

Classification of five-point differential–difference equations II

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Abstract

Using the generalized symmetry method we finish a classification, started in article (Garifullin *et al* 2017 *J. Phys. A: Math. Theor.* **50** 125201), of integrable autonomous five-point differential–difference equations. The resulting list, up to autonomous point transformations, contains 14 equations some of which seem to be new. We have found non-autonomous or non-point transformations relating most of the obtained equations among themselves as well as their generalized symmetries.

Keywords: integrability, generalized symmetry, classification, non-invertible transformation

1. Introduction

Here we conclude a generalized symmetry classification started in article [20]. The generalized symmetry method uses the existence of generalized symmetries as an integrability criterion and allows one to classify integrable equations of a certain class. Using this method some important classes of partial differential equations [31, 32], of differential–difference equations (D Δ Es) [9, 43] and of partial difference equations (P Δ Es) [16, 30] have been classified.

Other integrability criteria have been introduced to classify integrable P Δ Es, see e.g. the consistency around the cube technique introduced in [10, 35, 36], whose results are presented, for example, in [5, 6, 13, 14].

A class of P Δ Es, particularly important among recently studied, is given by those equations which are defined on a square, i.e. which relate four neighboring points in the two-dimensional plane. The complete classification of the integrable P Δ Es defined on a square is very difficult to perform.

Almost all integrable known PΔEs have the lowest order associated generalized symmetries given by integrable evolutionary DΔEs which are defined on three-point lattices [21, 22, 27, 30, 41] and belong to the classification presented in [28, 43]. This is the classification of Volterra type equations

$$\dot{u}_n = \Phi(u_{n+1}, u_n, u_{n-1}) \tag{1}$$

presented in [42], and the resulting list of equations is quite big, see the details in the review article [43]. Here \dot{u}_n is the derivative of u_n with respect to a continuous variable t and n is discrete integer variable.

Recently one has obtained examples of PΔEs defined on the square which have the lowest order generalized symmetries defined on more than three-point lattices [1, 16, 33, 37]. So an alternative classification that seems easier to perform is that of integrable five-point DΔEs

$$\dot{u}_n = \Psi(u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}). \tag{2}$$

Few results in this line of research are already known, see e.g. [2–4, 19, 20]. The integrable PΔEs are then obtained as Bäcklund transformations of these DΔEs [17, 20, 25, 26, 29]. Their construction scheme is discussed in more detail in [17] and [20, appendix B]. The best known integrable example in this class is the Ito–Narita–Bogoyavlensky (INB) equation [11, 24, 34]:

$$\dot{u}_n = u_n(u_{n+2} + u_{n+1} - u_{n-1} - u_{n-2}). \tag{3}$$

The classification of five-point lattice equations of the form (2) will contain equations coming from the classification of Volterra type equation (1) in two ways. On one hand, they appear as equations of the form

$$\dot{u}_n = \Phi(u_{n+2}, u_n, u_{n-2}). \tag{4}$$

If u_n is a solution of (4), then the functions $\tilde{u}_k = u_{2k}$ and $\hat{u}_k = u_{2k+1}$ satisfy (1) with k instead of n . Equation (4) is in fact a three-point lattice equation equivalent to (1). On other hand they appear as generalized symmetries of (1). Any integrable Volterra type equation has a five-point symmetry of the form (2). See the explicit results for Volterra type equations presented for example in [23, 38, 39, 43].

To avoid those two cases, which are included in the classification of Volterra type equations, and to simplify the problem, we limit ourselves to consideration here of just equations of the form

$$\begin{aligned} \dot{u}_n = & A(u_{n+1}, u_n, u_{n-1})u_{n+2} + B(u_{n+1}, u_n, u_{n-1})u_{n-2} \\ & + C(u_{n+1}, u_n, u_{n-1}), \end{aligned} \tag{5}$$

where A , B and C are n -independent functions of their arguments. The majority of the examples of DΔEs of the form (2) known up to now belong to the class (5) [4, 8, 11, 12, 16, 20, 24, 33, 34, 40]. So the class (5) is not void.

However also few equations of the Volterra classification (1) are included in the five-point classification of the equations of the class (5). They are those polynomial equations which are linearly dependent on u_{n+1} and u_{n-1} [43]. Such equations, rewritten in the form (4), belong to the class (5). Also their five-point symmetries are of the form (5).

The theory of the generalized symmetry method is well-developed in case of Volterra type equations [43] and it has been modified for the case of DΔEs depending on 5 and more lattice points in [2, 3]. The classification problem of class (5) equations seems to be technically quite complicate. For this reason we use a simpler version of the method, compared with the one presented in [2, 3], which has been developed in [20]. For equations analogous to (3), which

are the first members of their hierarchies, the simplest generalized symmetry has the form [1, 16, 20, 33, 44]:

$$u_{n,\tau} = G(u_{n+4}, u_{n+3}, u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4}), \tag{6}$$

where $u_{n,\tau}$ denotes τ -derivative of u_n . We will use the existence of such symmetry as an integrability criterion.

