ISSN 2304-0122 Ufa Mathematical Journal. Volume 4. 3 (2012). Pp. 174-180.

UDC 517.929

EXAMPLES OF DARBOUX INTEGRABLE DISCRETE EQUATIONS POSSESSING FIRST INTEGRALS OF AN ARBITRARILY HIGH MINIMAL ORDER

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Abstract. We consider a discrete equation defined on the two-dimensional square lattice, which is linearizable, namely, of the Burgers type and depends on a parameter α. For any natural number N we choose α so that the equation becomes Darboux integrable and the minimal orders of its first integrals in both directions are greater or equal than N.

Keywords: discrete equation, Darboux integrability, first integral.

1. Introduction

In the discrete case Darboux integrable equations with first integrals of low orders are well-known. The existence for such equations of first integrals with arbitrarily high minimal orders was an open problem up to now. In this paper we give a positive answer to this question.

The most general form of the discrete Burgers equation introduced in [1] reads

$$(u_{n+1,m+1}-\beta_{n+1,m})(\alpha_{n,m}u_{n,m}+\gamma_{n,m})u_{n,m+1}=(u_{n,m+1}-\beta_{n,m}) \\ *(\alpha_{n+1,m}u_{n+1,m}+\gamma_{n+1,m})u_{n,m}, \quad \alpha_{n,m},\beta_{n,m},\gamma_{n,m}\neq 0. \tag{1}$$

Here n,m are arbitrary integers, $\alpha_{n,m},\beta_{n,m},\gamma_{n,m}$ are known complex parameters, while $u_{n,m}$ is an unknown complex-valued function. Equation (1) can be obtained by the discrete Hopf-Cole transformation (see e.g. [2])

$$u_{n,m}=\frac{v_{n+1,m}}{v_{n,m}} \tag{2}$$

from the following non-autonomous linear equation

$$v_{n+1,m+1}=\alpha_{n,m}v_{n+1,m}+\beta_{n,m}v_{n,m+1}+\gamma_{n,m}v_{n,m}. \tag{3}$$

In the completely autonomous case, the discrete Burgers equation (1) can be rewritten in the form

$$(u_{n+1,m+1}-\beta)(u_{n,m}+\gamma)u_{n,m+1} \\ =(u_{n,m+1}-\beta)(u_{n+1,m}+\gamma)u_{n,m}. \tag{4}$$

This equation was known [3] earlier than (1). It has been noticed in [4] that there is one more autonomous particular case of eq. (1), namely, the equation

$$(u_{n+1,m+1}-\beta)(u_{n,m}+\gamma)u_{n,m+1} \\ =\alpha(u_{n,m+1}-\beta)(u_{n+1,m}+\gamma)u_{n,m} \tag{5}$$

R.N. Garifullin and R.I. Yamilov, Examples of Darboux Integrable Discrete Equations Possessing First Integrals of an Arbitrarily High Minimal Order.

The work was supported by the Russian Foundation for Basic Research (grant numbers 10-01-00088-a, 11-01-97005-r-povolzhie-a, 12-01-31208-mol-a) and by Federal Targeted Program (agreement 8499).

Submitted July 12, 2012.

generalizing (4). Unlike eq. (4), the last equation (5) is related by (2) to a non-autonomous linear equation.

In this paper we consider the particular case $\beta = \gamma$ of eq. (5) that can be expressed in the following form after a rescale,

$$(u_{n+1,m+1} + 1)(u_{n,m} - 1)u_{n,m+1} = \alpha(u_{n,m+1} + 1)(u_{n+1,m} - 1)u_{n,m}. \tag{6}$$

Here $\alpha \neq 0$ is a complex parameter. A particular case of the linear equation (3) corresponding to eq. (6) is

$$v_{n+1,m+1} + v_{n,m+1} = \alpha^n(v_{n+1,m} - v_{n,m}). \tag{7}$$

Our aim is to show that there exist equations of the form (6) possessing first integrals of an arbitrarily high minimal order.

Discrete equations of the form

$$u_{n+1,m+1} = f(u_{n,m}, u_{n+1,m}, u_{n,m+1}) \tag{8}$$

defined on the two-dimensional square lattice are analogues of the hyperbolic equations

$$u_{xy} = F(x, y, u, u_x, u_y). \tag{9}$$

There is an example similar to eq. (6) in the class (9), see [5] and a discussion at the very end of the present paper.

