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MULTI-COMPONENT INTEGRABLE SYSTEMS AND NONASSOCIATIVE STRUCTURES

Dedicated to the memory of S.I. Svinolupov

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Some classification results concerning integrable multi-component evolution equations are presented. They are naturally formulated in terms of nonassociative algebraic structures and their deformations. A number of new examples of integrable evolution, hyperbolic and differential-difference multi-component equations and integrable boundary conditions for them are given.

1 Introduction

This survey contains all main results of Sergey Svinolupov concerning the classification of integrable multi-component systems. He intended to write a review article on the subject for the Proceedings, but his tragic death destroyed the plan in embryo. We, being coauthors of some of his works, are trying to do something instead of him.

Sometime, we formulate less general results than in original papers. Some results of Sections 3 and 5 are published for the first time.

The symmetry approach to the classification of integrable equations is based on observation (see [62,63,68]) that both linearizable equations and equations, integrable by the inverse scattering method, possess higher symmetries. The approach turns out to be most efficient for evolution equations of the form

$$\vec{u}_t = A\vec{u}_n + \vec{F}(\vec{u}, \vec{u}_x, \dots, \vec{u}_{n-1}), \qquad (1)$$

where A is a constant matrix, $\vec{u} = (u^1, \dots, u^N)^t$, $\vec{u}_i = \partial^i \vec{u}/\partial x^i$.

The simplest case $N = 1$ was intensively investigated in 1979–1985 t [64,67,1,65,3,4,5,71]. It turns out that the existence of the higher symmetry

$$u_\tau = G(u, u_x, \dots, u_m), \qquad m > n,$$

implies the solvability of a triangular chain of equations of the form

$$D_x(X_{i+1}) = f_i(F, X_1, \dots, X_i), \qquad i = 1, \dots, m - n, \qquad (2)$$

where D_x is the total derivative operator for equation (1 and X_i are local functions (i.e. functions depending on a finite number of variables $u, u_1, u_2, \ldots$). As the equation $D_x(X) = f$ has a local solution only if $\delta f/\delta u = 0$, the chain (2) leads to a set of conditions to which the rhs of (1) must satisfy. Using these integrability conditions, Svinolupov and Sokolov (see [4,12,6,3,66]) have found all integrable equations of the form $u_t = F(t, x, u, u_x, u_{xx})$ and scalar integrable equations (1) for $n = 3, 4, 5$.

R.I.Yamilov generalized the symmetry approach to the case of evolution differential-difference equations and classified integrable Volterra and Toda type lattices [48,49].

In the paper [50], higher symmetries were involved by Habibullin to investigate initial boundary value problems for integrable equations. An effective algorithm for describing integrable boundary conditions was presented. A number of new such conditions for nonlinear equations of mathematical physics were obtained in the paper [53].

Let us discuss multi-component equations (1). It is clear that the matrix A in (1) can be reduced to the Jordan form by a linear transformation of $\vec{u}$. The case with A being degenerate or nondiagonalizable has not been investigated yet. Let A be a diagonal nondegenerate matrix. Using a diagonalization procedure (see [72,70]), one can prove that it is sufficient to consider two opposite cases. First of them, when A has different eigenvalues, has been considered by Mikhailov, Shabat and Yamilov [69,70]. Svinolupov [11] investigated the second one in which A is the unity matrix. It turns out that this case is much more similar to the scalar case than the first one. The system (2) is replaced by a system of the form

$$D_x(X_{i+1}) + [F_{n-1}, X_{i+1}] = f_i(\vec{F}, X_1, \ldots, X_i), \qquad i = 1, \ldots, m-n, \qquad (3)$$

where X_i are local matrices, F_i denotes the Jacobi matrix of $\vec{F}$ with respect to $\vec{u}_i$. For example, if we restrict ourselves to the equations

$$\vec{u}_t = \vec{u}_{xxx} + \vec{F}(\vec{u}, \vec{u}_x, \vec{u}_{xx}), \qquad i = 1, \ldots, N, \qquad (4)$$

then first two equations of (3) take the form

$$D_x(X_1) + [F_2, X_1] = D_t(F_2), \qquad (5)$$

$$D_x(X_2) + [F_2, X_2] = [X_1, F_1 - F_2^2] + D_t(F_1 - F_2^2). \qquad (6)$$

Using classifying conditions (3), Svinolupov investigated second order systems and some of third order ones. The results obtained will be formulated in Section 2.

The main technical problem in the classification of multi-component integrable equations is that any straightforward calculation leads to enormous expressions quite impossible to deal with. The second point is that, in the case of polynomial equations, integrability conditions yield an overdetermined system of algebraic equations for coefficients of the rhs. As a rule, it is very difficult to understand how many solutions such a system has. The most essential problem is the third one. One should expect that the classification problem for equations with arbitrary many unknown variables contains, as a subproblem, a classical "unsolvable" classification problem of algebra, such as, for instance, the description of all finite dimensional Lie algebras.

In order to illustrate all above points, let us consider Svinolupov's result [13] concerning the multi-component generalizations

$$u_t^i = u_{xxx}^i + C_{jk}^i u^j u_x^k \tag{7}$$

of the Korteweg-de Vries equation. Here and below we assume that the summation is carried out over repeated indices. Since any linear transformation of $\vec{u}$ preserves the class (7), the description of integrable cases has to be invariant under these transformations.

To solve the problem of complication of computations, Svinolupov interpreted C_{jk}^i as the structure constants of an (noncommutative and nonassociative) algebra J and rewrote (7) in the form

$$U_t = U_{xxx} + U \circ U_x, \tag{8}$$

where $U(x,t)$ is a J-valued function. It is easy to see that equations related by linear transformations correspond to isomorphic algebras.

It turns out that if J is commutative, then (8) is integrable iff J is the Jordan algebra (see Appendix). Although there is no description of all the Jordan algebras this result allows one:

i) to check the integrability of a given system (7);

ii) to classify all integrable cases for small dimensions;

iii) to construct the most interesting examples of an arbitrary high dimension.

Let us explain what the term "most interesting" means. A system of equations (7) is called irreducible if it cannot be reduced to the block-triangular form by an appropriate linear transformation (in the case of the block-triangular system, the functions $u^1, \ldots, u^M$ $(M < N)$ satisfy an autonomous system of the form (7), and remaining equations are linear in $u^{M+1}, \ldots, u^N$). It turns out that irreducible systems are associated with the simple algebras. Thus, one can use a well-known algebraic result [36], namely, the exhaustive description of all the simple Jordan algebras to construct all irreducible systems. They are nothing but so-called vector and matrix Korteweg-de Vries equations [43,24].

The most interesting example here is the following vector KdV equation [24]

$$u_t = u_{xxx} + < C, u > u_x + < C, u_x > u - < u, u_x > C, \tag{9}$$

where $< \, , \, >$ is the standard scalar product, C is a given constant vector. Usually (see [43]), one refers to the system

$$u_t = u_{xxx} + < C, u > u_x + < C, u_x > u \tag{10}$$

as the vector KdV equation (cf. (9) and (10) with (96) and (97)). However, we recommend (9) as a claimant upon this role, since the system (10) is reducible unlike (9).

