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Letter

Non-invertible transformations of differential–difference equations

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Abstract

We discuss aspects of the theory of non-invertible transformations of differential–difference equations and, in particular, the notion of Miura type transformation. We introduce the concept of non-Miura type linearizable transformation and we present techniques that allow one to construct simple linearizable transformations and might help one to solve classification problems. This theory is illustrated by the example of a new integrable differential–difference equation depending on five lattice points, interesting from the viewpoint of the non-invertible transformation, which relate it to an Itoh–Narita–Bogoyavlensky equation.

Keywords: differential–difference equations, transformations, integrable equations, linearizable equations

1. Introduction

The generalized symmetry method uses the existence of generalized symmetries as an integrability criterion and allows one to classify certain integrable equations. Using this method, the classification problem has been solved for some important classes of Partial Differential equations (PDEs) [21, 22], of differential–difference equations [8, 37], and of Partial Difference equations (PΔEs) [14, 20]. Classification is usually carried out in two steps: at first one finds all integrable equations of a certain class up to (usually point) invertible

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transformations, then one searches for non-invertible transformations that relate different resulting equations. For this reason, a theory of non-invertible transformations is necessary.

This is not the only integrability criterion introduced to produce integrable partial difference equations. Using the Compatibility Around the Cube technique introduced in [9, 26, 27], Adler *et al* [5] obtained a class of integrable equations on a quad graph. More results on this line of research can be found in [6, 12].

Let us consider autonomous differential–difference equations of the form:

$$\dot{u}_n = f(u_{n+k}, u_{n+k-1}, \ldots, u_{n+m}), \quad k > m, \tag{1}$$

where $n \in \mathbb{Z}$ is a discrete variable, $u_n = u_n(t)$ is the unknown function, $\dot{u}_n$ is its derivative with respect to the continuous variable t. In (1) we can find integrable equations of Volterra type, corresponding to the case $k = -m = 1$. They have been well-studied and a complete list of such equations has been obtained—see e.g. the review article [37]. In other cases only some integrable examples are known [1, 7, 10, 11, 14, 23, 28].

Let us consider the existence of explicit in one direction non-invertible transformations of the form:

$$v_n = \phi(u_{n+q}, u_{n+q-1}, \ldots, u_{n+s}), \ \ +\infty > q > s > -\infty, \tag{2}$$

which relate two equations of the form (1). An example of such a transformation is provided by the following well-known relation [32]

$$v_n = (1 + u_n)(1 - u_{n+1}), \tag{3}$$

which transforms the modified Volterra equation

$$\dot{u}_n = (1 - u_n^2)(u_{n+1} - u_{n-1}) \tag{4}$$

into the Volterra one

$$\dot{v}_n = v_n(v_{n+1} - v_{n-1}). \tag{5}$$

This is a discrete analogue of the Miura transformation [24]

$$v = u_x - u^2, \tag{6}$$

which relates the Kortevege-de Vries (KdV) equation to the modified KdV one:

$$v_t = v_{xxx} + 6vv_x, \quad u_t = u_{xxx} - 6u^2u_x,$$

where the indices t and x denote t- and x-derivatives. The Miura transformation (6) is locally non-invertible.

Differential substitutions in the class $v = \phi(u, u_x)$ of Miura type for evolutionary scalar PDEs have been classified by Startsev [30] and are, up to point transformations

$$v = u_x, \qquad v = u_x + u^2, \qquad v = u_x + e^u, \qquad v = u_x + e^u + e^{-u}, \tag{7}$$

see also [31, 35]. The first of the differential substitutions presented by Startsev in (7) is in effect a potentiation, locally non-invertible, but whose inverse is given by an integration. Transformations involving potentiations and point transformations can be very involved and sometimes difficult to distinguish from Miura transformations.

The notion of Miura type transformation on the lattice is more complicated, not yet well understood, and it is difficult to decide whether a given transformation is of Miura type. Sometimes in the literature complicated transformations (2) are called of Miura type even if their inversion requires just discrete potentiations. On the other hand, there are simple non-invertible transformations that look of Miura type.

To be able to define a Miura type transformation on the lattice we introduce and discuss the rather wide concept of *linearizable transformation* not of Miura type. We also present some techniques to construct simple linearizable transformations which help us to solve the problem of recognizing Miura type transformations.