The problem naturally splits into two cases depending on the form of the functions A and B of (5), see an explanation in section 2. In [20] we studied the case when the functions A and B in (5) satisfied the conditions:

$$A \neq \alpha(u_{n+1}, u_n)\alpha(u_n, u_{n-1}), \quad B \neq \beta(u_{n+1}, u_n)\beta(u_n, u_{n-1}) \tag{7}$$

for any functions α and β of their arguments. This class is called the *Class I*, and it includes such well-known examples as the INB equation (3) and the discrete Sawada–Kotera equation [40]. In this case the following simple criterion for checking conditions (7) takes place:

$$\frac{\partial}{\partial u_n} \frac{a_{n+1}a_{n-1}}{a_n} \neq 0, \quad \frac{\partial}{\partial u_n} \frac{b_{n+1}b_{n-1}}{b_n} \neq 0, \tag{8}$$

where

$$a_n = A(u_{n+1}, u_n, u_{n-1}), \quad b_n = B(u_{n+1}, u_n, u_{n-1}),$$

see [20].

In [20] it was presented, as a result of the classification of the Class I equations, a novel equation:

$$\dot{u}_n = (u_n^2 + 1) \left(u_{n+2} \sqrt{u_{n+1}^2 + 1} - u_{n-2} \sqrt{u_{n-1}^2 + 1} \right). \tag{9}$$

In [18] it is shown that in the continuous limit (9) goes into the well-known Kaup–Kupershmidt equation, and its integrability has been proved by constructing an L–A pair and conservation laws of sufficiently high order.

In this paper we consider the case when

$$A = \alpha(u_{n+1}, u_n)\alpha(u_n, u_{n-1}) \quad \text{or} \quad B = \beta(u_{n+1}, u_n)\beta(u_n, u_{n-1}) \tag{10}$$

for some functions α and β of their arguments, i.e. when

$$\frac{\partial}{\partial u_n} \frac{a_{n+1}a_{n-1}}{a_n} = 0 \quad \text{or} \quad \frac{\partial}{\partial u_n} \frac{b_{n+1}b_{n-1}}{b_n} = 0. \tag{11}$$

We will call this case *Class II*. A known representative of Class II is given in [16]:

$$\dot{u}_n = (u_{n+1}u_n - 1)(u_n u_{n-1} - 1)(u_{n+2} - u_{n-2}). \tag{12}$$

In this article we present a complete list of equations of the Class II possessing a generalized symmetry of the form (6). In this way we complete the classification of integrable equation (5) started in [20]. Among them there are a few probably new integrable examples. Then we find the non-point autonomous or point non-autonomous transformations relating most of resulting equations among themselves.

In section 2 we discuss a theory of the generalized symmetry method suitable to solve our specific problem for Class II equations. In particular, in section 2.2 some integrability conditions are derived and criteria for checking those conditions are proved. In section 3 we present the list of integrable equations and the relations between those equations expressed as autonomous non-point or non-autonomous point transformations. In section 4 the generalized symmetries of the resulting equations are discussed. Section 5 is devoted to some concluding remarks.

2. Theory

Here we briefly repeat the theory, presented in the previous paper [20] and necessary for our present work, as well as derive few additional results related just to the Class II case.

To simplify the notation let us represent (5) as:

$$\dot{u}_n = a_n u_{n+2} + b_n u_{n-2} + c_n \equiv f_n, \tag{13}$$

where

$$\begin{aligned} a_n &= A(u_{n+1}, u_n, u_{n-1}), & b_n &= B(u_{n+1}, u_n, u_{n-1}), \\ c_n &= C(u_{n+1}, u_n, u_{n-1}). \end{aligned} \tag{14}$$

In (13) we require

$$a_n \neq 0, \quad b_n \neq 0. \tag{15}$$

For convenience we denote the symmetry (6) as:

$$u_{n,\tau} = g_n, \tag{16}$$

with the restriction:

$$\frac{\partial g_n}{\partial u_{n+4}} \neq 0, \quad \frac{\partial g_n}{\partial u_{n-4}} \neq 0. \tag{17}$$

The compatibility condition for (13) and (16) is

$$u_{n,\tau,t} - u_{n,t,\tau} \equiv D_t g_n - D_\tau f_n = 0. \tag{18}$$

Here D_t and D_τ are the operators of total differentiation with respect to t and τ given respectively by:

$$D_t = \sum_{k \in \mathbb{Z}} f_k \frac{\partial}{\partial u_k}, \quad D_\tau = \sum_{k \in \mathbb{Z}} g_k \frac{\partial}{\partial u_k}. \tag{19}$$

As (13) and (16) as well as the compatibility condition (18) are autonomous, their form do not explicitly depend on the point n . For this reason, we write down for short below the equations and the compatibility condition (18) at the point $n = 0$: $\dot{u}_0 = f_0$, $u_{0,\tau} = g_0$.

$$D_t g_0 = D_\tau f_0. \tag{20}$$

We assume as independent variables the functions

$$u_0, u_1, u_{-1}, u_2, u_{-2}, u_3, u_{-3} \dots \tag{21}$$

The condition (20) must be identically satisfied for all values of the independent variables (21). Equation (20) depends on the variables $u_{-6}, u_{-5}, \dots, u_5, u_6$ and it is an overdetermined system of equations for the unknown function g_0 , for any given f_0 . Using a standard technique of the generalized symmetry method [43], we can calculate g_0 step by step, obtaining some necessary conditions for the function f_0 .

