tag{64}$$

Proof. The compatibility condition $u_{n,xy\lambda} = u_{n,\lambda xy}$ in this case can be written as

$$\sum_{k=-1}^{1} \frac{\partial f_n}{\partial u_{n+k}} \psi_{n+k} + \frac{\partial f_n}{\partial u_{n,x}} D_x \psi_n + \frac{\partial f_n}{\partial u_{n,y}} D_y \psi_n - D_x D_y \psi_n = 0 \tag{65}$$

with ψ_n as in equation (48*b*); D_x and D_y are the total derivative operators. The terms $u_{n\pm1,x}$, $u_{n\pm1,y}$ only figure in $\psi_{n\pm1}$ and in $D_x D_y \psi_n$ where we have

$$D_x D_y \psi_n = -\xi_n(D_x f_n) - \eta_n(D_y f_n) + \cdots$$

with

$$D_x f_n = \frac{\partial f_n}{\partial u_{n-1}} u_{n-1,x} + \frac{\partial f_n}{\partial u_{n+1}} u_{n+1,x} + \cdots$$
$$D_y f_n = \frac{\partial f_n}{\partial u_{n-1}} u_{n-1,y} + \frac{\partial f_n}{\partial u_{n+1}} u_{n+1,y} + \cdots.$$

Substituting into (65) and setting the coefficients of $u_{n\pm1,x}$ and $u_{n\pm1,y}$ equal to zero separately, we obtain

$$\begin{aligned} &(\xi_{n-1} - \xi_n)\frac{\partial f_n}{\partial u_{n-1}} = 0, \qquad (\eta_{n-1} - \eta_n)\frac{\partial f_n}{\partial u_{n-1}} = 0, \\ &(\xi_{n+1} - \xi_n)\frac{\partial f_n}{\partial u_{n+1}} = 0, \qquad (\eta_{n+1} - \eta_n)\frac{\partial f_n}{\partial u_{n+1}} = 0. \end{aligned} \tag{66}$$

Thus, under the assumption (63) we obtain (64) and this completes the proof. □

4. Examples

Let us now consider examples of each of the classes of differential–difference equations discussed in section 3.

4.1. The YdKN equation

The Krichever–Novikov equation [6]

$$\dot{u} = \frac{1}{4} u_{xxx} - \frac{3}{8}\frac{u_{xx}^2}{u_x} + \frac{3}{2}\frac{P(u)}{u_x}, \tag{67}$$

where $P(u)$ is an arbitrary fourth degree polynomial with constant coefficients, is an integrable PDE with many interesting properties [1, 4, 6–9, 17–19, 23, 24].

Yamilov and collaborators have proposed integrable discretizations of equation (67) [15, 26, 28]. The original form of the YdKN equation [26, 28] is

$$u_{n,t} = \frac{P_n u_{n+1} u_{n-1} + Q_n (u_{n+1} + u_{n-1}) + R_n}{u_{n+1} - u_{n-1}}, \tag{68}$$

$$\begin{aligned} P_n &= \alpha u_n^2 + 2\beta u_n + \gamma, \\ Q_n &= \beta u_n^2 + \lambda u_n + \delta, \\ R_n &= \gamma u_n^2 + 2\delta u_n + \omega, \end{aligned} \tag{69}$$

where $\alpha, \ldots, \omega$ are pure constants.

A complete symmetry analysis of this equation and its generalizations is in preparation [14]. Here we will just consider one special case as an example of a Volterra-type equation. Let us set $\alpha = 1$, $\beta = \cdots = \omega = 0$ in (69). The YdKN equation reduces to

$$u_{n,t} = \frac{u_n^2 u_{n+1} u_{n-1}}{u_{n+1} - u_{n-1}}. \tag{70}$$

According to theorem 3.1 a compatible flow corresponding to a point symmetry will have the form

$$u_{n,\lambda} = \Phi_n(t, u_n) - \tau(t) u_{n,t}. \tag{71}$$

We replace $u_{n,t}$ in (71) using (70) and then impose the compatibility condition $u_{n,t\lambda} = u_{n,\lambda t}$. First of all, from terms containing u_{n+2} and u_{n-2} we find that Φ_n and τ must satisfy

$$\begin{aligned} &\tau = \tau_0 + \tau_1 t, \qquad \Phi_n = a_n + b_n u_n + c_n u_n^2, \\ &a_n = a + \hat{a}(-1)^n, \qquad b_n = b + \hat{b}(-1)^n, \qquad c_n = c + \hat{c}(-1)^n, \end{aligned} \tag{72}$$