2. Definitions

An equation of the form (8) is Darboux integrable if it has two first integrals W_1, W_2,

$$(T_1 - 1)W_2 = 0, \quad W_2 = w^{(2)}_{n,m}(u_{n,m+l_2}, u_{n,m+l_2+1}, \dots, u_{n,m+k_2}), \tag{10}$$

$$(T_2 - 1)W_1 = 0, \quad W_1 = w^{(1)}_{n,m}(u_{n+l_1,m}, u_{n+l_1+1,m}, \dots, u_{n+k_1,m}). \tag{11}$$

Here l_1, l_2, k_1, k_2 are integers, such that $l_1 < k_1,\ l_2 < k_2$, and T_1, T_2 are operators of the shift in the first and second directions, respectively, $T_1 h_{n,m} = h_{n+1,m}$, $T_2 h_{n,m} = h_{n,m+1}$. We suppose that relations (10, 11) are satisfied identically on the solutions of corresponding equation (8). The form of W_1, W_2 given in (10,11) is the most general possible form. The functions $u_{n+i,m+j}$, with $i, j \neq 0$, are expressed in terms of $u_{n+i,m}$,$u_{n,m+j}$ by using eq. (8). The dependence of W_1 on $u_{n,m+j}$, $j \neq 0$ and of W_2 on $u_{n+i,m}$, $i \neq 0$ is impossible. We will call W_1 a first integral in the first (or n) direction, and W_2 will be called a first integral in the second (or m) direction.

It is obvious that for any first integrals W_1, W_2 arbitrary functions Ω_1, Ω_2 of the form

$$\begin{aligned} \Omega_1 &= \Omega_1(W_1, T_1^{\pm 1}W_1, \dots, T_1^{\pm j_1}W_1), \\ \Omega_2 &= \Omega_2(W_2, T_2^{\pm 1}W_2, \dots, T_2^{\pm j_2}W_2) \end{aligned} \tag{12}$$

are also the first integrals. In particular, using the shifts, we can rewrite the first integrals W_1, W_2 of (10, 11) as

$$\begin{aligned} W_1 &= w^{(1)}_{n,m}(u_{n,m}, u_{n+1,m}, \dots, u_{n+k,m}), \quad k \geq 1, \\ W_2 &= w^{(2)}_{n,m}(u_{n,m}, u_{n,m+1}, \dots, u_{n,m+l}), \quad l \geq 1. \end{aligned} \tag{13}$$

We assume here that $\frac{\partial W_1}{\partial u_{n,m}} \neq 0, \frac{\partial W_1}{\partial u_{n+k,m}} \neq 0, \frac{\partial W_2}{\partial u_{n,m}} \neq 0, \frac{\partial W_2}{\partial u_{n,m+l}} \neq 0$ for at least some n, m. The numbers k, l are called the orders of these first integrals W_1, W_2, respectively,

$$\operatorname{ord} W_1 = k, \qquad \operatorname{ord} W_2 = l.$$

Due to (12) it is clear that the orders of first integrals of a given equation are unbounded from above. We are going to construct the examples for which the minimal (or lowest) orders of their first integrals may be arbitrarily high.

3. Transition to the linear equation (7)

In spite of the fact that the transform (2) is not invertible, sometimes such transformations allow one to rewrite conservation laws and generalized symmetries from one equation to another, see e.g. [6] for the discrete-differential case. We are going to transfer first integrals from (6) to (7) and backward and to reduce in this way the problem to the case of the linear equation (7).