2.1. General case

The first steps for the calculation of g_0 can be carried out with no restriction on the form of f_0 .

In fact, differentiating (20) with respect to u_6 , we obtain as before (see [20]) up to a τ -scaling in (16):

$$\frac{\partial g_0}{\partial u_4} = a_0 a_2. \tag{22}$$

By differentiating (20) with respect to u_5 and taking into account (22), we can define

$$h_0^+ = \frac{\partial g_0}{\partial u_3} - a_1 \frac{\partial f_0}{\partial u_1} - a_0 \frac{\partial f_2}{\partial u_3}, \tag{23}$$

and we can state the following lemma [20]:

Lemma 1. *If $h_0^+ \neq 0$, then there exists $\hat{\alpha}_n = \alpha(u_n, u_{n-1})$, such that $a_0 = \hat{\alpha}_1 \hat{\alpha}_0$.*

As a consequence of lemma 1 there are two possibilities:

- Case 1. Let $a_0 \neq \hat{\alpha}_1 \hat{\alpha}_0$ for any $\hat{\alpha}_n = \alpha(u_n, u_{n-1})$, see (7). Then $h_0^+ = 0$ due to lemma 1.
- Case 2. Let $a_0 = \hat{\alpha}_0 \hat{\alpha}_1$ for some $\hat{\alpha}_n = \alpha(u_n, u_{n-1})$. Then we can find:

$$h_0^+ = \mu^+ \hat{\alpha}_0 \hat{\alpha}_1 \hat{\alpha}_2 \tag{24}$$

with an arbitrary constant μ^+ .

In both cases (23) gives us $\frac{\partial g_0}{\partial u_3}$.

In quite similar way, differentiating (20) with respect to u_{-6} and u_{-5} , we get a set of relations analogous to (22) and (23). Namely,

$$\frac{\partial g_0}{\partial u_{-4}} = \nu b_0 b_{-2}, \tag{25}$$

where ν is an arbitrary nonzero constant, and

$$h_0^- = \frac{\partial g_0}{\partial u_{-3}} - \nu b_{-1} \frac{\partial f_0}{\partial u_{-1}} - \nu b_0 \frac{\partial f_{-2}}{\partial u_{-3}}. \tag{26}$$

We can state a lemma similar to lemma 1 and, as a consequence, we get again two cases:

1. Let $b_0 \neq \hat{\beta}_0 \hat{\beta}_{-1}$ for any $\hat{\beta}_n = \beta(u_{n+1}, u_n)$, then $h_0^- = 0$.
2. Let $b_0 = \hat{\beta}_0 \hat{\beta}_{-1}$, then we can find:

$$h_0^- = \mu^- \hat{\beta}_0 \hat{\beta}_{-1} \hat{\beta}_{-2} \tag{27}$$

with an arbitrary constant μ^- .

In both cases (26) provides us $\frac{\partial g_0}{\partial u_{-3}}$.

So the results presented in this subsection provide a natural frame for splitting further calculation into several different cases. Obviously, any of the equation (5) belongs either to Class I given by (7) or to Class II given by (10). In the following in this paper we consider the Class II defined in the introduction by (10).

2.2. Integrability conditions for equations of Class II

We can construct two types of integrability conditions for equations of Class II. The first of them is obtained when one of the conditions (10) is not satisfied. Up to the involution $n \rightarrow -n$, this corresponds to the case

$$A \neq \alpha(u_{n+1}, u_n) \alpha(u_n, u_{n-1}), \quad B = \beta(u_{n+1}, u_n) \beta(u_n, u_{n-1}), \tag{28}$$

i.e.

$$h_0^+ = 0, \quad h_0^- = \mu^- \hat{\beta}_0 \hat{\beta}_{-1} \hat{\beta}_{-2}.$$

As we know, in this case the partial derivatives $\frac{\partial g_0}{\partial u_3}$, $\frac{\partial g_0}{\partial u_{-3}}$, $\frac{\partial g_0}{\partial u_4}$ and $\frac{\partial g_0}{\partial u_{-4}}$ are given by (22), (23), (25) and (26).

Differentiating (20) with respect to u_4 and u_{-4} and introducing the functions:

$$q_0^+ = \frac{1}{a_0} \frac{\partial g_0}{\partial u_2} - D_t \log a_0 - \frac{\partial f_0}{\partial u_0} - \frac{\partial f_2}{\partial u_2} - \frac{1}{a_0} \frac{\partial f_0}{\partial u_1} \frac{\partial f_1}{\partial u_2}, \quad (29)$$

$$q_0^- = \frac{1}{\nu b_0} \frac{\partial g_0}{\partial u_{-2}} - D_t \log b_0 - \frac{\partial f_0}{\partial u_0} - \frac{\partial f_{-2}}{\partial u_{-2}} - \frac{1}{b_0} \frac{\partial f_0}{\partial u_{-1}} \frac{\partial f_{-1}}{\partial u_{-2}}, \quad (30)$$

we obtain two relations. The first of them has the form of a conservation law [20]:

$$2D_t \log a_0 = q_2^+ - q_0^+. \quad (31)$$

The second one is more complicated:

$$2D_t \log b_0 + (T^{-3} - 1) \left(\frac{\mu^-}{\nu \hat{\beta}_0} \frac{\partial f_0}{\partial u_{-1}} \right) = q_{-2}^- - q_0^-. \quad (32)$$

Here T is the shift operator, such that $Th_n = h_{n+1}$. Relations (31) and (32) provide necessary conditions for the integrability. If (13) is integrable in the sense that a symmetry (16) exists, then there must exist the functions q_n^+ , q_n^- depending on a finite number of independent variables (21), such that the relations (31) and (32) are satisfied.