In this paper we consider an example which belongs to the class (1) with $k = -m = 2$,

$$\dot{u}_n = f(u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}). \tag{8}$$

Equations of this form (8) are currently relevant, as their Bäcklund transformations are completely discrete partial difference equations [1, 14, 17, 18, 23, 28] whose classification is very difficult to perform. Study of equations (8) is in progress, see e.g. [2–4, 13].

Let us consider here the equation

$$\dot{u}_n = (u_{n+1} - u_n)(u_n - u_{n-1})\left(\frac{u_{n+2}}{u_{n+1}} - \frac{u_{n-2}}{u_{n-1}}\right). \tag{9}$$

It has been found by generalized symmetry classification of some particular cases of (8). Currently this classification is in progress, see [13]. It turns out that (9) is transformed by

$$v_n = u_{n+1} + u_{n-1} - u_n - u_{n+1}u_{n-1}/u_n \tag{10}$$

into the equation

$$\dot{v}_n = v_n(v_{n+2} + v_{n+1} - v_{n-1} - v_{n-2}), \tag{11}$$

a well-known integrable equation of the Itoh–Narita–Bogoyavlensky class (8) [10, 16, 25].

In section 2 we will present the notions of Miura type and linearizable transformations and present some techniques to construct simple linearizable transformations. Then in section 3 we will apply our results to the case of (10), showing that it is linearizable and, therefore, not of Miura type. Section 4 is devoted to some concluding remarks.

2. Theory

In this section we discuss the notions of Miura type and linearizable transformations and present some techniques necessary to the construction of simple linearizable transformations.

2.1. Miura type and linearizable transformations

The Miura transformation (6) is a Riccati equation if we consider (6) as an equation for the unknown function u with v a given function. As is known, the Riccati equation with x-dependent coefficients cannot be solved by quadrature, i.e. by a simple integration. Inversion of the discrete Miura transformation (3) is also equivalent to solving the discrete Riccati equation [15]. From our point of view, Miura type transformations must be of the same type, i.e. their inversion is somehow reduced to solving a Riccati equation with x-dependent coefficients and cannot be done by a simple integration.

On the other hand, in case of KdV type partial differential equations, we find many transformations of the form

$$v = \phi(u, u_x, u_{xx}, \ldots), \tag{12}$$

which are a superposition of point transformations $v = \psi(u)$ and a potentiation $v = u_x$. In this case finding u in (12) is reduced to integrations.

In the case of Volterra type equations (1) with $k = -m = 1$, many transformations of the form of (2) are superpositions [37] of linear transformations of the form

$$v_n = u_{n+1} - u_n, \qquad v_n = u_{n+1} + u_n \tag{13}$$

which are solved by a summation, a *discrete potentiation* and point transformations as $v_n = \psi(u_n)$. Transformations of the form

$$v_n = u_{n+1}u_n, \qquad v_n = u_{n+1}/u_n \tag{14}$$

are also obviously transformed into equations of the form (13), as

$$v_n = (\exp \circ (T \pm 1) \circ \log) u_n,$$

where T is the shift operator $Th_n = h_{n+1}$. So (14) are solved by a summation and point transformations. Thus, in these cases, finding the unknown function u_n is reduced to solving a number of linear equations with constant coefficients, i.e. it contains discrete potentiations and point transformations.

Let us pass to the general case of (1) and the non-invertible transformations (2). We see that such transformations are explicit in one direction. If an equation A is transformed into B by a transformation (2), we will say that this transformation has the *direction* from A to B and we will write $A \to B$ as in this direction it is explicit.

In the general case (1) let us consider the most general form of *linear transformations*

$$v_n = \nu_k u_{n+k} + \nu_{k-1}u_{n+k-1} + \ldots + \nu_m u_{n+m} + \nu, \quad k > m, \tag{15}$$

with constant coefficients. Then we can introduce the following definition:

Definition 2.1. A transformation of the form (2) is called *linearizable* if it can be represented as a superposition of linear transformations (15) and point transformations $v_n = \psi(u_n)$. In this superposition we allow linear transformations acting in different directions.