Here we present four lemmas on the transfer of first integrals from (6) to (7) and backwards together with the corresponding explicit formulae. More precisely, we discuss some relations between the first integrals (13) of eq. (6) and first integrals $\hat{W}_1, \hat{W}_2$ of eq. (7),

$$\begin{aligned} \hat{W}_1 &= \hat{w}^{(1)}_{n,m}(v_{n,m}, v_{n+1,m}, \ldots, v_{n+\hat{k},m}), \quad \hat{k} \geq 1, \\ \hat{W}_2 &= \hat{w}^{(2)}_{n,m}(v_{n,m}, v_{n,m+1}, \ldots, v_{n,m+\hat{l}}), \quad \hat{l} \geq 1. \end{aligned} \tag{14}$$

Lemma 1. *If eq. (6) possesses a first integral W_1 of an order k, then eq. (7) has a first integral $\hat{W}_1$ of the order $\hat{k} = k + 1$.*

Proof. Using the transform (2), we obtain

$$\hat{W}_1 = w^{(1)}_{n,m}\left(\frac{v_{n+1,m}}{v_{n,m}}, \frac{v_{n+2,m}}{v_{n+1,m}}, \ldots, \frac{v_{n+k+1,m}}{v_{n+k,m}}\right).$$

It is clear that $\frac{\partial \hat{W}_1}{\partial v_{n,m}} \neq 0$, $\frac{\partial \hat{W}_1}{\partial v_{n+k+1,m}} \neq 0$ for at least some n, m, and therefore $\hat{W}_1$ is nontrivial (i.e., cannot depend on n, m only) but also it has the order $k+1$. Due to (12) this order may become not minimal.

Lemma 2. *If eq. (6) possesses a first integral W_2 of an order l, then eq. (7) has a first integral $\hat{W}_2$ of an order $\hat{l}$, such that $1 \leqslant \hat{l} \leqslant l$.*

Proof. The transformation (2) leads us to

$$\hat{W}_2 = w^{(2)}_{n,m}\left(\frac{v_{n+1,m}}{v_{n,m}}, \frac{v_{n+1,m+1}}{v_{n,m+1}}, \ldots, \frac{v_{n+1,m+l}}{v_{n,m+l}}\right).$$

By induction on l we can prove that

$$v_{n+1,m+l} = \alpha^{nl} v_{n+1,m} - v_{n,m+l} + V_{n,l}(v_{n,m}, v_{n,m+1}, \ldots, v_{n,m+l-1}).$$

It is clear that

$$\frac{\partial \hat{W}_2}{\partial v_{n,m+l}} = \frac{\partial w^{(2)}_{n,m}}{\partial u_{n,m+l}} \frac{\partial}{\partial v_{n,m+l}}\left(\frac{\alpha^{nl} v_{n+1,m} - v_{n,m+l} + \ldots}{v_{n,m+l}}\right) \neq 0$$

for some n, m, and therefore $\hat{W}_2$ is nontrivial. Its order is not greater than l.

Lemma 3. *If eq. (7) has a first integral $\hat{W}_1$ of an order $\hat{k}$, and $\hat{W}_1$ is linear w.r.t. $v_{n,m}, v_{n+1,m}, \ldots, v_{n+\hat{k},m}$, then eq. (6) possesses a first integral W_1 of the order $k = \hat{k}$.*

Proof. Using the transform (2) and the property (12), we obtain the relations

$$\begin{aligned} W_1 = \frac{T_1 \hat{W}_1}{\hat{W}_1} &= \frac{a_{0,n+1,m} v_{n+1,m} + a_{1,n+1,m} v_{n+2,m} + \ldots + a_{\hat{k},n+1,m} v_{n+\hat{k}+1,m}}{a_{0,n,m} v_{n,m} + a_{1,n,m} v_{n+1,m} + \ldots + a_{\hat{k},n,m} v_{n+\hat{k},m}} \\ &= \frac{a_{0,n+1,m} v_{n+1,m}/v_{n,m} + a_{1,n+1,m} v_{n+2,m}/v_{n,m} + \ldots + a_{\hat{k},n+1,m} v_{n+\hat{k}+1,m}/v_{n,m}}{a_{0,n,m} + a_{1,n,m} v_{n+1,m}/v_{n,m} + \ldots + a_{\hat{k},n,m} v_{n+\hat{k},m}/v_{n,m}} \\ &= \frac{a_{0,n+1,m} u_{n,m} + \ldots + a_{\hat{k},n+1,m} u_{n+\hat{k},m} u_{n+\hat{k}-1,m} \cdots u_{n,m}}{a_{0,n,m} + \ldots + a_{\hat{k},n,m} u_{n+\hat{k}-1,m} u_{n+\hat{k}-2,m} \cdots u_{n,m}}. \end{aligned} \tag{15}$$

One can readily verify that $\frac{\partial W_1}{\partial u_{n,m}} \neq 0$, $\frac{\partial W_1}{\partial u_{n+\hat{k},m}} \neq 0$ for at least some values of n, m.