When both conditions (10) are satisfied, i.e.

$$A = \alpha(u_{n+1}, u_n) \alpha(u_n, u_{n-1}), \quad B = \beta(u_{n+1}, u_n) \beta(u_n, u_{n-1}), \quad (33)$$

then differentiating (20) with respect to u_4 we obtain instead of (31) the integrability condition

$$2D_t \log a_0 + (T^3 - 1) \left(\frac{\mu^+}{\hat{\alpha}_0} \frac{\partial f_0}{\partial u_1} \right) = q_2^+ - q_0^+, \quad (34)$$

which is similar to (32).

The relation (31) has the form of conservation law. Due to (33), i.e $a_0 = \hat{\alpha}_1 \hat{\alpha}_0$, $b_0 = \hat{\beta}_0 \hat{\beta}_{-1}$, (32) and (34) can also be represented as conservation laws:

$$D_t \log \hat{\alpha}_0 = (T - 1) Q_0^+, \quad D_t \log \hat{\beta}_0 = (T^{-1} - 1) Q_0^-, \quad (35)$$

where

$$Q_0^+ = \frac{1}{4}(T + 1)q_0^+ - \frac{1}{4}(T^2 + T + 1) \left(\frac{\mu^+}{\hat{\alpha}_0} \frac{\partial f_0}{\partial u_1} \right) - \frac{1}{2}D_t \log \hat{\alpha}_0,$$

$$Q_0^- = \frac{1}{4}(T^{-1} + 1)q_0^- - \frac{1}{4}(T^{-2} + T^{-1} + 1) \left(\frac{\mu^-}{\nu \hat{\beta}_0} \frac{\partial f_0}{\partial u_{-1}} \right) - \frac{1}{2}D_t \log \hat{\beta}_0.$$

Equations (31) and (35) are conservation laws of the minimum possible order [43]. Consequently any integrable equation under consideration must have two conservation laws of the forms (31) or (35).

If, for a given equation (13), conditions (31) and (32) in the case (28) or (32) and (34) in the case (33) are satisfied and the functions q_0^\pm and Q_0^\pm are known, then partial derivatives $\frac{\partial g_0}{\partial u_2}$, $\frac{\partial g_0}{\partial u_{-2}}$ can be obtained from (29) and (30). So the right hand side of symmetry (16) is defined up to one unknown function of three variables:

$$\psi(u_{n+1}, u_n, u_{n-1}). \quad (36)$$

This function can be found directly from the compatibility condition (18).

In this way we carry out the classification of the equations of Class II. At first we use the integrability conditions (31) and (32) or (32) and (34). Then we define the symmetry up to a function (36) and then try to derive it implementing the compatibility condition (18).

To derive simpler relations to check the integrability conditions (31), (32) and (34) we use the variational derivatives considered in [20].

For any function

$$\varphi = \varphi(u_{m_1}, u_{m_1-1}, \dots, u_{m_2}), \quad m_1 \geq m_2, \tag{37}$$

we define the formal variational derivative through the formula:

$$\frac{\delta\varphi}{\delta u_0} = \sum_{k=m_2}^{m_1} T^{-k} \frac{\partial\varphi}{\partial u_k}, \tag{38}$$

see e.g. [43], and its adjoint version [20]:

$$\frac{\bar{\delta}\varphi}{\bar{\delta} u_0} = \sum_{k=m_2}^{m_1} (-1)^k T^{-k} \frac{\partial\varphi}{\partial u_k}. \tag{39}$$

Then we can state the following lemma:

Lemma 2. *The equations $\frac{\delta\varphi}{\delta u_0} = 0$ and $\frac{\bar{\delta}\varphi}{\bar{\delta} u_0} = 0$ hold iff φ can be represented in the form*

$$\varphi = \kappa + (T^2 - 1)\omega, \tag{40}$$

where κ is a constant, while ω is a function of a finite number of independent variables (21).

To check if a given function φ is of the form $\varphi = (T^2 - 1)\omega$, we have at first to check the conditions of lemma 2. Then we can represent φ in the form (40) and check if $\kappa = 0$.