The linearizable transformation so defined is decomposed into linear transformations (up to point ones) and thus *it is not of Miura type*. In an example below we will show in (51) that a decomposition of the transformation $A \to B$ of the form (2) is possible

$$A \leftarrow C \to D \to B,$$

and it consists of transformations acting in different directions.

There are two other representations for linearizable transformations that may sometimes be useful. For complex constants and functions entering in the transformations we can represent (15) as:

$$v_n - \nu = \nu_k (T - \eta_1)(T - \eta_2) \ldots (T - \eta_{k-m+1}) T^m u_n.$$

So, any linearizable transformation is a superposition of elementary transformations of the form:

$$v_n = \psi(u_n), \quad v_n = (T - \eta)u_n, \quad v_n = T^m u_n. \tag{16}$$

The second of the transformations presented in (16) can be simplified further as

$$v_n = (T - \eta)u_n = \eta^{n+1}(T - 1)\eta^{-n}u_n,$$

i.e. it can be reduced to a superposition of transformations of the form:

$$v_n = \eta u_n, \quad v_n = \alpha^n u_n, \quad v_n = (T - 1)u_n.$$

In conclusion any linearizable transformation is reduced, up to a shift, to a combination of autonomous point transformations, elementary non-autonomous point transformations and only one concrete non-invertible linear transformation solvable by discrete potentiation.

Let us consider as an example the equation

$$\dot{u}_n = (u_n^2 + au_n)(u_{n+2}u_{n+1} - u_{n-1}u_{n-2}), \tag{17}$$

with a constant. Equation (17) is transformed into (11) by the transformation

$$v_n = (u_n + a)u_{n+1}u_{n+2}. \tag{18}$$

When $a = 0$ the transformation (18) is linearizable, as it is analogous to (14):

$$v_n = (\exp \circ (T^2 + T + 1) \circ \log)u_n.$$

In the case when $a = 0$ (17) is a well-known modification of (11), see e.g. [11].

The ambiguity in the use of the wording *Miura transformation* is present in many researches in this field. For example in [23] the transformation

$$v_n = \frac{1}{u_{n+1}u_nu_{n-1} - 1} \tag{19}$$

is called a Miura tranformation. However, up to a point transformation and a shift, (19) coincides with (18) with $a = 0$.

In case $a = 1$, (17) and the transformation (18) are considered in [7, 23]. It is shown in [7] that the transformation (18) is an analogue, from the viewpoint of the L– A pair, to (3) and therefore it is a transformation of Miura type.

There is the following analogy between the discrete transformations (3), (18) and the Miura transformation (6). It is known that the Riccati equation (6) can be rewritten by using the transformation $u = -z_x/z$ as a second order homogeneous linear equation with a non-constant coefficient v:

$$z_{xx} + vz = 0. \tag{20}$$

In case of (3) we introduce z_n by the definition $u_n = 1 - z_n/z_{n+1}$ and obtain a discrete linear equation of the same type as (20):

$$v_nz_{n+2} - 2z_{n+1} + z_n = 0.$$

Equation (18) with $a = 1$ can be written in terms of z_n, by the definition $u_n = z_n/z_{n+1}$. Equation (18) with $a = 1$ then takes the form of a third order linear discrete equation:

$$v_nz_{n+3} - z_{n+1} - z_n = 0.$$

So the transformation (18) with $a = 1$ is not linearizable, in the sense of definition 2.1.

As far we know, the equations produced by linearizable transformations and Miura type transformations do not have essentially different properties. There is, however, a difference in the ease of the construction of solutions by using these different types of transformations. In case of a linearizable transformation, one get solutions of a modified equation by solving a number of linear problems while in case of a Miura type transformation we have to solve a Riccati equation.

We do not know of any algorithmic way to establish the nature of a given non-invertible lattice transformation, i.e. to distinguish linearizable and Miura type transformations. We can only assert that a transformation is not of Miura type if we manage to show that it is linearizable.