Section 3 is devoted to multi-component systems for which classification parameters are not constants, as above, but functions of $\vec{u} = (u^1, \ldots, u^N)^t$. One of the most interesting classes of such systems are hyperbolic systems of the form

$$u^i_{xy} = \alpha^i_{jk}(\vec{u}) u^j_x u^k_y. \tag{11}$$

This class is important for applications in the string theory. For instance, the chiral field models belong to it. The papers [24,30] contain new examples of integrable systems (11).

The class (11) is invariant under point transformations: $\vec{v} = \vec{\Psi}(\vec{u})$. The vector functions $\alpha^i_{jk}(\vec{u})$ are transformed as components of an affine connection Γ under these transformations. It is clear that an invariant description of integrable cases must be reduced to some conditions on the connection.

There exist the "geometric" classes of equations in the evolution case too. The simplest of them are

$$u^i_t = \Lambda u^i_{xx} + \alpha^i_{jk}(\vec{u}) u^j_x u^k_x, \tag{12}$$

with $\Lambda^2 = 1$, and

$$u^i_t = u^i_{xxx} + \alpha^i_{jk}(\vec{u}) u^j_x u^k_{xx} + \beta^i_{jks}(\vec{u}) u^j_x u^k_x u^s_x. \tag{13}$$

Subsection 3.2 contains some examples of integrable equations (12) and (13)

The classification of integrable equations (13) has been given by Svinolupov and Sokolov (see Subsection 3.3). In the process of classification, a new class of affinely connected spaces associated with integrable equations (13) has been found.

Apparently for any integrable system of the geometric type one can find a preferred system of coordinates which is singled out by the fact that the functions $\alpha^i_{jk}(\vec{u})$ are the structure constants of an N-parameter family $J(u)$

of nonassociative algebras like the Jordan or left-symmetric ones. One can regard such a family of nonassociative algebras, with the structure constants depending on parameters, as a deformation of an algebra with the structure constants $\alpha^i_{jk}(0)$. Deformations of nonassociative algebras defined by the overdetermined consistent system

$$\frac{\partial \alpha^i_{jk}}{\partial u^m} = \alpha^i_{rk}\alpha^r_{mj} + \alpha^i_{jr}\alpha^r_{mk} - \alpha^i_{mr}\alpha^r_{jk} \tag{14}$$

were investigated in [19,30]. All the examples of integrable systems of the geometric type known to the authors are associated with the deformation (14).

There are several papers of Svinolupov, Yamilov and Adler devoted to multi-component integrable generalizations of evolution differential-difference equations (chains) [14,23,47,32]. Unlike first Svinolupov's papers, the purpose of these ones was not to classify integrable cases, but only to construct integrable multi-component examples and to show that the algebraic approach gives results in the differential-difference case as well.

Results of Svinolupov and Yamilov obtained in the papers [14], [23] are discussed in Section 4. It is well-known [57] that Bäcklund transformations of integrable partial differential equations generate integrable differential-difference equations, and using some special Bäcklund transformations, we can obtain *evolution* differential-difference equations [44,45,46]. For example, the nonreduced Schrödinger equation (or ZS-AKNS equation)

$$u_t = u_{xx} + 2u^2 v, \qquad -v_t = v_{xx} + 2v^2 u \tag{15}$$

admits two special Bäcklund transformations. One of them is of the form

$$\tilde{u}_x = u + \tilde{u}^2 v, \qquad -v_x = \tilde{v} + v^2 \tilde{u}, \tag{16}$$

the second one is the following explicit invertible auto-transformation:

$$\tilde{u} = u_{xx} - u_x^2/u + u^2 v, \qquad \tilde{v} = 1/u. \tag{17}$$

In the first case, if we consider a chain of Bäcklund transformations which links together solutions $(u, v) = (u_{n+1}, v_n)$ and $(\tilde{u}, \tilde{v}) = (u_n, v_{n-1})$ of (15), we come to the system of differential-difference equations

$$u_{nx} = u_{n+1} + u_n^2 v_n, \qquad -v_{nx} = v_{n-1} + v_n^2 u_n, \tag{18}$$

where n is discrete integer variable. By carrying out the continuous limit, as one does for the Volterra equation to obtain the Korteweg-de Vries equation, we get (15). In the second case, a chain of Bäcklund transformations which links together solutions $(u,v) = (u_n, v_n)$ and $(\tilde{u}, \tilde{v}) = (u_{n+1}, v_{n+1})$ of (15) is a system of the form:

$$u_{nxx} = u_{nx}^2/u_n + u_{n+1} - u_n^2/u_{n-1}. \tag{19}$$

Introducing $u_n = \exp q_n$, we obtain the classical Toda model

$$q_{nxx} = \exp(q_{n+1} - q_n) - \exp(q_n - q_{n-1}). \tag{20}$$

According to [23], conservation laws and higher symmetries of (18) and (20) can be constructed using ones of (15).

In the same manner as above, multi-component generalizations of the differential-difference Schrödinger equation (18) and the Toda chain (20) possessing higher order symmetries and conservation laws will be constructed in Section 4. We will use the fact that there is a multi-component integrable analog of the system (15) (see [16] and Section 2).

In the next Section, we present results obtained by Svinolupov, Yamilov and Adler [32]. The paper [32] contains not only generalizations of the Volterra equation but also examples of multi-component local master symmetries (both differential-difference and partial differential ones). Such master symmetries exemplify local evolution equations explicitly depending on the spatial variable and integrable by the inverse scattering method.

The concept of the master symmetry was introduced in [60,61]. As it is known, for the first time the master symmetries have arisen and were investigated as integrable equations with the spectral problem in which the spectral parameter depends on the time (see e.g. [59,74]). Later it was observed that these equations generate higher symmetries for usual integrable equations, and a possibility appeared to give an algebraic definition for them.

The well-known Volterra equation

$$v_{nx} = v_n(v_{n+1} - v_{n-1}) \tag{21}$$

can be rewritten in the form

$$u_{nx} = u_n^2(u_{n+1} - u_{n-1}) \tag{22}$$

by introducing u_n such that $u_{n+1}u_n = v_n$. The form (22) is more convenient to construct multi-component generalizations. The chain (22) possesses the following local master symmetry

$$u_{n\tau} = u_n^2[(n+1)u_{n+1} - (n-1)u_{n-1}] \tag{23}$$

obtained for the first time in [58] as an integrable chain. We see that (23) is the local evolution equation, the rhs of which does not contain any integrals or their difference analogs unlike, for example, the master symmetry of the KdV equation

$$u_\tau = x(u_{xxx} + 6uu_x) + 4(u_{xx} + 2u^2) + 2u_x \partial_x^{-1}(u).$$

This is the reason why the master symmetry (23) is called local. This master symmetry admits an $L - A$ pair with the spectral parameter depending on the time [58].