So the criteria for checking (31) are of the form:

$$\frac{\delta}{\delta u_0} D_t \log a_0 = 0, \quad \frac{\bar{\delta}}{\bar{\delta} u_0} D_t \log a_0 = 0, \tag{41}$$

see [20]. In the case of the integrability conditions (32) and (34), we first get

$$D_t \log a_0 = (T + 1)D_t \log \hat{\alpha}_0, \quad D_t \log b_0 = (T^{-1} + 1)D_t \log \hat{\beta}_0$$

from

$$a_0 = \hat{\alpha}_1 \hat{\alpha}_0, \quad b_0 = \hat{\beta}_0 \hat{\beta}_{-1}. \tag{42}$$

Then from (32) and (34) we derive the following criteria for checking them:

$$\frac{\delta}{\delta u_0} D_t \log \hat{\beta}_0 = 0, \quad \mu^- \frac{\bar{\delta}}{\bar{\delta} u_0} \left(\frac{1}{\hat{\beta}_0} \frac{\partial f_0}{\partial u_{-1}} \right) = 0, \tag{43}$$

$$\frac{\delta}{\delta u_0} D_t \log \hat{\alpha}_0 = 0, \quad \mu^+ \frac{\bar{\delta}}{\bar{\delta} u_0} \left(\frac{1}{\hat{\alpha}_0} \frac{\partial f_0}{\partial u_1} \right) = 0. \tag{44}$$

2.3. The classification

Using the method described in the previous subsections, we carry out the classification of integrable equations belonging to the Class II. It is easy to proof that there are no integrable equations in the asymmetric case (28). So here we discuss only the symmetric case (33).

Let us recall that the partial derivatives $\frac{\partial g_0}{\partial u_{\pm 4}}$ and $\frac{\partial g_0}{\partial u_{\pm 3}}$ are given by (22), (23) and (25) and (26), and that we have implicit definitions for $\frac{\partial g_0}{\partial u_{\pm 2}}$ given by the relations (29), (30), (32) and (34).

Applying $\frac{\partial}{\partial u_3}$ and $\frac{\partial}{\partial u_{-3}}$ to (32) and (34) respectively, we get:

$$(\nu + 1) \frac{\partial \hat{\alpha}_0}{\partial u_{-1}} = 0, \quad (\nu + 1) \frac{\partial \hat{\beta}_0}{\partial u_1} = 0, \tag{45}$$

where $\hat{\alpha}_0, \hat{\beta}_0$ are defined in (42), while the parameter ν has been introduced in (25). By using (45) we find that if $\nu \neq -1$, i.e. when $\frac{\partial \hat{\alpha}_0}{\partial u_{-1}}$ and $\frac{\partial \hat{\beta}_0}{\partial u_1}$ are zero, there are no integrable equations.

In the symmetric case with $\nu = -1$, from (32) and (34) we can derive some simple integrability conditions, see also (43) and (44). Differentiating (32) and (34), we can find the second derivatives of g_0 , i.e. $\frac{\partial^2 g_0}{\partial u_2^2}, \frac{\partial^2 g_0}{\partial u_2 \partial u_{-2}}, \frac{\partial^2 g_0}{\partial u_{-2}^2}$. Then using the following consequences of the compatibility condition (18):

$$\begin{aligned} \frac{\partial}{\partial u_2} \left(\frac{1}{a_1} \frac{\partial^3}{\partial u_3 \partial u_{-2}^2} (D_t g_0 - D_\tau f_0) \right) &= 0, \\ \frac{\partial}{\partial u_{-2}} \left(\frac{1}{b_{-1}} \frac{\partial^3}{\partial u_{-3} \partial u_2^2} (D_t g_0 - D_\tau f_0) \right) &= 0, \end{aligned}$$

together with formulae for the partial derivatives g_0 with respect to u_2 and u_{-2} , we get the conditions:

$$\frac{\partial \hat{\alpha}_0}{\partial u_{-1}} \frac{\partial^2 \hat{\alpha}_{-1}}{\partial u_{-2}^2} = 0, \quad \frac{\partial \hat{\beta}_0}{\partial u_1} \frac{\partial^2 \hat{\beta}_1}{\partial u_2^2} = 0. \tag{46}$$

From (46) we derive the simple integrability conditions:

$$\frac{\partial^2 \alpha(u_0, u_{-1})}{\partial u_{-1}^2} = 0, \quad \frac{\partial^2 \beta(u_1, u_0)}{\partial u_1^2} = 0, \tag{47}$$

where the functions α, β are defined in (33). In such a way we have reduced the classification to the calculation of four unknown functions of one variable instead of two functions α, β of two variables.

In next section we will get all integrable equations of the form (5) and (33) with α, β satisfying (47). Since $\nu = -1$, from (22) and (25) we get that the generalized symmetry (6) will have the form:

$$u_{0,\tau} = \hat{\alpha}_0 \hat{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3 u_4 - \hat{\beta}_0 \hat{\beta}_{-1} \hat{\beta}_{-2} \hat{\beta}_{-3} u_{-4} + \hat{G}(u_3, u_2, u_1, u_0, u_{-1}, u_{-2}, u_{-3}). \tag{48}$$

3. Complete list of integrable equations of Class II

In this section we present the complete list of integrable equations of Class II together with the non-point autonomous or point non-autonomous relations between them. These equations are referred by the numbers (E1)–(E14). Some of the obtained equations seem to be new.