2.2. Linearizable transformations, point symmetries and conservation laws

In this subsection we will explain how to construct simple autonomous linearizable—according to definition 2.1—transformations of the form

$$v_n = \phi(u_{n+1}, u_n) \tag{21}$$

by using point symmetries and conservation laws. This result may be useful in solving classification problems and will allow us to analyze in detail in the case of example (9) the transformation (10). In the case of KdV type partial differential equations, such a theory exists, see e.g. [29]. For both partial differential and differential–difference equations a different theory for the construction of non-invertible transformations, in particular of Miura type, has been developed in [34–36].

For an equation of the form (1) we can describe all non-autonomous point symmetries of the form

$$\partial_\tau u_n = \sigma_n(u_n), \quad \sigma_n(u_n) \neq 0, \ \forall n \tag{22}$$

by solving the determining equation:

$$\sigma_n'(u_n) f = \sum_{j=m}^{k} \frac{\partial f}{\partial u_{n+j}} \sigma_{n+j}(u_{n+j}), \tag{23}$$

where by a $'$ we mean the derivative of the function with respect to its argument. If we introduce the point transformation:

$$\hat{u}_n = \eta_n(u_n), \quad \eta_n'(u_n) = \frac{1}{\sigma_n(u_n)},$$

then the point symmetry (22) turns into

$$\partial_\tau \hat{u}_n = 1 \tag{24}$$

and (1) into an equation of the form:

$$\partial_t \hat{u}_n = \hat{f}_n(\hat{u}_{n+k}, \hat{u}_{n+k-1}, \dots, \hat{u}_{n+m}). \tag{25}$$

As (25) admits the symmetry (24) the determining equation (23) reduces to

$$\sum_{j=m}^{k} \frac{\partial \hat{f}_n}{\partial \hat{u}_{n+j}} = 0. \tag{26}$$

This shows that

$$\hat{f}_n = g_n(\hat{u}_{n+k} - \hat{u}_{n+k-1}, \hat{u}_{n+k-1} - \hat{u}_{n+k-2}, \dots, \hat{u}_{n+m+1} - \hat{u}_{n+m}),$$

as (26) is a simple first order linear PDE for the function $\hat{f}_n$ which can be solved on the characteristics. So we can use the non-invertible linearizable transformation $v_n = \hat{u}_{n+1} - \hat{u}_n$, to reduce (1) to the equation

$$\dot{v}_n = (T - 1) g_n(v_{n+k-1}, v_{n+k-2}, \dots, v_{n+m}). \tag{27}$$

We can summarize the previous results in the following theorem:

Theorem 2.1. *If (1) has a point symmetry (22) with $\sigma_n(u_n) \neq 0$ for all n, then it admits the following non-invertible and linearizable transformation*

$$v_n = (T - 1)\eta_n(u_n), \quad \eta_n'(u_n) = \frac{1}{\sigma_n(u_n)} \tag{28}$$

which allows us to construct a modified equation (27).

We are mainly interested in autonomous linearizable transformations of the form (2), therefore, primarily in autonomous point symmetries (22). However, sometimes a non-

Table 1. Examples of point symmetries (22), corresponding autonomous linearizable transformations (21) and point transformations (29).

$\sigma_n(u_n)$	1	$(-1)^n$	u_n	$(-1)^n u_n$
$\phi(u_{n+1}, u_n)$	$u_{n+1} - u_n$	$u_{n+1} + u_n$	u_{n+1}/u_n	$u_{n+1}u_n$
$\psi_n(v_n)$	v_n	$(-1)^{n+1}v_n$	$\exp v_n$	$\exp[(-1)^{n+1}v_n]$

autonomous point symmetry of the form (22) may also lead to an autonomous result. This is the case when there exists a non-autonomous point transformation

$$\tilde{v}_n = \psi_n(v_n) \tag{29}$$

which turns the transformation (28) into an autonomous one. In this case the resulting equation for $\tilde{v}_n$ turns out also to be autonomous.

Indeed, the autonomous equation (1) is transformed by the autonomous transformation (21) into (27), i.e.

$$\dot{v}_n = G_n(v_{n+k}, v_{n+k-1}, \ldots, v_{n+m}). \tag{30}$$

Differentiating (21) with respect to the continuous variable t we get

$$G_n(\phi(u_{n+k+1}, u_{n+k}), \phi(u_{n+k}, u_{n+k-1}), \ldots, \phi(u_{n+m+1}, u_{n+m})) \\ = \frac{\partial\phi(u_{n+1}, u_n)}{\partial u_n} f(u_{n+k}, \ldots, u_{n+m}) + \frac{\partial\phi(u_{n+1}, u_n)}{\partial u_{n+1}} f(u_{n+k+1}, \ldots, u_{n+m+1}).$$

This shows that G_n cannot have any explicit dependence on n, i.e. the equation for v_n is autonomous.

