Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes
Volume **9**, 1996

Local Master Symmetries of Differential-Difference Equations

Igor Cherdantsev and Ravil Yamilov

ABSTRACT. It is demonstrated that in case of integrable differential-difference equations like the well-known Volterra equation and Toda model (unlike the Korteweg-de Vries and nonlinear Schrödinger equations) there are many instances in which local evolution master symmetries can be found. These master symmetries generate not only higher symmetries but also conservation laws and additional Hamiltonian operators. Also, they provide new local evolution chains that depend explicitly on time and the space variable and are, in a specific sense, integrable. We construct examples of master symmetries, using Miura-type transformations and local conservation laws of the zeroth order.

1. Introduction

The concept of the master symmetry was introduced by Fokas, Fuchssteiner [**7**]. At present, there is a well developed theory for master symmetries of "continuous" integrable equations like the Kadomtsev-Petviashvili, Korteweg-de Vries, and nonlinear Schrödinger equations (see, e.g., the review article [**6**]). It is not surprising that this theory turns out to be more satisfactory in the case of equations in one space and one time dimension. In this case, many algebraic properties of integrable equations can be described in terms of differential operators and functions of a finite number of dynamical variables; there are local symmetries, conserved quantities, and (in many instances) Hamiltonian operators (there exists a close and simple connection between master symmetries and recursion operators [**6**]). It is natural to expect that the theory of master symmetries will be even more satisfactory and elementary if not only the equations, higher symmetries, and conserved quantities, but also the master symmetries, are local. We have discovered that there are many instances in which integrable $1+1$ dimensional differential-difference equations, analogous to the Toda and Volterra chains, possess local evolution master symmetries, and we discuss in the present paper just such equations.

1991 *Mathematics Subject Classification*. Primary 22E70; Secondary 35A30.

The authors are grateful to V. E. Adler, D. Levi, and P. M. Santini for useful discussions.

This work has been carried out with the financial support of the International Science Foundation under grant # MLY000 and the Russian Foundation for Fundamental Researches under grant # 93 011 16088.

This is the final form of the paper.

Let us consider differential-difference equations (chains) of the form

$$\partial_t(U_n) = F_n, \quad F_n = F(U_{n+1}, U_n, U_{n-1}), \tag{1}$$

where $U_n = (u_n^1, \dots, u_n^N)^T$ are vector dynamical variables, $F_n = (f_n^1, \dots, f_n^N)^T$ are vector functions, and n is an arbitrary integer. It is easy to see that the Volterra equation

$$\partial_t(u_n) = u_n(u_{n+1} - u_{n-1}) \tag{2}$$

and the Toda model, written in the form

$$\begin{aligned} \partial_t(u_n) &= u_n(v_{n+1} - v_n), \\ \partial_t(v_n) &= u_n - u_{n-1} \end{aligned} \tag{3}$$

are chains of this kind.

We are interested in master symmetries which have the form

$$\partial_\tau(U_n) = G_n, \quad G_n = G(\tau, n, U_{n+1}, U_n, U_{n-1}). \tag{4}$$

The Volterra equation (2) and the Toda model (3) possess such master symmetries, which are given, respectively, by the following chains:

$$\partial_\tau(u_n) = u_n[(\varepsilon + n + 2)u_{n+1} + u_n - (\varepsilon + n - 1)u_{n-1}], \tag{5}$$

$$\begin{aligned} \partial_\tau(u_n) &= u_n[(\varepsilon + 2n + 2)v_{n+1} - (\varepsilon + 2n - 2)v_n], \\ \partial_\tau(v_n) &= (\varepsilon + 2n + 1)u_n - (\varepsilon + 2n - 3)u_{n-1} + v_n^2, \end{aligned} \tag{6}$$

where ε is an arbitrary constant. Eq. (5) has been derived in [**10**] and eq. (6) has been introduced in [**14**] as new examples of integrable chains (see also [**1, 11, 12**]).

If one compares (5), (6) with the master symmetry [**6**]

$$u_\tau = x(u_{xxx} + 6uu_x) + 4(u_{xx} + 2u^2) + 2u_x\partial_x^{-1}(u) \tag{7}$$

of the KdV equation $u_t = u_{xxx} + 6uu_x$, one can see that (5) and (6) do not contains any difference analogue of the integral operator ∂_x^{-1}. This is the reason why (5) and (6) are, unlike (7), local evolution equations.