Following [32], we point out in Section 5 the multi-component generalizations of (22) and (23) corresponding to an arbitrary Jordan triple system. Moreover, multi-component "continuous" examples of local master symmetries will arise there in a natural way.

In the last Section 6, we generalize results of [51,52,50] concerning the phenomena of integrable boundary conditions, to the case of the multi-component Burgers and nonlinear Schrödinger equations described in Section 2. Note that the use of algebraic notation allows one to express not only equations but also integrable boundary conditions in a compact elegant form.

Results of this Section have been obtained by Svinolupov in collaboration with Habibullin [28,29].

Let us remind the definition of the integrable boundary condition. A boundary condition

$$u_x = f(u,t)|_{x=0} \tag{24}$$

for an integrable equation

$$u_t = F(u, u_1, \ldots, u_n) \tag{25}$$

is said to be integrable if it is compatible with higher symmetries

$$u_\tau = G(u, u_1, \ldots, u_m) \tag{26}$$

of (25).

In order to explain the definition given above, we consider boundary conditions of the form

$$u_x = f(u)|_{x=0} \tag{27}$$

for the classical Burgers equation

$$u_t = u_{xx} + 2uu_x, \tag{28}$$

compatible with the fourth order symmetry

$$u_\tau = u_4 + 4uu_3 + 10u_1u_2 + 6u^2u_2 + 12uu_1^2 + 4u^3u_1 \tag{29}$$

of (28).

Differentiating (27) with respect to t and using the Burgers equation one gets the expression for the variable u_3 in terms of u, u_2:

$$u_3 = f'(u)(u_2 + 2uf(u)) - 2uu_2 - 2f(u)^2 = f_1(u, u_2).$$

After one more step of this kind, one can derive a formula for the variable u_5 : $u_5 = f_2(u, u_2, u_4)$. Then, differentiating the constraint (27) with respect to τ in virtue of the higher symmetry (29), we obtain the equality $u_{\tau x} = f_u G$, where G is the rhs of (29). Using expressions given above, we can evaluate from this equality the τ-derivative and the variables u_1, u_3, u_5, and then, by definition, the equality must be satisfied identically. After a calculation, one is led to the following condition which is necessary and sufficient for the boundary condition (27) to be compatible with the symmetry (29): $f_{uu} = -2$, i.e. $f = -u^2 + C_1 u + C_2$. It has been proved in [53] that this boundary condition is compatible with all the even order homogeneous symmetries of the Burgers equation. Furthermore, if (27) is compatible with at least one higher symmetry, then it is of the form $u_x = -u^2 + C_1 u + C_2$.

2 Polynomial systems

One of the most remarkable observations of Svinolupov is the discovery of the fact that polynomial multi-component integrable equations are closely connected to the well-known nonassociative algebraic structures as the left-symmetric algebras, Jordan algebras, Jordan triple systems, etc. This connection allows one to clarify the nature of known vector and matrix generalizations (see, for instance [41,42,43]) of classical scalar integrable equations and to construct some new examples of this kind [24].

In this Section, we will consider some classes of polynomial integrable systems of evolution equations generalizing the following famous scalar integrable equations: the Burgers equation

$$u_t = u_{xx} + 2uu_x,$$

the modified Korteweg-de Vries equation

$$u_t = u_{xxx} + u^2 u_x,$$

the nonlinear Schrödinger equation

$$u_t = u_{xx} + 2u^2 v, \qquad -v_t = v_{xx} + 2v^2 u, \tag{30}$$

and the nonlinear derivative Schrödinger equation

$$u_t = u_{xx} + 2(u^2v)_x, \qquad -v_t = v_{xx} + 2(v^2u)_x.$$

Class 2.1. The multi-component Burgers equation introduced by Svinolupov in [11] is of the form

$$u^i_t = u^i_{xx} + 2a^i_{jk}u^j u^k_x + b^i_{jkm}u^j u^k u^m, \qquad i = 1, \ldots, N, \tag{31}$$

where $u^i = u^i(t,x)$, and the parameters a^i_{jk}, b^i_{jkm} are constant. The summation on the repeated indices is assumed.

The following classification statement holds.

Theorem 2.1. In order to the system of equations (31) possesses at least one higher symmetry, it is necessary and sufficient that the set of parameters a^i_{jk}, b^i_{jkm} satisfy the constraints

$$b^i_{jkm} = \frac{1}{3}(a^i_{jr}a^r_{km} + a^i_{kr}a^r_{mj} + a^i_{mr}a^r_{jk} - a^i_{rj}a^r_{km} - a^i_{rk}a^r_{mj} - a^i_{rm}a^r_{jk}), \tag{32}$$

$$a^i_{jr}a^r_{km} - a^i_{kr}a^r_{jm} = a^r_{jk}a^i_{rm} - a^r_{kj}a^i_{rm}. \tag{33}$$

The relation (33) means that a^i_{jk} are the structure constants of a left-symmetric algebra A (see Appendix). Let $e_1, \ldots, e_N$ be a basis of A and $u = u^i e_i$. Then (31) can be written in the following simple form

$$u_t = u_{xx} + 2u \circ u_x + u \circ (u \circ u) - (u \circ u) \circ u, \tag{34}$$

where $\circ$ denotes the multiplication in A. Let us consider two simplest examples of the systems (34).

Example 2.1. The set of all quadratic matrices forms an associative (and, therefore, left-symmetric) algebra. The corresponding equation (34) is the matrix Burgers equation

$$u_t = u_{xx} + 2uu_x. \tag{35}$$

Example 2.2. (V.V. Sokolov) The left-symmetric algebra (89) generates the following vector Burgers equation

$$u_t = u_{xx} + 2 < u, u_x > C + 2 < u, C > u_x + \\ + \|u\|^2 < u, C > C - \|C\|^2\|u\|^2 u. \tag{36}$$

Every equation of the classes presented below has infinitely many higher symmetries and a zero-curvature representation of the form $U_t - V_x = [U, V]$, where U and V belong to the superstructure Lie algebra of a Jordan algebra or a Jordan triple system (see [38,40]).

Class 2.2. Multi-component generalizations of the nonlinear Schrödinger equation (see [16]) are given by the systems of $2N$ equations of the form

$$\begin{aligned} u_t^i &= u_{xx}^i + 2a_{jkm}^i u^j v^k u^m \\ v_t^i &= -v_{xx}^i - 2a_{jkm}^i v^j u^k v^m, \end{aligned} \tag{37}$$

where $i = 1, \dots, N$, and a_{jkm}^i are constants. Without loss of generality, we assume that $a_{jkm}^i = a_{mkj}^i$.

The following statement has been proved in [16].