where $\tau_0, \tau_1, a, \hat{a}, b, \hat{b}$, and $c, \hat{c}$ are pure constants. This is actually the case for the general YdKN equation (68). Substituting (72) into the compatibility condition we obtain an equation that is polynomial in u_{n+k}. Setting coefficients of $u_{n-1}^a u_n^b u_{n+1}^c$ equal to zero for each independent term we obtain the following basis of the Lie point symmetry algebra of equation (70):

$$\begin{aligned} &X_1 = \partial_t, \qquad X_2 = u_n^2 \partial_{u_n}, \qquad X_3 = (-1)^n u_n^2 \partial_{u_n}, \\ &X_4 = t\partial_t - \tfrac{1}{2} u_n \partial_{u_n}, \qquad X_5 = (-1)^n u_n \partial_{u_n}. \end{aligned} \tag{73}$$

This is a solvable Lie algebra with $\{X_1, X_2, X_3\}$ its Abelina nilradical. The two nonnilpotent elements satisfy $[X_4, X_5] = 0$ and their action on the nilradical is given by

$$\begin{aligned} &\begin{pmatrix} [X_4, X_1] \\ [X_4, X_2] \\ [X_4, X_3] \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \\ &\begin{pmatrix} [X_5, X_1] \\ [X_5, X_2] \\ [X_5, X_3] \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \end{aligned} \tag{74}$$

4.2. The Toda lattice

The Toda lattice itself,

$$u_{n,tt} = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1}), \tag{75}$$

is the best known example of an equation of the type (46). According to theorem 3.3 the flow corresponding to its point symmetries will satisfy (71). From the compatibility condition $u_{n,tt\lambda} = u_{n,\lambda tt}$ we obtain the Lie point symmetry algebra

$$X_1 = \partial_t, \qquad X_2 = t\partial_{u_n}, \qquad X_3 = \partial_{u_n}, \qquad X_4 = t\partial_t + 2n\partial_{u_n}. \tag{76}$$

This Lie algebra is solvable, its nilradical $\{X_1, X_2, X_3\}$ is isomorphic to the Heisenberg algebra. We note that the ansatz made in [11] was not correct and lead to $X_3 = q(n)\partial_{u_n}$ in (76) with $q(n)$ arbitrary. It was, however, noted there that a closed Lie algebra is obtained only for $q(n) = \text{const}$.

4.3. *The two-dimensional Toda lattice equation*

The equation to be considered [5, 16] is

$$u_{n,xy} = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1}). \tag{77}$$

According to theorem 3.4 the flow corresponding to point symmetries will take the form

$$u_{n,\lambda} = \phi_n(x, y, u_n) - \xi(x, y)u_{n,x} - \eta(x, y)u_{n,y}. \tag{78}$$

The Lie point symmetry algebra obtained from the compatibility condition $u_{n,xy\lambda} = u_{n,\lambda xy}$ is infinite dimensional and depends on four arbitrary functions of one variable each:

$$\begin{aligned} X(f) &= f(x)\partial_x + f'(x)n\partial_{u_n}, \qquad & U(k) &= k(x)\partial_{u_n}, \\ X(g) &= g(y)\partial_y + g'(y)n\partial_{u_n}, \qquad & W(\ell) &= \ell(y)\partial_{u_n}. \end{aligned} \tag{79}$$

This algebra happens to coincide with the one found in [12] though the prolongation formula used there was incorrect. This is a Kac–Moody–Virasoro algebra as is typical for integrable equations with more than two independent variables (in this case x, y and n).

5. Conclusions

The main results of the present communication are as follows.

1. The prolongation formulas (41)–(43) for evolutionary and ordinary vector fields generating commuting flows and Lie point symmetry transformation for differential–difference equations. These are viewed here as differential equations on fixed non-transforming lattices.
2. The prolongation formulas and the corresponding algorithm for calculating Lie point symmetries of differential–difference equations are greatly simplified for three rather general classes of equations (including the Toda lattice, the two-dimensional Toda lattice and the Volterra equations). The results are summed up in theorems 3.1–3.4.
3. We have presented an example of each class of equations covered by the above theorems and identified a class of equations depending on six parameters (with generalizations depending on nine parameters). These are the Yamilov discretizations of the Krichever–Novikov equation (68, 69).

A complete analysis of the symmetries of the YdKN equation and its generalizations will be published separately.

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