An equation of the form (4) is called a master symmetry of a chain (1) if it enables one to construct higher symmetries

$$\partial_{t_i}(U_n) = F_n^{(i)}, \quad F_n^{(i)} = F^{(i)}(U_{n+i}, U_{n+i-1}, \dots, U_{n-i}), \qquad i = 1, 2, \dots, \tag{8}$$

of the chain (1) using the following recursive relation:

$$[G_n, F_n^{(i)}] = F_n^{(i+1)}, \quad i = 0, 1, 2, \dots, \tag{9}$$

where $F_n^{(0)} = F_n$. In eq. (9) the commutator of two vector functions is defined, as usual, by the formula: $[G_n, F_n] = \partial_\tau(F_n) - \partial_t(G_n)$. As, by definition, higher symmetries satisfy the condition $[F_n^{(i)}, F_n] = 0$, it follows that a master symmetry cannot be a higher symmetry. It is easy to check that, in the case of the Volterra equation (2),

$$[G_n, F_n] = u_n[u_{n+1}(u_{n+2} + u_{n+1} + u_n) - u_{n-1}(u_n + u_{n-1} + u_{n-2})],$$

and this is the right hand side of the simplest higher symmetry of (2).

In Section 2, it is explained why the chains (5), (6) are master symmetries, and how to construct conservation laws and additional Hamiltonian operators with the help of these master symmetries. In Section 3, we obtain other examples by an approach using Miura-type transformations [**16, 17**]. In Section 4, we discuss briefly the problem of the construction of exact solutions for the new master symmetries.

2. Master symmetries of the Volterra and Toda chains

Local master symmetries enable one to construct not only higher symmetries but also local conservation laws, i.e., relations of the form $\partial_t(p_n) = (D-1)(q_n)$, where D is the shift operator, p_n and q_n are scalar local functions of a finite number of dynamical variables. Two conservation laws (let the second one be $\partial_t(\tilde{p}_n) = (D-1)(\tilde{q}_n)$) are considered to be the same if their conserved densities p_n and $\tilde{p}_n$ are equivalent ($p_n \sim \tilde{p}_n$ if $p_n - \tilde{p}_n \in \mathrm{Im}(D-1)$).

Let us consider the general case when p_n is a common conserved density for a given chain (1) and its higher symmetries (8): $\partial_{t_i}(p_n) = (D-1)(q_n^{(i)})$, where $t_0 = t$. Formula (9) implies the following formula for the time derivatives:

$$[\partial_\tau, \partial_{t_i}] = \partial_{t_{i+1}}. \tag{10}$$

Differentiating the conservation laws w.r.t. τ and using (10), we are led to local conservation laws $\partial_{t_i}\partial_\tau(p_n) = (D-1)\big[\partial_\tau(q_n^{(i)}) - q_n^{(i+1)}\big]$ with a new common density $\partial_\tau(p_n)$. It is easy to see that, in the case of the Volterra equation (2), we have $\partial_\tau(u_n) \sim u_n^2 + 2u_{n+1}u_n$, where u_n and $u_n^2 + 2u_{n+1}u_n$ are conserved densities. Note that such a beautiful formula for the construction of conservation laws does not hold in the case of the master symmetry (7), and we cannot use above explanations if we deal with master symmetries similar to (7).

In the case of the Volterra and Toda chains, we first shall prove that (5), (6) generate conservation laws and then that they are master symmetries. As is known, (2) admits Lax representation:

$$\begin{gathered}\partial_t(L_n) = [B_n, L_n],\\ L_n = u_n^{1/2}D + u_{n-1}^{1/2}D^{-1}, \quad 2B_n = u_{n+1}^{1/2}u_n^{1/2}D^2 - u_{n-1}^{1/2}u_{n-2}^{1/2}D^{-2}.\end{gathered} \tag{11}$$

This representation allows one to obtain conserved densities as follows: $p_n^{(i)} = \mathrm{res}(L_n^{2i})$ (i.e. a coefficient of L_n^{2i} at D^0). For example, $p_n^{(1)} = u_n + u_{n-1} \sim 2u_n$, $p_n^{(2)} \sim 2(u_n^2 + 2u_{n+1}u_n)$. On the other hand, the corresponding master symmetry (5) has the representation

$$\begin{gathered}\partial_\tau(L_n) = [A_n, L_n] + (1/2)L_n^3,\\ 2A_n = (\varepsilon + n + 1)u_{n+1}^{1/2}u_n^{1/2}D^2 - (\varepsilon + n - 1)u_{n-1}^{1/2}u_{n-2}^{1/2}D^{-2}\end{gathered} \tag{12}$$

with the same L_n [**1**]. Since $\partial_\tau(L_n^{2i}) = [A_n, L_n^{2i}] + iL_n^{2i+2}$, we are led to the following relation:

$$\partial_\tau(p_n^{(i)}) \sim ip_n^{(i+1)}. \tag{13}$$

For the Toda chain (3) and its master symmetry (6), operator representations are of the form