Lemma 4. *If eq.* (7) *has a first integral* $\hat{W}_2$ *of an order* $\hat{l}$, *and* $\hat{W}_2$ *is linear w.r.t.* $v_{n,m}, v_{n,m+1}, \ldots, v_{n,m+\hat{l}}$, *then eq.* (6) *possesses a first integral* W_2 *of the order* $l = \hat{l} + 1$.

Proof. By using the property (12) we obtain

$$W_2 = \frac{T_2\hat{W}_2}{\hat{W}_2} = \frac{b_{0,n,m+1}v_{n,m+1} + b_{1,n,m+1}v_{n,m+2} + \ldots + b_{\hat{l},n,m+1}v_{n,m+1+\hat{l}}}{b_{0,n,m}v_{n,m} + b_{1,n,m}v_{n,m+1} + \ldots + b_{\hat{l},n,m}v_{n,m+\hat{l}}}$$
$$= \frac{b_{0,n,m+1}v_{n,m+1}/v_{n,m} + b_{1,n,m+1}v_{n,m+2}/v_{n,m} + \ldots + b_{\hat{l},n,m+1}v_{n,m+1+\hat{l}}/v_{n,m}}{b_{0,n,m} + b_{1,n,m}v_{n,m+1}/v_{n,m} + \ldots + b_{\hat{l},n,m}v_{n,m+\hat{l}}/v_{n,m}} \tag{16}$$
$$= \frac{b_{0,n,m+1}Z_{n,m} + \ldots + b_{\hat{l},n,m+1}Z_{n,m+\hat{l}}Z_{n,m+\hat{l}-1}\ldots Z_{n,m}}{b_{0,n,m} + \ldots + b_{\hat{l},n,m}Z_{n,m+\hat{l}-1}Z_{n,m+\hat{l}-2}\ldots Z_{n,m}}.$$

It follows from eqs. (2) and (7) that

$$Z_{n,m} = \frac{v_{n,m+1}}{v_{n,m}} = \alpha^n \frac{u_{n,m} - 1}{u_{n,m+1} + 1}.$$

4. First integrals of linear equation (7)

We shall use some necessary conditions of the Darboux integrability deduced for the discrete case in [7]. Those conditions were formulated there for autonomous equations of the form (8). We can reformulate and prove those conditions for the case of the non-autonomous linear equation (7).

We can rewrite eq. (7) in the form

$$(T_2 - \alpha^n)(T_1 + 1)v_{n,m} + 2\alpha^n v_{n,m} = 0. \tag{17}$$

By using the discrete Laplace transformation [7]

$$v_{n,m,j} = (T_1 + \alpha^{j-1})v_{n,m,j-1}, \quad j \geq 1, \qquad v_{n,m,0} = v_{n,m}, \tag{18}$$

we can introduce a sequence of unknown functions $v_{n,m,j}$ satisfying the equations

$$(T_2 - \alpha^{n+j})(T_1 + \alpha^j)v_{n,m,j} + \alpha^{n+j}(1 + \alpha^j)v_{n,m,j} = 0. \tag{19}$$

The last relations are proved by induction on j. One of the necessary conditions is formulated in terms of functions $K_{n,j}$,

$$K_{n,j} = \alpha^{n+j}(1 + \alpha^j), \quad j \geq 1, \qquad K_{n,0} = 2\alpha^n.$$

The following theorem has been taken from [7].

Theorem 1. *If eq.* (17) *possesses a first integral* $\hat{W}_1$ *of an order* $\hat{k}$, *then there exists* $\tilde{k}$, $0 \leqslant \tilde{k} < \hat{k}$, *such that* $K_{n,\tilde{k}} = 0$.