The classification is carried out in two steps: at first one finds all integrable equations of a certain class up to point transformations, then one searches for non-point transformations which link the different resulting equations. In this paper we use autonomous point transformations which, because of the specific form (5) of the equations, are linear transformations with constant coefficients:

$$\hat{u}_0 = c_1 u_0 + c_2, \quad \hat{t} = c_3 t, \quad c_1 c_3 \neq 0. \tag{49}$$

The non-point transformations linking the different resulting equations are transformations of the form

$$\hat{u}_0 = \varphi(u_k, u_{k-1}, \dots, u_m), \quad k > m, \tag{50}$$

and their compositions. Some of the resulting equations are related among each other by point non-autonomous transformations.

Equations (50) transforms (2) into

$$\hat{u}_{0,t} = \hat{\Psi}(\hat{u}_2, \hat{u}_1, \hat{u}_0, \hat{u}_{-1}, \hat{u}_{-2}). \tag{51}$$

For any solution u_n of (2), formula (50) provides a solution \hat{u}_n of (51).

The transformation (50) is explicit in one direction. If an equation A is transformed into B by a transformation (50), then this transformation has the *direction* from A to B , and we will write in diagrams below $A \longrightarrow B$, so indicating the direction in which it is explicit. Non-autonomous point transformations

$$\hat{u}_n = \xi_n u_n, \quad \hat{t} = ct, \quad c \neq 0, \quad \xi_n \neq 0, \quad \forall n, \tag{52}$$

which are invertible, will be denoted by $A \longleftarrow\text{----}\longrightarrow B$.

The classification result is formulated in the following theorem:

Theorem 1. *If a nonlinear equation of the form (13)–(15) belongs to Class II (10) and has a generalized symmetry (6), (16) and (17), then up to an autonomous point transformation (49) it is equivalent to one of the following equations (E1)–(E14). Any equation in (E1)–(E14) has a generalized symmetry of the form (6), (16) and (17).*

For a better understanding of the results, we split the complete list into the Lists 1–4, where the equations are related among themselves either by autonomous non-point transformations or by non-autonomous point ones. For each of these lists we show the relations between the equations by a diagram, where the transformations (50) or (52) are shown by arrows and are denoted by the numbers (T1)–(T8).

All necessary transformations are given in the List T. Autonomous non-point transformations, which are linearizable [19], were constructed by using the transformation theory presented in [19]. A shorter version of this theory with some modifications can be found in [20, appendix A].

The generalized symmetries of (E1)–(E14) are discussed in section 4.

List 1. *Equations related to the double Volterra equation*

$$\dot{u}_0 = u_0 [u_1(u_2 - u_0) + u_{-1}(u_0 - u_{-2})] \tag{E1}$$

$$\dot{u}_0 = u_1 u_0^2 u_{-1} (u_2 - u_{-2}). \tag{E2}$$

The equation (E2) has been presented in [7]. Both equations of List 1 are transformed into the equation

$$\dot{u}_0 = u_0 (u_2 - u_{-2}) \tag{53}$$

as shown in diagram (54).

$$(E2) \xrightarrow{(T4)} (E1) \xrightarrow{(T2)} (53). \tag{54}$$

The non-invertible transformations (T2) and (T4) are given in the List T below. Transformations (T2) and (T4) are of the linearizable class, i.e. are not of Miura type (see a discussion of this notion in [19, 20]). Equation (53) is called the double Volterra equation, as two transformations $\tilde{u}_k = u_{2k}$ and $\hat{u}_k = u_{2k+1}$ turn it into the standard form of the Volterra equation [43].

List 2. Equations related to a generalized symmetry of the Volterra equation

$$\dot{u}_0 = u_0 [u_1(u_2 + u_1 + u_0) - u_{-1}(u_0 + u_{-1} + u_{-2})] + cu_0(u_1 - u_{-1}) \tag{E3}$$

$$\dot{u}_0 = (u_0^2 - a^2) [(u_1^2 - a^2)(u_2 + u_0) - (u_{-1}^2 - a^2)(u_0 + u_{-2})] + c(u_0^2 - a^2)(u_1 - u_{-1}) \tag{E4}$$

$$\dot{u}_0 = (u_1 - u_0 + a)(u_0 - u_{-1} + a)(u_2 - u_{-2} + 4a + c) + b \tag{E5}$$

$$\dot{u}_0 = u_0[u_1(u_2 - u_1 + u_0) - u_{-1}(u_0 - u_{-1} + u_{-2})] \tag{E6}$$

$$\dot{u}_0 = (u_0^2 - a^2) [(u_1^2 - a^2)(u_2 - u_0) + (u_{-1}^2 - a^2)(u_0 - u_{-2})] \tag{E7}$$

$$\dot{u}_0 = (u_1 + u_0)(u_0 + u_{-1})(u_2 - u_{-2}). \tag{E8}$$

The equations of the List 2 are related among themselves by the transformations shown in the following diagrams (55).

$$\begin{aligned} (E5) &\xrightarrow{(T1)} (E3) \xleftarrow{(T5)} (E4) \\ (E6) &\xleftarrow{(T7)} (E3, c = 0) \\ (E7) &\xleftarrow{(T8)} (E4, c = 0) \\ (E8) &\xleftarrow{(T7)} (E5, a = b = c = 0). \end{aligned} \tag{55}$$

Equations (E3)–(E5) are the generalized symmetries of the equations [43]:

$$\dot{u}_0 = u_0(u_1 - u_{-1}), \tag{56}$$

$$\dot{u}_0 = (u_0^2 - a^2)(u_1 - u_{-1}), \tag{57}$$

$$\dot{u}_0 = (u_1 - u_0 + a)(u_0 - u_{-1} + a). \tag{58}$$

These Volterra type equations are related among themselves by the same transformations (T1) and (T5) as their symmetries. The transformations (T1) and (T5) are well-known, see e.g. [43]. Equation (56) is the Volterra equation itself, (57) is the modified Volterra

equation, and the transformation (T5) with $a \neq 0$ is of Miura type [19]. The transformation (T1) is linear.