$$\begin{gathered}\partial_t(L_n) = [B_n, L_n],\\ L_n = u_n^{1/2}D + v_n + u_{n-1}^{1/2}D^{-1}, \quad 2B_n = u_n^{1/2}D - u_{n-1}^{1/2}D^{-1},\end{gathered} \tag{14}$$

$$\begin{gathered}\partial_\tau(L_n) = [A_n, L_n] + L_n^2,\\ 2A_n = (\varepsilon + 2n)u_n^{1/2}D - (\varepsilon + 2n - 2)u_{n-1}^{1/2}D^{-1},\end{gathered} \tag{15}$$

respectively ((15) can be found, e.g., in [**1**]). In this case, conserved densities are defined by $p_n^{(i)} = \mathrm{res}(L_n^i)$, and using (15), we obtain the formula (13) again.

The master symmetries of the Volterra and Toda chains generate additional local Hamiltonian structures. The Volterra and Toda chains (2), (3) are Hamiltonian, since they can be expressed in the form

(16) $$\partial_t(U_n) = F_n = H_n\left(\frac{\delta h_n}{\delta U_n}\right).$$

Here H_n is a Hamiltonian operator in powers of D with matrix coefficients, and h_n is a Hamiltonian density. The formal variational derivative is defined by

$$\frac{\delta h_n}{\delta U_n} = \sum_k \frac{\partial h_k}{\partial U_n}, \quad \frac{\partial h_k}{\partial U_n} = \left(\frac{\partial h_k}{\partial u_n^1}, \dots, \frac{\partial h_k}{\partial u_n^N}\right)^T.$$

For the Volterra equation (2), for instance, the simplest Hamiltonian operator is $H_n = u_n(D - D^{-1})u_n$, and $h_n = u_n$.

Let us introduce an operator G_n^+ (G_n is the right hand side of a master symmetry (4)):

$$G_n^+(H_n) = H_n(\partial_\tau + G_n^{*T}) - (\partial_\tau - G_n^*)H_n,$$

$$G_n^* = \sum_k \frac{\partial G_n}{\partial U_{n+k}} D^k, \quad G_n^{*T} = \sum_k \left(\frac{\partial G_{n+k}}{\partial U_n}\right)^T D^k.$$

Here $\partial G_m/\partial U_k$ are matrices of partial derivatives $(\partial g_m^i/\partial u_k^j)$, and $(\partial G_m/\partial U_k)^T$ are transposed matrices. There is a remarkable formula for the construction of additional local Hamiltonian operators:

(17) $$H_n^{(i)} = (G_n^+)^i(H_n), \quad i = 1, 2, \dots.$$

For the Volterra equation (2),

$$H_n^{(1)} = u_n[u_{n+1}D^2 + (u_{n+1} + u_n)D - (u_n + u_{n-1})D^{-1} - u_{n-1}D^{-2}]u_n,$$

and the function $h_n^{(1)} = (1/2)\log u_n$ plays the role of the corresponding Hamiltonian density. Since the Volterra equation is bi-Hamiltonian (there are only two local Hamiltonian operators), and only local operators can be obtained by the formula (17), it is natural that $H_n^{(2)} = 0$. As is known, the Toda chain (3) is tri-Hamiltonian [**9**]. The two simplest Hamiltonian operators are of the form

$$H_n = \begin{pmatrix} 0 & u_n(D-1) \\ (1-D^{-1})u_n & 0 \end{pmatrix},$$

$$H_n^{(1)} = 2\begin{pmatrix} u_n(D-D^{-1})u_n & u_n(D-1)v_n \\ v_n(1-D^{-1})u_n & u_nD - D^{-1}u_n \end{pmatrix},$$

and $h_n = u_n + v_n^2/2$, $h_n^{(1)} = v_n/2$ are the corresponding Hamiltonian densities. Formula (17) yields additional Hamiltonian operators again ($H_n^{(3)} = 0$).

In order to prove that (5) is the master symmetry of the Volterra equation (in the case of the Toda chain the proof will be the same), let us introduce functions $\widetilde{F}_n^{(i)} = H_n(g_n^{(i)})$, where $g_n^{(i)} = \delta p_n^{(i)}/\delta u_n$, and $p_n^{(i)}$ are conserved densities for (2) constructed above. Note that $\widetilde{F}_n^{(1)} = 2F_n$, all the other functions $\widetilde{F}_n^{(i)}$ are higher symmetries of (2). The second Hamiltonian structure allows us to obtain another formula $H_n^{(1)}(g_n^{(i)}) = (i/(i+1))\widetilde{F}_n^{(i+1)}$. We also shall use the identity