Theorem 2.2. In order to the system (37) possesses at least one non-degenerate higher symmetry, it is necessary and sufficient that the constants a_{jkm}^i satisfy the following relation:

$$a_{jkn}^i a_{msp}^n - a_{msn}^i a_{jkp}^n - a_{nsp}^i a_{jkm}^n + a_{mnp}^i a_{kjs}^n = 0. \tag{38}$$

The relation (38) just means that a_{jkm}^i are the structure constants of a Jordan triple system. Using this fact, one can write down integrable systems of the form (37) in an invariant compact form. Setting $u = u^i e_i$ and $v = v^i e_i$, we can see that (37) is equivalent to

$$u_t = u_{xx} + 2\{u, v, u\}, \qquad v_t = -v_{xx} - 2\{v, u, v\}. \tag{39}$$

The formulas (94)-(97) given in Appendix yield examples of integrable matrix and vector Schrödinger equations.

Example 2.3. The well-known vector Schrödinger equation [54]

$$u_t = u_{xx} + 2 < u, v > u, \qquad v_t = -v_{xx} - 2 < v, u > v \tag{40}$$

corresponds to the Jordan triple system (97).

Example 2.4. A new integrable vector nonlinear Schrödinger equation

$$\begin{aligned} u_t &= u_{xx} + 4 < u, v > u - 2\|u\|^2 v, \\ v_t &= -v_{xx} - 4 < v, u > v + 2\|v\|^2 u \end{aligned} \tag{41}$$

contained in [24] corresponds to the Jordan triple system (96). Note that (41) looks very similar to equations of the paper [73] devoted to the fibre optics.

Example 2.5. The well-known matrix generalization of the Schrödinger equation

$$u_t = u_{xx} + 2uvu, \qquad v_t = -v_{xx} - 2vuv, \tag{42}$$

where v and v are $m \times m$ matrices, is associated with the Jordan triple system (94).

For the next two types of multi-component equations, we do not formulate any classification results (see [13,16,21]), but write down classes of equations containing all irreducible systems.

Class 2.3. The following generalization of the derivative nonlinear Schrödinger equation

$$u_t = u_{xx} + 2\{v, u, v\}_x, \qquad v_t = -v_{xx} - 2\{u, v, u\}_x \tag{43}$$

is integrable for any Jordan triple system. Matrix and vector examples are constructed using formulae (94)-(97).

Class 2.4. In Introduction we described the multi-component KdV equations. Here we present equations of the MKdV type. Equations of the form

$$u_t = u_{xxx} + \{u, u, u_x\} \tag{44}$$

are integrable for any Jordan triple system. Matrix and vector examples can be obtained in the standard way (see [24]).

3 Geometric type systems

Results of this Section have been obtained by Svinolupov and Sokolov.

3.1 Deformations of the Jordan algebras

It can be shown that for any initial data

$$\alpha^i_{jk}(0) = a^i_{jk}, \tag{45}$$

where a^i_{jk} are the structure constants of a Jordan algebra J_0, a solution $\alpha^i_{jk}(u)$ of (14) exists (for a sufficiently small u) and is unique. Moreover, $\alpha^i_{jk}(u)$ turn out to be the structure constants of a Jordan algebra J_u for any u. Let us denote by $\sigma^i_{jkm}(u)$ the structure constants of a Jordan triple system σ_u generated by the algebra J_u by means of the formula (99). There are two important cases in which the deformation equation can be solved explicitly [19,30].

Construction 3.1. If J_0 possesses the unity element e, then the multiplication $\alpha_u(X,Y)$ in the J_u is given by the formula

$$\alpha_u(X,Y) = -(e-u)^{-1}\circ(X\circ Y)+(X\circ(e-u)^{-1})\circ Y+(Y\circ(e-u)^{-1})\circ X. \quad (46)$$

The definition of the inverse element is contained in Appendix. For every simple Jordan algebra the inverse element can be explicitly found. For example, the simple algebra of the type A_n is defined by the multiplication (91). The inverse of v coincides with the standard matrix inverse v^{-1}. For the algebra of the type D_n (see formula (92)),

$$v^{-1} = \frac{2 < C, v > C - ||C||^2 v}{||C||^4 ||v||^2}.$$

Construction 3.2. Let $\{X,Y,Z\}$ be a Jordan triple system, $\phi(u)$ be a solution of the following overdetermined consistent system

$$\frac{\partial \phi}{\partial u^k} = -\{\phi,\ e_k,\ \phi\}, \quad (47)$$

$k = 1, \ldots, N$. Then the structure constants of $J(u)$ with the multiplication

$$\alpha_u(X,Y) = \{X, \phi, Y\} \quad (48)$$

satisfy the deformation equation (14). The Jordan triple system corresponding to (48) is of the form

$$\sigma_u(X,Y,Z) = \{X, \{\phi, Y, \phi\}, Z\}. \quad (49)$$

If $\{X,Y,Z\}$ is given by (94), then one of the solutions of (47) is

$$\phi(u) = u^{-1}. \quad (50)$$

An analog of u^{-1} is well-known in the theory of the Jordan triple systems. Let us define a linear operator P_X by the formula $P_X(Y) = \{X,Y,X\}$. Then, by definition, $u^{-1} = P_u^{-1}(u)$.

Assume that there is a vector X, such that P_X is nondegenerate. Then $\phi(u) = P_{X+u}^{-1}(X+u)$ exists for small u and satisfies (47). Without loss of generality, we assume in this case that

$$\phi(u) = P_u^{-1}(u). \quad (51)$$

In particular, one can choose

$$\phi(u) = \frac{u}{||u||^2} \quad (52)$$

for the Jordan triple system (96).

Let us take (97) for the Jordan triple system σ. It is easy to see that the operator P_X is degenerate for any X, and we must solve (47) straightforwardly. The general solution is

$$\phi(u) = \frac{C}{2 < C, u >}, \tag{53}$$

where C is an arbitrary constant vector.

The formula (53) is a special case of the following formula

$$\phi(u) = C(C^t u)^{-1}, \tag{54}$$

where C is a constant $n \times m$ matrix, corresponding to the Jordan triple system (95)

3.2 Examples of geometric type integrable equations generated by the deformation

We present here some classes of integrable equations closely related to the deformation (14). A class of integrable chains generated by (14) is contained in Section 4. Formulas (50), (52), (53) allow one to build up one matrix and two vector equations for every class, using Construction 3.2. We will write down some of them explicitly. The paper [24] contains examples of integrable equations corresponding to (54).

Class 3.1. Let $J(u)$ be the deformation of a Jordan algebra. Consider the equation

$$u_{xy} = \alpha_u(u_x, u_y), \tag{55}$$

where α_u is the multiplication in $J(u)$. In the matrix case, (55) coincides with the equation of the principal chiral field

$$u_{xy} = \frac{1}{2}(u_x u^{-1} u_y + u_y u^{-1} u_x). \tag{56}$$

For this reason we will call (55) the Jordan chiral field equation.