In a similar way, we can rewrite eq. (7) in the form

$$(T_1 + 1)(T_2 - \alpha^{n-1})v_{n,m} + \alpha^{n-1}(1 + \alpha)v_{n,m} = 0. \tag{20}$$

Using the second discrete Laplace transformation

$$\check{v}_{n,m,j} = (T_2 - \alpha^{n-j})\check{v}_{n,m,j-1}, \quad j \geq 1, \qquad \check{v}_{n,m,0} = v_{n,m}, \tag{21}$$

we can define a sequence of unknown functions $\check{v}_{n,m,j}$ which satisfy the equations

$$(T_1 + 1)(T_2 - \alpha^{n-j-1})\check{v}_{n,m,j} + \alpha^{n-j-1}(1 + \alpha^{j+1})\check{v}_{n,m,j} = 0. \tag{22}$$

The last relations are proved by induction on j. The second of the necessary conditions is formulated in terms of functions $H_{n,j}$,

$$H_{n,j} = \alpha^{n-j-1}(1+\alpha^{j+1}), \quad j \ge 0.$$

The following theorem has been taken from [7].

Theorem 2. *If eq. (20) possesses a first integral $\hat{W}_2$ of an order $\hat{l}$, then there exists $\tilde{l}$, $0 \leqslant \tilde{l} < \hat{l}$, such that $H_{n,\tilde{l}} = 0$.*

Let α_N be a root of -1, such that

$$\alpha_N^N = -1, \qquad \alpha_N^j \ne -1, \quad 1 \leqslant j < N. \tag{23}$$

For any $N \ge 1$, such a root always exists, for example, $\alpha_N = \exp(i\pi/N)$. Let us now consider eq. (7) with $\alpha = \alpha_N$. It follows from Theorems 1 and 2 for this equation that the orders of its any first integrals in the first and second directions must be such that

$$\operatorname{ord} \hat{W}_1 \ge N+1, \qquad \operatorname{ord} \hat{W}_2 \ge N. \tag{24}$$

On the other hand, we can construct first integrals for eq. (7) with $\alpha = \alpha_N$ of such orders in an explicit form. Eqs. (19) and (22) with $j = N$ and $j = N-1$, respectively, cast into the form

$$\begin{aligned} (T_2 + \alpha_N^n)(T_1 - 1)v_{n,m,N} &= 0, \\ (T_1 + 1)(T_2 + \alpha_N^n)\check{v}_{n,m,N-1} &= 0. \end{aligned} \tag{25}$$

So we can find first integrals for these equations (25) in the first and second directions, respectively,

$$\begin{aligned} \hat{W}_1 &= (-1)^m \alpha_N^{-nm}(T_1 - 1)v_{n,m,N}, \\ \hat{W}_2 &= (-1)^n (T_2 + \alpha_N^n)\check{v}_{n,m,N-1}. \end{aligned} \tag{26}$$

By using the Laplace transformations (18) and (21), we obtain first integrals for eq. (7) in the following explicit form,

$$\begin{aligned} \hat{W}_1 &= (-1)^m \alpha_N^{-nm}(T_1 - 1)(T_1 + \alpha_N^{N-1})(T_1 + \alpha_N^{N-2})\dots(T_1+1)v_{n,m}, \\ \hat{W}_2 &= (-1)^n (T_2 + \alpha_N^n)(T_2 - \alpha_N^{n-N+1})(T_2 - \alpha_N^{n-N+2})\dots(T_2 - \alpha_N^{n-1})v_{n,m}. \end{aligned} \tag{27}$$

We can see that both these first integrals are nontrivial, $\hat{W}_1$ is expressed in terms of $v_{n,m}, v_{n+1,m}, \dots, v_{n+N+1,m}$, while $\hat{W}_2$ is expressed in terms of $v_{n,m}, v_{n,m+1}, \dots, v_{n,m+N}$. Both $\hat{W}_1$ and $\hat{W}_2$ are linear w.r.t. $v_{n,m}$ and its shifts. We deduce that

$$\operatorname{ord} \hat{W}_1 = N+1, \qquad \operatorname{ord} \hat{W}_2 = N \tag{28}$$

and that these orders are minimal, taking into account the property (24).