Even if (27) is autonomous we can also simplify the result by applying (29) with an autonomous function ψ.

Let us write down in table 1 the four most typical examples of point symmetries, two of which are not autonomous, together with corresponding autonomous transformations (21).

We can construct simple autonomous linearizable transformations also starting from conservation laws. Let us now show how. From (1) we can look for conservation laws of the form

$$\partial_t \rho_n(u_n) = (T-1)h_n, \quad \rho_n'(u_n) \neq 0, \ \ \forall n, \tag{31}$$

where $h_n = h_n(u_{n+k-1}, \ldots, u_{n+m})$ with $k > m$. The conserved density ρ_n is found using the criterion introduced in [19]. A function $\rho_n(u_n)$ is a conserved density of (1) iff

$$\frac{\delta(\partial_t \rho_n(u_n))}{\delta u_n} = \frac{\delta(\rho_n'(u_n) f)}{\delta u_n} \equiv \sum_{j=m}^{k} T^{-j} \frac{\partial(\rho_n'(u_n) f)}{\partial u_{n+j}} = 0. \tag{32}$$

Let us introduce a variable w_n, so that $\rho_n(u_n) = w_{n+1} - w_n$. It follows from (31) that

$$\dot{w}_n = h_n(\rho_{n+k-1}^{-1}(w_{n+k} - w_{n+k-1}), \ldots, \rho_{n+m}^{-1}(w_{n+m+1} - w_{n+m})) \tag{33}$$

and we can enunciate the following theorem:

Theorem 2.2. *If (1) has a conservation law (31) with $\rho_n'(u_n) \neq 0$ for all n, then it admits the following non-autonomous non-invertible linearizable transformation:*

$$u_n = \rho_n^{-1}(w_{n+1} - w_n), \tag{34}$$

which allows us to construct a modified equation (33).

As we are looking for linearizable transformations of form (2) then we need

$$u_n = \psi(w_{n+1}, w_n), \tag{35}$$

i.e. we are concerned primarily with autonomous conservation laws. However, sometimes a non-autonomous conservation law may also lead to an autonomous result. This is the case when a non-autonomous point transformation

$$\tilde{w}_n = \mu_n(w_n) \tag{36}$$

makes transformation (34) autonomous, i.e. of form (35). In this case introducing (36) into (33) and renaming $\tilde{w}_n$ as w_n to simplify the notation, we get:

$$\dot{w}_n = H_n(w_{n+k}, w_{n+k-1}, \dots, w_{n+m}). \tag{37}$$

In this case we can only show that (37) for w_n is close to an autonomous one as we have:

$$\partial_t w_n = \tilde{H}(w_{n+k}, \dots, w_{n+m}) + Q_n(w_n). \tag{38}$$

In all cases we have considered up to now we can pass to the autonomous equation

$$\partial_t w_n = \tilde{H}(w_{n+k}, \dots, w_{n+m}) \tag{39}$$

as $\partial_\tau w_n = Q_n(w_n)$ is a point symmetry of (38), and (35) transforms both (38) and (39) into the same equation (1).

Let us see the passage from (37) to (38) in more detail: let us assume that (37) is transformed into (1) by the transformation (35). Then we can differentiate (35) with respect to the continuous variable t and we get:

$$\begin{aligned} &f(\psi(w_{n+k+1}, w_{n+k}), \psi(w_{n+k}, w_{n+k-1}), \dots, \psi(w_{n+m+1}, w_{n+m})) \\ &\quad = \frac{\partial\psi(w_{n+1}, w_n)}{\partial w_n} H_n + \frac{\partial\psi(w_{n+1}, w_n)}{\partial w_{n+1}} H_{n+1}. \end{aligned} \tag{40}$$