It is easy to verify that (55) admits the following zero-curvature representation

$$\Psi_x = \frac{2}{(1-\lambda)} L_{u_x} \Psi, \qquad \Psi_y = \frac{2}{(1+\lambda)} L_{u_y} \Psi.$$

Here and below we denote by L_X the left multiplication operator: $L_X(Y) = \alpha_u(X, Y)$. Note that this formula gives us a zero-curvature representation for (56):

$$\Psi_x = \frac{1}{(1-\lambda)} M \Psi, \qquad \Psi_y = \frac{1}{(1+\lambda)} N \Psi,$$

where Ψ is a matrix and

$$M\Psi = -u_x u^{-1}\Psi - \Psi u^{-1}u_x, \qquad N\Psi = -u_y u^{-1}\Psi - \Psi u^{-1}u_y,$$

different from the standard one. It should also be remarked that (55) is linearizable if J_u is the deformation of a left-symmetric algebra (see [24,30]).

All equations of classes presented below have higher symmetries and zero-curvature representations in the superstructure Lie algebra of J_u (see [38]).

Class 3.2. The following equation

$$u_t = u_{xxx} - 3\alpha_u(u_x, u_{xx}) + \frac{3}{2}\sigma_u(u_x, u_x, u_x) \tag{57}$$

of the form (13) is integrable if it corresponds to the deformation of a Jordan algebra. Matrix and vector equations have the following form:

$$u_t = u_{xxx} - \frac{3}{2}u_x u^{-1}u_{xx} - \frac{3}{2}u_{xx}u^{-1}u_x + \frac{3}{2}u_x u^{-1}u_x u^{-1}u_x, \tag{58}$$

where $u(x,t)$ is an $m \times m$ matrix,

$$\begin{aligned} u_t \;=\; & u_{xxx} - 3\frac{< u, u_x >}{||u||^2}u_{xx} - 3\frac{< u, u_{xx} >}{||u||^2}u_x + 3\frac{< u_x, u_{xx} >}{||u||^2}u \\ & -\frac{3}{2}\frac{||u_x||^2}{||u||^2}u_x + 6\frac{< u, u_x >^2}{||u||^4}u_x - 3\frac{< u, u_x > ||u_x||^2}{||u||^4}u, \end{aligned} \tag{59}$$

and

$$u_t = u_{xxx} - \frac{3}{2}\frac{< C, u_x >}{< C, u >}u_{xx} - \frac{3}{2}\frac{< C, u_{xx} >}{< C, u >}u_x + \frac{3}{2}\frac{< C, u_x >^2}{< C, u >^2}u_x.$$

Class 3.3. The following integrable equations

$$v_t = v_{xxx} - \frac{3}{2}\alpha_{v_x}(v_{xx}, v_{xx}) \tag{60}$$

are related to ones of Class 3.2 by the potentiation $u = v_x$. Vector equations are of the form:

$$u_t = u_{xxx} - 3\frac{< u_x, u_{xx} >}{||u_x||^2}u_{xx} + \frac{3}{2}\frac{||u_{xx}||^2}{||u_x||^2}u_x,$$

$$u_t = u_{xxx} - \frac{3}{2}\frac{< C, u_{xx} >}{< C, u_x >}u_{xx}.$$

Class 3.4. The scalar representative of this class is given by the Heisenberg model (see [70])

$$u_t = u_{xx} - \frac{2}{u+v} u_x^2, \qquad v_t = -v_{xx} + \frac{2}{u+v} v_x^2.$$

The following coupled equation

$$u_t = u_{xx} - 2\alpha_{u+v}(u_x, u_x), \qquad v_t = -v_{xx} + 2\alpha_{u+v}(v_x, v_x) \tag{61}$$

is integrable if it is associated with the deformation of a Jordan algebra. The equation (61) has a higher symmetry of the form

$$\begin{aligned} u_t &= u_{xxx} - 6\alpha_{u+v}(u_x, u_{xx}) + 6\sigma_{u+v}(u_x, u_x, u_x), \\ v_t &= v_{xxx} - 6\alpha_{u+v}(v_x, v_{xx}) + 6\sigma_{u+v}(v_x, v_x, v_x). \end{aligned} \tag{62}$$

Note that (62) belongs to Class 3.2. After the reduction $u = v$ and the scaling $2u \to u$, it turns into (57). A matrix equation (61) is of the form

$$u_t = u_{xx} - 2u_x(u+v)^{-1}u_x, \qquad v_t = -v_{xx} + 2v_x(u+v)^{-1}v_x. \tag{63}$$

One of vector equations is

$$u_t = u_{xx} - 4\frac{< u_x, u+v >}{||u+v||^2} u_x + 2\frac{||u_x||^2}{||u+v||^2}(u+v),$$

$$v_t = -v_{xx} + 4\frac{< v_x, u+v >}{||u+v||^2} v_x - 2\frac{||v_x||^2}{||u+v||^2}(u+v).$$

3.3 Classification of integrable equations (13)

Let us consider the systems of the type (13). It is convenient to rewrite (13) in the following way

$$u_t^i = u_{xxx}^i + 3\alpha_{jk}^i u_x^j u_{xx}^k + \left(\frac{\partial \alpha_{km}^i}{\partial u^j} + 2\alpha_{jr}^i \alpha_{km}^r - \alpha_{rj}^i \alpha_{km}^r + \beta_{jkm}^i\right) u_x^j u_x^k u_x^m, \tag{64}$$

where $\beta_{jkm}^i = \beta_{kjm}^i = \beta_{mkj}^i$, i.e.

$$\beta(X, Y, Z) = \beta(Y, X, Z) = \beta(X, Z, Y)$$

for any vectors X, Y, Z.

The class (64) is invariant under arbitrary point transformations $u \to \vec{\Phi}(u)$, where $u = (u^1, \dots, u^N)^t$. It is easy to see that under such a change of coordinates, α^i_{jk} and β^i_{jkm} are transformed just as components of an affine connection Γ and a tensor, respectively. Let R and T be the curvature and torsion tensors of Γ.

In order to formulate classification results, we introduce the following tensor:

$$\sigma(X,Y,Z) = \beta(X,Y,Z) - \frac{1}{3}\delta(X,Y,Z) + \frac{1}{3}\delta(Z,X,Y),$$

where

$$\delta(X,Y,Z) = T(X,T(Y,Z)) + R(X,Y,Z) - \nabla_X(T(Y,Z)).$$

Using the Bianchi's identity, one can find that

$$\sigma(X,Y,Z) = \sigma(Z,Y,X). \tag{65}$$

Theorem 3.1. Eq. (64) possesses a higher symmetry of the form

$$u_\tau = u_n + \vec{G}(u, u_x, \cdots, u_{n-1}), \qquad n > 3,$$

iff

$$\nabla_X(R(Y,Z,V)) = R(Y,X,T(Z,V)), \tag{66}$$

$$\nabla_X\left(\nabla_Y(T(Z,V)) - T(Y,T(Z,V)) - R(Y,Z,V)\right) = 0, \tag{67}$$

$$\nabla_X(\sigma(Y,Z,V)) = 0, \tag{68}$$

$$\begin{aligned} &T(X,\sigma(Y,Z,V)) + T(Z,\sigma(Y,X,V)) + \\ &+T(Y,\sigma(X,V,Z)) + T(V,\sigma(X,Y,Z)) = 0, \end{aligned} \tag{69}$$

and

$$\begin{aligned} &\sigma(X,\sigma(Y,Z,V),W) - \sigma(W,V,\sigma(X,Y,Z)) + \\ &+\sigma(Z,Y,\sigma(X,V,W)) - \sigma(X,V,\sigma(Z,Y,W)) = 0. \end{aligned} \tag{70}$$

The identities (65) and (70) mean that $\sigma^i_{jkm}(u)$ are the structure constants of a Jordan triple system for any u.