For example, if $N = 1$, then $\alpha_N = -1$, and we have first integrals of the minimal orders,

$$\hat{W}_1 = (-1)^{(1-n)m}(v_{n+2,m} - v_{n,m}), \qquad \hat{W}_2 = (-1)^n v_{n,m+1} + v_{n,m}.$$

If $N = 2$, then $\alpha_N = \pm i$, and the first integrals read

$$\begin{aligned} \hat{W}_1 &= (-1)^m \alpha_N^{-nm}(v_{n+3,m} - v_{n+1,m} + \alpha_N(v_{n+2,m} - v_{n,m})), \\ \hat{W}_2 &= (-1)^n (v_{n,m+2} + \alpha_N^{n-1}(\alpha_N - 1)v_{n,m+1} - \alpha_N^{2n-1}v_{n,m}). \end{aligned}$$

5. First integrals of eq. (6)

We consider here the equation (6) with $\alpha = \alpha_N$, where α_N is defined by (23). Using Lemmas 3 and 4 together with the formulae (15) and (16), we construct first integrals W_1 and W_2 for eq. (6), starting from the first integrals (27). Their orders are

$$\operatorname{ord} W_1 = \operatorname{ord} W_2 = N + 1. \tag{29}$$

From Lemmas 1 and 2 and the relations (28) it follows that the minimal orders of first integrals of eq. (6) in both directions must be greater than or equal to N. We are led to the following theorem.

Theorem 3. *Eq.* (6) *with* $\alpha = \alpha_N$ *is Darboux integrable. The minimal orders of its first integrals in both directions must be equal to* N *or* $N+1$.

It follows from this theorem that there exist Darboux integrable discrete equations among equations of the form (6), such that the minimal orders of their first integrals in both directions are arbitrarily high.

For eq. (6) with $\alpha = \alpha_1 = -1$, we obtain the following first integrals,

$$W_1 = (-1)^{-m} \frac{u_{n,m}(u_{n+2,m}u_{n+1,m} - 1)}{u_{n+1,m}u_{n,m} - 1}, \quad W_2 = (-1)^n \frac{(u_{n,m+1} + u_{n,m+2})(u_{n,m} - 1)}{(u_{n,m} + u_{n,m+1})(u_{n,m+2} + 1)}.$$

In the case when $\alpha = \alpha_2 = \pm i$, the first integrals read,

$$W_1 = (\alpha_2)^{-m} \frac{u_{n,m}(u_{n+3,m}u_{n+2,m}u_{n+1,m} + \alpha_2 u_{n+2,m}u_{n+1,m} - u_{n+1,m} - \alpha_2)}{u_{n+2,m}u_{n+1,m}u_{n,m} + \alpha_2 u_{n+1,m}u_{n,m} - u_{n,m} - \alpha_2},$$

$$W_2 = (\alpha_2)^n \frac{(u_{n,m} - 1)(\alpha_2(u_{n,m+3} + u_{n,m+2})(u_{n,m+1} - 1) - (u_{n,m+3} + 1)(u_{n,m+2} + u_{n,m+1})}{(1 + u_{n,m+3})(\alpha_2(u_{n,m+2} + u_{n,m+1})(u_{n,m} - 1) - (u_{n,m+2} + 1)(u_{n,m+1} + u_{n,m}))}.$$

In the paper [4] a method was presented which used the so-called annihilation operators [8] and which allowed one to construct first integrals of low orders and to show that those orders are minimal. By using this method we have checked that four first integrals given just above have the minimal orders. We believe that all first integrals of eq. (6) with $\alpha = \alpha_N$ which can be constructed by the scheme presented in this paper have the minimal orders.

As it was said above, there is a hyperbolic equation [5] of the form (9), namely,

$$u_{xy} = \frac{2N}{x+y}\sqrt{u_x u_y} \tag{30}$$

which is analogues to eq. (6) with $\alpha = \alpha_N$ in the sense that the minimal orders of first integrals of such equations (30) may be arbitrarily high. Unlike eq. (30), which is symmetric under the involution $x \leftrightarrow y$, the discrete equation (6) is not symmetric under $n \leftrightarrow m$, and its first integrals in different directions have quite different forms. The second difference is that eq. (6) is of the Burgers type with linearizing transformation (2), while linearizing transformation for eq. (30) has the form

$$v = \sqrt{u_y},$$

where $v(x, y)$ is a solution of a hyperbolic linear equation.

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