Transformations (T7) and (T8) are non-autonomous invertible point transformations. So (E6)–(E8) are equivalent to (E3)–(E5), as shown in diagrams (55). The transformation (T8) is nontrivial, see a comment after List T. It can be shown that (E6)–(E8) are also generalized symmetries of some simpler non-autonomous equations of the Volterra type.

List 3. Equations related to the INB equation (3)

$$\dot{u}_0 = u_0(u_2u_1 - u_{-1}u_{-2}) \tag{E9}$$

$$\dot{u}_0 = (u_1 - u_0 + a)(u_0 - u_{-1} + a)(u_2 - u_1 + u_{-1} - u_{-2} + 2a) + b \tag{E10}$$

$$\dot{u}_0 = u_0(u_1u_0 - a)(u_0u_{-1} - a)(u_2u_1 - u_{-1}u_{-2}) \tag{E11}$$

$$\dot{u}_0 = (u_1 + u_0)(u_0 + u_{-1})(u_2 + u_1 - u_{-1} - u_{-2}). \tag{E12}$$

Equation (E9) is a well-known modification of INB (3), see [11]. Equation (E11) with $a = 0$ has been considered in [7] and with $a = 1$ in [37]. The equations of this list are related among themselves and to

$$\dot{u}_0 = (u_0^2 + au_0)(u_2u_1 - u_{-1}u_{-2}) \tag{59}$$

as shown in the following diagrams:

$$\begin{array}{ccccc} (E10) & \xrightarrow{(T1)} & (E9) & \xrightarrow{(T2)} & (3) \\ & & & & \uparrow (T6) \\ & & (E11) & \xrightarrow{(T3)} & (59) \\ (E12) & \xleftarrow{(T7)} & (E10, a = b = 0) & & \end{array} \tag{60}$$

Equation (59) is presented in [11] in the case $a = 0$ and in [8, 33] in the case $a \neq 0$. The transformation (T7) is invertible, while all the other transformations are non-invertible. All the transformations present in the diagram (60) are linearizable except for (T6) with $a \neq 0$ which is of Miura type, see a comment in [19].

List 4. The remaining equations

$$\dot{u}_0 = (u_1u_0 - 1)(u_0u_{-1} - 1)(u_2 - u_{-2}) \tag{E13}$$

$$\dot{u}_0 = u_1u_0^3u_{-1}(u_2u_1 - u_{-1}u_{-2}) - u_0^2(u_1 - u_{-1}). \tag{E14}$$

Equation (E13) is known (see [15, 16]), while (E14) is a simple modification of

$$\dot{u}_0 = u_0^2(u_2u_1 - u_{-1}u_{-2}) - u_0(u_1 - u_{-1}) \tag{61}$$

as shown in diagram (62).

$$(E14) \xrightarrow{(T2)} (61). \tag{62}$$

Equation (61) has been found in [40] and can be called the discrete Sawada–Kotera equation [1, 40].

All transformations relating the equations of Lists 1–4 are presented in the List T.

List T. *List of used transformations*

$$\hat{u}_0 = u_1 - u_0 + a \tag{T1}$$

$$\hat{u}_0 = u_1 u_0 \tag{T2}$$

$$\hat{u}_0 = u_1 u_0 - a \tag{T3}$$

$$\hat{u}_0 = u_1 u_{-1} \tag{T4}$$

$$\hat{u}_0 = (u_1 - a)(u_0 + a) \text{ or } \hat{u}_0 = (u_1 + a)(u_0 - a) \tag{T5}$$

$$\hat{u}_0 = (u_2 + a)u_1 u_0 \text{ or } \hat{u}_0 = u_2 u_1 (u_0 + a) \tag{T6}$$

$$\hat{u}_n = (-1)^n u_n, \quad \hat{t} = -t \tag{T7}$$

$$\hat{u}_n = \kappa_n u_n, \quad \kappa_n = \frac{1}{2}(1 - i)[i^n + i(-i)^n], \quad \hat{t} = -t. \tag{T8}$$

Here transformation (T1) is linear, while transformations (T2)–(T4) and (T5), (T6) with $a = 0$ are linearizable. Transformations (T5) and (T6) with $a \neq 0$ are of Miura type. Transformations (T7) and (T8) are invertible and non-autonomous. The function κ_n appearing in (T8) is four-periodic, i.e. $\kappa_{n+4} = \kappa_n$ for all n . It can be defined by the following initial conditions:

$$\kappa_0 = \kappa_1 = 1, \quad \kappa_2 = \kappa_3 = -1 \tag{63}$$

and satisfy the relations:

$$\kappa_{n+2} = -\kappa_n, \quad \kappa_n^2 = 1. \tag{64}$$

4. Generalized symmetries

There is no need to write down here the generalized symmetries (48) for equations (E1)–(E14) explicitly, as symmetries of key equations are known, while the symmetries for the other equations can be constructed by the transformations (T1)–(T8).