It follows from (66) that any free-torsion space of this kind is the symmetric one. In the case $T \neq 0$, a generalization of the symmetric spaces gives rise. We don not know if such affine connected spaces have been considered by geometers.

Theorem 3.2. For every Jordan triple system $\{\cdot,\cdot,\cdot\}$ with the structure constants s^i_{jkm}, there exists a unique (up to point transformations) integrable equation (64), such that $T = 0$ and

$$\sigma^i_{jkm}(0) = s^i_{jkm}. \tag{71}$$

In the case $T = 0$, there is a class of integrable equations (64) generated by the deformation (14). If the structure constants $\alpha^i_{jk}(u)$ of a family of Jordan algebras satisfy (14) and $\sigma^i_{jkm}(u)$ are given by (99), then all the conditions (65)-(69) turn out to be satisfied. Corresponding examples are presented above (see Class 3.2). This class contains all integrable equations whose initial Jordan triple system $\{\cdot,\cdot,\cdot\}$ (see Theorem 3.2) can be obtained from a Jordan algebra by the triple product (99).

Equations with initial Jordan triple systems (96) and (97) cannot be obtained in this way. Using some tricks, we have found such equations, solving directly (66) and (68). They are of the form

$$u_t = u_{xxx} + \frac{3}{2}\left(P(u,u_x)(C - \|C\|^2 u)\right)_x + 3\frac{\lambda - 1}{\lambda + 1}\|C\|^2 P(u,u_x)u_x,$$

where $\lambda = 1$ or $\lambda = 0$, and

$$P(u,u_x) = \|u_x + \frac{< C, u_x > u}{1 - (C,u)}\|^2.$$

4 Multi-component generalizations of the differential-difference Schrödinger equation and the Toda model

In this Section we, following the papers [14,23] by Svinolupov and Yamilov, discuss integrable generalizations of the scalar chains (18) and (19). As it has been said above (see Section 2), for any Jordan triple system we can construct the multi-component Schrödinger system (39).

Theorem 4.1. The system of equations (39) admits a Bäcklund transformation of the form

$$\tilde{u}_x = u + \{\tilde{u}, v, \tilde{u}\}, \qquad -v_x = \tilde{v} + \{v, \tilde{u}, v\} \tag{72}$$

iff $\{\cdot,\cdot,\cdot\}$ is a product in the Jordan triple system.

In Theorem 2.2 the condition that a triple algebra associated with (39) must be the Jordan triple system was derived by assuming that (39) possesses higher order symmetries. Now we have obtained the same condition, using quite different assumption.

As in the scalar case (see Introduction), let us consider a chain of Bäcklund transformations which links together solutions $(u, v) = (u_{n+1}, v_n)$ and $(\tilde{u}, \tilde{v}) = (u_n, v_{n-1})$ of (39). Such a chain is equivalent to the following system of differential-difference equations:

$$u_{nx} = u_{n+1} + \{u_n, v_n, u_n\}, \qquad -v_{nx} = v_{n-1} + \{v_n, u_n, v_n\}. \qquad (73)$$

It can be verified that (73) is a differential-difference approximation of the system (39). According to [23], a chain obtained in this way is integrable: conservation laws and higher symmetries of (73) can be constructed using the known ones of (39) (see [16]).

Two vector and one matrix examples of chains of the form (73) can be constructed as in the continuous case. For instance, the matrix example corresponding to (95) is of the form

$$u_{nx} = u_{n+1} + u_n v_n^t u_n, \qquad -v_{nx} = v_{n-1} + v_n u_n^t v_n,$$

where u_n and v_n are $N \times M$ matrices. Vector chains (73) can be written as follows:

$$u_{nx} = u_{n+1} + < u_n, v_n > u_n, \qquad -v_{nx} = v_{n-1} + < u_n, v_n > v_n;$$

and

$$\begin{aligned} u_{nx} &= u_{n+1} + 2 < u_n, v_n > u_n - < u_n, u_n > v_n, \\ - \quad v_{nx} &= v_{n-1} + 2 < u_n, v_n > v_n - < v_n, v_n > u_n. \end{aligned}$$

For all vector and matrix chains there are zero curvature representations [47]; for any integrable chain of the form (73) there is a recursion operator [14]. In the case when the Jordan triple system is generated by a simple Jordan algebra, the corresponding chain is Hamiltonian [14]. The last statement is valid for all chains and partial differential systems we discuss in Sections 2-5.

Theorem 4.2. If a system of the form (39) corresponds to the Jordan triple system $\{\cdot, \cdot, \cdot\}$ generated by (99) from a Jordan algebra with unity element, then this system is invariant under the following transformation:

$$\tilde{u} = u_{xx} - \{u_x, u^{-1}, u_x\} + \{u, v, u\}, \qquad \tilde{v} = u^{-1}, \qquad (74)$$

where u^{-1} is the inverse of u.

A chain of transformations (74) which links together solutions $(u,v) = (u_n, v_n)$ and $(\tilde{u},\tilde{v}) = (u_{n+1}, v_{n+1})$ of (39) can be written in the form

$$u_{nxx} = \{u_{nx}, u_n^{-1}, u_{nx}\} + u_{n+1} - \{u_n, u_{n-1}^{-1}, u_n\}. \qquad (75)$$

The chain (75) can be regarded as a Jordan generalization of the Toda model. In particular, in the matrix case we have the well-known matrix Toda chain:

$$(u_{nx}u_n^{-1})_x = u_{n+1}u_n^{-1} - u_n u_{n-1}^{-1}.$$

As Jordan triple system from Theorem 4.2 has to be generated by the Jordan algebra with unity element, we can construct only one vector example

$$u_{nxx} = 2\frac{< u_n, u_{nx} >}{||u_n||^2}u_{nx} - \frac{||u_{nx}||^2}{||u_n||^2}u_n + u_{n+1} -$$

$$-2\frac{< u_n, u_{n-1} >}{||u_{n-1}||^2}u_n + \frac{||u_n||^2}{||u_{n-1}||^2}u_{n-1}.$$

Conservation laws and higher symmetries of the Jordan Toda model (75) can be constructed using ones of (39) [23].