If $k > 0$ and $m < 0$ then we can differentiate with respect to w_{n+k+1} and w_{n+m} and we get:

$$\frac{\partial f}{\partial u_{n+k}} T^k \frac{\partial\psi}{\partial w_{n+1}} = \frac{\partial\psi}{\partial w_{n+1}} \frac{\partial H_{n+1}}{\partial w_{n+k+1}}, \tag{41}$$

$$\frac{\partial f}{\partial u_{n+m}} T^m \frac{\partial\psi}{\partial w_n} = \frac{\partial\psi}{\partial w_n} \frac{\partial H_n}{\partial w_{n+m}}. \tag{42}$$

From (41), (42) we deduce that $\frac{\partial H_n}{\partial w_{n+k}}$ and $\frac{\partial H_n}{\partial w_{n+m}}$ are autonomous and consequently

$$H_n = \tilde{H}(w_{n+k}, \dots, w_{n+m}) + \hat{H}_n(w_{n+k-1}, \dots, w_{n+m+1}). \tag{43}$$

Substituting (43) into (40) we get an expression indicating that $\frac{\partial\psi}{\partial w_n}\hat{H}_n + \frac{\partial\psi}{\partial w_{n+1}}\hat{H}_{n+1}$ is also autonomous and we can repeat the procedure. At the end we get

$$H_n = \tilde{H}(w_{n+k}, \dots, w_{n+m}) + Q_n(w_n), \tag{44}$$

where Q_n is such that $\frac{\partial\psi}{\partial w_n} Q_n + \frac{\partial\psi}{\partial w_{n+1}} Q_{n+1} = 0$. This shows that on the rhs of (40) H_n can be replaced by $\tilde{H}$. Consequently (37) and $\dot{w}_n = \tilde{H}$ are transformed by the same transformation (35) into the same equation (1).

In the particular case when

$$\rho_n(u_n) = (-1)^n p(u_n) \tag{45}$$

Table 2. Examples of conserved densities of (31) and corresponding autonomous linearizable transformations (35), up to autonomous point transformations of w_n.

$\rho_n(u_n)$	u_n	$(-1)^n u_n$	$\log u_n$	$(-1)^n \log u_n$
$\psi(w_{n+1}, w_n)$	$w_{n+1} - w_n$	$w_{n+1} + w_n$	w_{n+1}/w_n	$w_{n+1}w_n$

this procedure is more obvious. In this case we have the representation

$$\partial_t p(u_n) = (T+1)\hat{h}_n, \tag{46}$$

where $\hat{h}_n = (-1)^{n+1}h_n$. As the left-hand side of (46) is autonomous, we can easily prove that

$$\hat{h}_n = \tilde{h}(u_{n+k-1}, \dots, u_{n+m}) + (-1)^n c, \quad c \in \mathbb{C},$$

and $(-1)^n c$ is in the null space of $T+1$. So, there is a representation (46) with the autonomous function $\tilde{h}$ instead of $\hat{h}_n$. In this case the autonomous linearizable transformation and the new equation can be defined as follows:

$$p(u_n) = w_{n+1} + w_n, \quad \partial_t w_n = \tilde{h}.$$

Let us write down in table 2 the four most typical examples of conserved densities together with the corresponding autonomous transformations (35). Two of these conserved densities are non-autonomous and have the special form (45).

For a given conserved density $\rho_n(u_n)$, if the representation (31) contains instead of $T-1$ a higher order linear difference operator with constant coefficients, we can get more linearizable transformations. For instance, for the Volterra equation given in (5), one has the following conservation law

$$\partial_t \log v_n = (T^2-1)v_{n-1} = (T-1)(T+1)v_{n-1}.$$

In this case we can construct three linearizable transformations in a quite similar way:

$$v_n = \frac{u_{n+1}}{u_n}, \quad v_n = u_{n+1}u_n, \quad v_n = \frac{u_{n+2}}{u_n}, \tag{47}$$

with the corresponding modified Volterra equations. For instance, from the last of the transformations in (47) we get the following modified Volterra equation

$$\dot{u}_n = \frac{u_{n+1}u_n}{u_{n-1}}.$$

A further example is provided by (11). Considering the conserved density $\log v_n$ we get:

$$\partial_t \log v_n = (T^4 + T^3 - T - 1)v_{n-2} = (T-1)(T+1)(T-c_1)(T-c_2)v_{n-2},$$

where $c_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}\mathrm{i}}{2}$. Here we have more possibilities. If we start from the operator $T-c_1$ instead of $T-1$, we can construct the transformation $v_n = u_{n+1}u_n^{-c_1}$. Another possibility is provided by the operator

$$(T-c_1)(T-c_2) = T^2 + T + 1$$

which leads to the known transformation (18) with $a=0$. It is clear that this transformation is the superposition of the following ones:

$$v_n = z_{n+1}z_n^{-c_1}, \quad z_n = u_{n+1}u_n^{-c_2}.$$

As a final example we consider (17). In this case one has

$$\partial_t \int \frac{\mathrm{d}u_n}{u_n^2 + au_n} = (T^3 - 1)u_{n-1}u_{n-2}. \tag{48}$$

Equation (48) provides a number of possibilities, too.

3. Discussion of example (9)

A transformation of the form (2) transforming (9) into (11) can be found by a straightforward investigation [37]. We can fix $s = -1$ and $q = 1$ in (2) and then we find the explicit transform (10).

Let us try to find (10) as a chain of linearizable transformations relating (9) and (11) by applying the theory presented in section 2. Let us start from (9) and find a conservation law of density $\rho_n(u_n) = (-1)^n \log u_n$. According to table 2 this gives the transformation $u_n = w_{n+1}w_n$ which relates (9) to

$$\dot{w}_n = (w_{n+2} - w_n)(w_{n+1} - w_{n-1})(w_n - w_{n-2}). \tag{49}$$

We can now apply the obvious transformation $z_n = w_{n+1} - w_{n-1}$. This is the superposition of the first two transformations of table 1, which are provided by the point symmetry

$$\partial_\tau w_n = \alpha + \beta(-1)^n.$$

The resulting equation reads:

$$\dot{z}_n = z_n(z_{n+2}z_{n+1} - z_{n-1}z_{n-2}). \tag{50}$$

This is the well-known modification of (11). The transformation of (50) into (11) is $v_n = z_{n+1}z_n$ and it corresponds to the symmetry $\partial_\tau z_n = (-1)^n z_n$.

We can construct the same chain of linearizable transformations, moving in the opposite direction, i.e. starting from (11). We are led to the following picture:

$$\begin{array}{ccc} C\colon(49) & \xrightarrow{\eta_2\colon\, z_n = w_{n+1} - w_{n-1}} & D\colon(50) \\ \Big\downarrow{\scriptstyle \eta_1\colon\, u_n = w_{n+1}w_n} & & \Big\downarrow{\scriptstyle \eta_3\colon\, v_n = z_{n+1}z_n} \\ A\colon(9) & \xrightarrow{\eta\colon\,(10)} & B\colon(11) \end{array} \tag{51}$$

It turns out that the superposition of the three linearizable transformations shown in the picture can be rewritten as (10) which is of form (2): $\eta = \eta_3 \circ \eta_2 \circ \eta_1^{-1}$.

We see that in order to find the unknown function u_n in the non-autonomous nonlinear discrete equation (10), one has to find w_n by solving two linear problems $v_n = z_{n+1}z_n$ and $z_n = w_{n+1} - w_{n-1}$ and then to construct u_n using the explicit formula $u_n = w_{n+1}w_n$.

4. Conclusions

In this article we discuss the transformations necessary to classify differential–difference equations. By a definition 2.1 we introduce what we call a *linearizable transformation*, e.g. a transformation, linear in one direction, which in the other direction corresponds to one of the many kinds of discrete potentiations. This is not a transformation of Miura type as its inversion is also simple, being usually of discrete potentiation type. Using this definition we are able to introduce two theorems, 2.1 and 2.2, which allow us to construct transformations to a new differential–difference equation, if the first has point symmetries or conservation laws. These two theorems are used in section 3 to show that (9) and (11) are related by a linearizable transformation.

This result is the starting point for a subsequent work [13] in progress on the classification of a class of differential–difference equations of form (1) with $k = 2$ and $m = -2$ which extends the classification of Volterra type equations, corresponding to $k = 1$ and $m = -1$, carried out with great success in [33].

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