In order to construct generalized symmetries of the form (48), we need symmetries of equations (3), (53), (59) and (61) which are given explicitly in [20, section 4]. Originally, the generalized symmetry of (3) has been presented in [44], of (59) with $a = 1$ in [33], of (59) with $a = 0$ in [44] and of (61) in [1]. The generalized symmetry of (E13) can be found in

[15, 16]. Symmetries of (E3) and (E4) belong to the hierarchies of the Volterra and modified Volterra equations which are well-known, see [43].

The generalized symmetries for (E6)–(E8) and (E12) can be constructed easily with the help of the invertible transformations (T7) and (T8). These transformations are non-autonomous, but in this case they allow us to construct autonomous symmetries of the form (48).

The generalized symmetries for the remaining equations of Lists 1–4, namely (E1), (E2), (E5), (E9)–(E11), (E14), can be constructed by using the non-invertible transformations (T1)–(T4), where (T2) is a particular case of (T3). For the sake of clarity we present here the construction scheme for the generalized symmetries obtained by the transformations (T1)–(T4), see more details in [19, 20].

Let us first consider the case when an equation A is transformed into an equation B by transformation (T1): $A \xrightarrow{(T1)} B$ If B has a symmetry

$$\hat{u}_{0,\tau} = \hat{G}(\hat{u}_4, \hat{u}_3, \hat{u}_2, \dots, \hat{u}_{-4}), \tag{65}$$

then we look for a symmetry of the form (6) for A . As $\hat{u}_{0,\tau} = (T - 1)u_{0,\tau}$, we should represent the function \hat{G} in the form

$$\hat{G} = (T - 1)H(\hat{u}_3, \hat{u}_2, \dots, \hat{u}_{-4}) \tag{66}$$

and then we immediately get (6):

$$u_{0,\tau} = G = H|_{\hat{u}_k = u_{k+1} - u_k + a}. \tag{67}$$

It is evident that G contains an arbitrary constant of integration, as it is obtained by solving a first order difference equation.

Let us now consider the case: $A \xrightarrow{(T4)} B$ As $(\log \hat{u}_0)_\tau = (T + T^{-1})(\log u_0)_\tau$, then for (65) we should get the representation

$$\hat{G}/\hat{u}_0 = (T + T^{-1})H(\hat{u}_3, \hat{u}_2, \dots, \hat{u}_{-3}), \tag{68}$$

and consequently (6) is given by:

$$u_{0,\tau} = G = u_0 H|_{\hat{u}_k = u_{k+1} u_{k-1}}. \tag{69}$$

No constant of integration arises in this case.

In the case $A \xrightarrow{(T3)} B$ we have: $(\log(\hat{u}_0 + a))_\tau = (T + 1)(\log u_0)_\tau$. Equation (65) turns out to be:

$$\hat{G}/(\hat{u}_0 + a) = (T + 1)H(\hat{u}_3, \hat{u}_2, \dots, \hat{u}_{-4}), \tag{70}$$

and (6) is given by:

$$u_{0,\tau} = G = u_0 H|_{\hat{u}_k = u_{k+1} u_k - a}. \tag{71}$$

No constant of integration arises here.

Now we consider the example: (E1) $\xrightarrow{(T2)}$ (53) The symmetry for (53) is:

$$\hat{u}_{0,\tau} = \hat{u}_0[\hat{u}_2(\hat{u}_4 + \hat{u}_2 + \hat{u}_0) - \hat{u}_{-2}(\hat{u}_0 + \hat{u}_{-2} + \hat{u}_{-4})],$$

see [20]. The transformation (T2) is (T3) with $a = 0$, and we need to get the representation (70). As

$$\hat{G}/\hat{u}_0 = (T^2 - T^{-2})\hat{u}_0(\hat{u}_2 + \hat{u}_0 + \hat{u}_{-2})$$

and

$$T^2 - T^{-2} = (T + 1)(T - 1)(1 + T^{-2}),$$

the generalized symmetry for (E1) is given by:

$$u_{0,\tau} = u_0(T - 1)(1 + T^{-2})u_1u_0(u_3u_2 + u_1u_0 + u_{-1}u_{-2}).$$

5. Conclusion

Here we have finished a generalized symmetry classification started in our previous article [20]. The resulting list contains 14 equations, some of which seem to be new. We have found non-autonomous or non-point transformations relating most of the resulting equations among themselves.

Using the obtained five-point integrable equation (5), we can construct integrable examples of partial difference equations, defined on a square lattice. Their construction scheme is discussed in [17] and [20, appendix B].

Connections between different equations obtained in this paper are simpler than in the previous article [20]. However, a transformation can relate two five-point equation (5), but this connection may collapse for the corresponding PΔEs. So it may happen that for one of the five-point equations the corresponding PΔE exists, while for another the corresponding PΔE becomes nonlocal. Therefore the problem of the construction of PΔEs remains nontrivial.

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