5 Jordan analogs of the Volterra equation, and multi-component local master symmetries

Here we present some results obtained by Svinolupov, Yamilov and Adler in the paper [32]. The Jordan analog of the Volterra equation is given by the following multi-component differential-difference system:

$$u_{nx} = \{u_n, u_{n+1}, u_n\} - \{u_n, u_{n-1}, u_n\}, \qquad (76)$$

where $\{\cdot,\cdot,\cdot\}$ is a Jordan triple system. The local master symmetry corresponding to (76) have the form:

$$u_{n\tau} = (n+1)\{u_n, u_{n+1}, u_n\} - (n-1)\{u_n, u_{n-1}, u_n\}. \qquad (77)$$

Theorem 5.1 The chain (77) is the master symmetry of (76) if the associated triple system $\{\cdot,\cdot,\cdot\}$ is the Jordan one.

Theoretically we, as usually, can construct one matrix and two vector examples of the Jordan Volterra equations. However, one of the vector examples is degenerate. In fact, in the case of the simplest vector triple system, we are led to the chain

$$u_{nx} =< u_{n+1} - u_{n-1}, u_n > u_n.$$

If $u_n = (u_n^1, \dots, u_n^N)^t$, the constraint $(\log(u_n^i/u_n^j))_x = 0$ holds for any i and j, and one can easily reduce the multi-component chain under consideration to the system consisting of N scalar equations (22).

It turns out that if the Jordan triple system is generated by a Jordan algebra with unity element, then the chain (76) generates an invertible auto-transformation

$$\tilde{u} = v - (u^{-1})_x, \qquad \tilde{v} = u \tag{78}$$

for the multi-component derivative Schrödinger equation (43) which also has the local master symmetry

$$u_\tau = x(u_{xx} + 2\{u, v, u\}_x) + (a + \frac{3}{2})u_x + 2\{u, v, u\},$$

$$v_\tau = x(-v_{xx} - 2\{v, u, v\}_x) + (a - \frac{3}{2})v_x + 2\{v, u, v\}, \tag{79}$$

where a is an arbitrary constant. As far as we know, equations (77) and (79) are first examples of multi-component local master symmetries. Two following examples explain why these equations are integrable and what is the difference between a usual integrable equation and it's master symmetry.

Example 5.1. In the matrix case the chain (76) admits the usual Lax representation $L_{nx} = [A_n, L_n]$, where

$$L_n = u_n(D + D^{-1}), \qquad 2A_n = u_n u_{n+1}(D^2 + 1) - u_n u_{n-1}(1 + D^{-2}),$$

and D is the shift operator. In the case of (77), there is the representation

$$L_{n\tau} = [B_n, L_n] + \frac{1}{2}L_n^3$$

with the same operator L_n and

$$2B_n = (n + \frac{1}{2})u_n u_{n+1}(D^2 + 1) - (n - \frac{1}{2})u_n u_{n-1}(1 + D^{-2}).$$

This means that in the case of (77) we have the spectral problem

$$L_n\psi_n = \lambda\psi_n, \qquad \psi_{n\tau} = B_n\psi_n$$

with the spectral parameter λ depending on the time $\tau : \lambda(\tau) = (\varepsilon - \tau)^{-1/2}$.

The transformation (78) allows one to construct exact solutions for the system (43), starting from a solution (u, v) such that $v = 0$ and u satisfies the multi-component heat equation $u_t = u_{xx}$. As for system (79), this transformation (78) changes the parameter a: $\tilde{a} = a + 1$. Consequently, to obtain exact solutions of (79), we must use not only (78) but also the Galilei transformation: $\tilde{x} = x + \tau$. As a starting point, we can take a solution (u, v) such that $v = 0$ and u satisfies the following linear equation:

$$u_\tau = xu_{xx} + (a + \frac{3}{2})u_x.$$

6 Integrable boundary conditions for multi-component equations

Here we present a class of integrable boundary conditions for multi-component equations (34), (39) obtained by Habibullin and Svinolupov [28,29].

6.1 Integrable boundary conditions for the multi-component Burgers equations

Theorem 6.1. The following boundary conditions for the equation (34):

$$(\sum_{i=0}^{m}(D_t + L(u_x + u \circ u))^i (K_i(u_x + u \circ u) + M_i u + c_i))|_{x=0} = 0 \qquad (80)$$

are integrable. Here K_i, M_i are arbitrary linear operators satisfying the identities

$$M_i(X \circ Y) - (X \circ M_i Y) = 0, \qquad K_i(X \circ Y) - (X \circ K_i Y) = 0, \qquad (81)$$

$\circ$ is the multiplication in a left-symmetric algebra A, c_i are constant vectors satisfying the condition

$$AS(X, Y, c) = 0. \qquad (82)$$

Both the equalities (81) and (82) must be held for all $X, Y \in A$. The operator D_t is the total t-derivative for (34), and the operator $L(X)$ is defined by $L(X)Y = X \circ Y$.

Since $M_i = 0$, $K_i = Id$, $c = 0$ satisfy (81), (82), the boundary condition $u_x + (u \circ u) = 0$ is integrable for any equation (34).

Example 6.1. The following boundary conditions

$$i) \quad u = 0,$$
$$ii) \quad u_x c_1 + u^2 c_1 + u c_2 + c_3 = 0$$

are integrable for the matrix Burgers equation (35) of Example 2.1. Here c_1, c_2, c_3 are arbitrary constant matrices.

Example 6.2. In the case of the equation of Example 2.2, only scalar (i.e. proportional to the identical) operators satisfy (81). The simplest integrable boundary conditions are

$$
\begin{aligned}
&i)\quad u = 0,\\
&ii)\quad u_x + < u, C > u + ||u||^2 C + \lambda u = 0,
\end{aligned}
$$

where λ is a scalar parameter.

6.2 Integrable boundary conditions for the multi-component nonlinear Schrödinger equations

For equations (39), the following boundary conditions are integrable:

$$
\begin{aligned}
&i)\qquad u = 0, \qquad v = 0;\\
&ii)\qquad u_x = cu, \qquad v_x = cv;\\
&iii)\qquad u_{xx} - cu_x + 2\{\bar{u}vu_x\} = 0, \qquad v_{xx} - cv_x + 2\{v\bar{u}v_x\} = 0, \quad (83)
\end{aligned}
$$

where c is an arbitrary constant, $\bar{u}$ is a solution of the equation

$$\{\bar{u}v\bar{u}\} + u - c\bar{u} = 0. \qquad (84)$$

These conditions generalize the known boundary conditions [51,52] for the nonlinear Schrödinger equation (30):

$$
\begin{aligned}
&i)\quad u_x = cu|_{x=0}, \qquad v_x = cv|_{x=0};\\
&ii)\quad u_{xx} = (c + uv)^{1/2} u_x|_{x=0}, \qquad v_{xx} = (c + uv)^{1/2} v_x|_{x=0}.
\end{aligned}
$$

Let us discuss the problem how to eliminate the extra variable $\bar{u}$ from the boundary condition. For a large class of Jordan triple systems (for instance, for those which are generated by a Jordan algebra with the unity, (see [56])), we have $\det N(v, v_x) \neq 0$, where $N(X,Y)Z = \{X, Y, Z\}$. This allows one to express the variable $\bar{u}$ from the second equation of (83):

$$\bar{u} = -\frac{1}{2} N(v, v_x)^{-1}(v_{xx} - cv_x). \qquad (85)$$

Substituting this expression for $\bar{u}$ to (83) and (84), one obtains a boundary condition in the usual form. But sometimes it's more convenient to express $\bar{u}$ from the equation (84). In the examples given below we just follow such a way.

Example 6.3. The following integrable boundary conditions specified at $x = x_0$ are compatible with the system of Example 2.3:

$$
\begin{aligned}
&i) && u = 0, \qquad v = 0; \\
&ii) && u_x = cu, \qquad v_x = cv; \\
&iii) && u_{xx} = (c - \frac{< u, v >}{Q})u_x - \frac{< v, u_x >}{Q}u, \\
& && v_{xx} = (c - \frac{< u, v >}{Q})v_x - \frac{< u, v_x >}{Q}v.
\end{aligned}
$$

Here Q is a solution of the equation $Q^2 - cQ + < u, v >= 0$, and c is a scalar parameter.

Example 6.4. The integrable boundary condition for the vector Schrödinger equation of Example 2.4 is of the form:

$$
\begin{aligned}
u_{xx} + Pu_x - 2\frac{< v, u_x >}{P}u + 2\frac{< u, u_x >}{P}v = 0, \\
v_{xx} + Pv_x - 2\frac{< u, v_x >}{P}v + 2\frac{< v, v_x >}{P}u = 0,
\end{aligned}
$$

where P is determined from the equation

$$P^4 + (4 < u, v > -c^2)P^2 + 4 < u, v >^2 - 4||u||^2||v||^2 = 0,$$

and c is a scalar parameter.

Example 6.5. Let us write down the integrable boundary condition of the second order for the matrix Schrödinger equation of Example 2.5. It follows from the general formulas (83), (84) that this condition has the form:

$$u_{xx} + Pu_x + u_xQ = 0, \quad v_{xx} + Qv_x + v_xP = 0.$$

Here P and Q satisfy the equations $P^2 = \frac{c^2}{4}Id - uv$ and $Q^2 = \frac{c^2}{4}Id - vu$, c is a scalar parameter, and Id is the unity matrix.

7 Appendix

We present here definitions and the simplest examples we need in the main body of the paper. We refer to [35,36,33,39,38,40,56] for more detail information.

Let $e_1, e_2, \ldots, e_N$ be a basis of a finite dimensional algebra J over $\mathbf{C}$. The multiplication in J is given by the formula

$$(e_j \circ e_k) = a^i_{jk}e_i, \tag{86}$$

where a^i_{jk} are the structure constants of J. If $a^i_{jk} = a^i_{kj}$ then J is commutative.

The formula

$$X \circ Y = \lambda < X, C > Y + \mu < Y, C > X + \nu < X, Y > C, \qquad (87)$$

where $< \, , \, >$ is the standard scalar product in a vector space J and C is a given vector, gives us for different λ, μ, ν a number of interesting examples of nonassociative algebras. The so called vector nonlinear differential equations are closely related to those.

We shall use the following notation:

$$AS(X, Y, Z) = (X \circ Y) \circ Z - X \circ (Y \circ Z),$$

$$[X, Y, Z] = AS(X, Y, Z) - AS(Y, X, Z).$$

Note that J is associative iff $AS(X, Y, Z) = 0$.

Definition 1. An algebra J is called leftsymmetric if

$$[X, Y, Z] = 0. \qquad (88)$$

Any associative algebra is leftsymmetric one. The formula

$$X \circ Y = < X, C > Y + < X, Y > C, \qquad (89)$$

gives us an example of leftsymmetric algebra of the type (87).

Definition 2. A commutative algebra J is said to be Jordan if the following identity is fulfilled

$$AS(X \circ X, Y, X) = 0. \qquad (90)$$

The set of all matrices is a Jordan algebra with respect to the anticommutator operation

$$X \circ Y = \frac{1}{2}(XY + YX). \qquad (91)$$

The formula

$$X \circ Y = < X, C > Y + < Y, C > X - < X, Y > C \qquad (92)$$

turns a vector space J to a Jordan algebra. For a Jordan algebra with the unity e the element X^{-1} is defined as a polynomial of X such that $X \circ X^{-1} = e$.

In the article [30] S.I. Svinolupov and V.V.Sokolov have introduced a class of nonassociative algebras defined by the identity

$$[V, X, Y \circ Z] - [V, X, Y] \circ Z - Y \circ [V, X, Z] = 0. \qquad (93)$$

The multiplication (87) satisfies (93) if $\nu = 0$. It is interesting to note that all nonassociative algebras naturally arising in connection with integrable systems (Lie algebras, Jordan algebras, left-symmetric algebras, LT-algebras [56]) satisfy the universal identity (93). Furthermore, the class of algebras with identity (93) is invariant with respect to the deformation (14).

While any collection of constants a^i_{jk} can be regarded as a set of structure constants of an algebra J, every collection σ^i_{jkm} defines a triple system $\{X, Y, Z\}$ by the formula

$$\{e_j, e_k, e_m\} = \sigma^i_{jkm} e_i.$$

Definition 3. A triple system $\{X, Y, Z\}$ is said to be Jordan if

$$\{X, Y, Z) = \{Z, Y, X\},$$

and

$$\{X, \{Y, Z, V\}, W\} - \{W, V, \{X, Y, Z\}\} + \{Z, Y, \{X, V, W\}\} - \{X, V, \{Z, Y, W\}\} = 0.$$

The set of $n \times n$-matrices equipped with the operation

$$\{X, Y, Z\} = \frac{1}{2}(XYZ + ZYX), \tag{94}$$

is a Jordan triple system. The vector space of all $n \times m$-matrices is a Jordan triple system with respect to operation

$$\{X, Y, Z\} = \frac{1}{2}(XY^tZ + ZY^tX), \tag{95}$$

where "t" stands for transposition. The following operations

$$\{X, Y, Z) = <X, Y> Z + <Y, Z> X - <X, Z> Y, \tag{96}$$

and

$$\{X, Y, Z\} = <X, Y> Z + <Y, Z> X \tag{97}$$

define two "vector" (cf. (87)) simple Jordan triple systems.

For each Jordan triple system $\{X, Y, Z\}$ and a given vector C the system

$$\sigma(X, Y, Z) = \{X, \{C, Y, C\}, Z\} \tag{98}$$

is also Jordan one.

There exists close relationships between the Jordan algebras and the Jordan triple systems. Namely, any Jordan algebra generates a triple system by the formula

$$\{X, Y, Z\} = (X \circ Y) \circ Z + (Z \circ Y) \circ X - Y \circ (X \circ Z). \tag{99}$$

Conversely, any Jordan triple system $\{X, Y, Z\}$ yields a family of Jordan algebras with the multiplication

$$X \circ Y = \{X, \phi, Y\}, \tag{100}$$

where ϕ is an arbitrary element.

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