$(\delta/\delta u_n)\partial_\tau(p_n^{(i)}) = (\partial_\tau + G_n^{*T})(g_n^{(i)})$. Now we can check (using also (13) and (17)) that

$$\begin{aligned} i\widetilde{F}_n^{(i+1)} &= iH_n\left(\frac{\delta p_n^{(i+1)}}{\delta u_n}\right) \\ &= H_n\left(\frac{\delta}{\delta u_n}\right)\partial_\tau(p_n^{(i)}) \\ &= H_n(\partial_\tau + G_n^{*T})(g_n^{(i)}) \\ &= [(\partial_\tau - G_n^*)H_n + H_n^{(1)}](g_n^{(i)}) \\ &= (\partial_\tau - G_n^*)(\widetilde{F}_n^{(i)}) + \frac{i}{i+1}\widetilde{F}_n^{(i+1)}. \end{aligned}$$

Since $(\partial_\tau - G_n^*)(\widetilde{F}_n^{(i)}) = [G_n, \widetilde{F}_n^{(i)}]$, we are led to the formula

$$[G_n, \widetilde{F}_n^{(i)}] = \frac{i^2}{i+1}\widetilde{F}_n^{(i+1)} \tag{18}$$

which shows that the chain (5) is the master symmetry of the Volterra equation (2).

3. Other examples of local master symmetries

An exhaustive list of scalar integrable chains of the form (1) has been obtained by one of the authors in [**15**]. All the chains of this list are reduced to the Volterra equation (2) or the Toda model (3) except for a complicated chain

$$\partial_t(u_n) = \frac{R(u_{n+1}, u_n, u_{n-1})}{u_{n+1} - u_{n-1}},$$

where R is the following polynomial with arbitrary constant coefficients:

$$R(x,y,z) = (\alpha y^2 + 2\beta y + \gamma)xz + (\beta y^2 + \lambda y + \delta)(x+z) + \gamma y^2 + 2\delta y + \varepsilon.$$

This chain is the difference analogue of the Krichever-Novikov equation [**8**]

$$u_t = u_{xxx} - \frac{3}{2}u_x^{-1}[u_{xx}^2 + P(u)]$$

(here $P(u)$ is an arbitrary polynomial of degree $N \le 4$ with constant coefficients). There are several chains in the list which are reduced to the Volterra and Toda chains by discrete Miura-type transformations (we shall call them key equations). In the case of the Volterra equation (2), for example, those Miura transformations have the form $\tilde{u}_n = \varphi(u_n, u_{n+1}, \dots, u_{n+m})$ (see (23) and (28) below). All the other chains are reduced to the key ones by simpler transformations like $\tilde{u}_n = u_{n+1} - u_n$. We can construct local evolution master symmetries for all the key chains and for some of the other ones. We apply a scheme which allows us to obtain new integrable equations, using Miura transformations. This scheme was presented in [**16**] (the case of partial differential equations) and [**17**] (the case of differential-difference ones). Also, we use local conservation laws of the zeroth order. In this section, it is explained how to construct integrable chains together with corresponding local master symmetries, and new examples are cited.

In order to obtain a master symmetry of the modified Volterra equation

$$\partial_t(v_n) = (\lambda - v_n)v_n(v_{n+1} - v_{n-1}), \tag{19}$$

let us consider the system

$$L_n(\psi_n) = \lambda\psi_n, \quad \partial_\tau(\psi_n) = (A_n + \alpha)(\psi_n). \tag{20}$$

where L_n and A_n are the operators appearing in eq. (11) and (12), $\alpha = \alpha(\tau)$, $\lambda = \lambda(\tau)$, and

$$\lambda' = \frac{\lambda^3}{2}. \tag{21}$$

This system is consistent (see (12)). It is easy to check that the function

$$v_n = u_n^{1/2}\frac{\psi_{n+1}}{\psi_n} \tag{22}$$

satisfies the following relationships:

$$u_n = (\lambda - v_{n+1})v_n, \tag{23}$$

$$\partial_\tau(v_n) = (\lambda - v_n)v_n[(\varepsilon + n + 1)v_{n+1} - (\varepsilon + n - 1)v_{n-1}] + \lambda v_n\left(v_n - \frac{\lambda}{2}\right). \tag{24}$$

Eq. (24) is a master symmetry of (19). This equation was first derived in [**2**] as an integrable chain. Transformation (23) is a difference analogue of the well-known Miura transformation $u = v_x - v^2$ [**13**] which reduces the modified KdV equation $v_t = v_{xxx} - 6v^2v_x$ to the KdV equation. It is important for us that (19) and (24) are reduced to (2) and (5) by the same transformation (23). For any solution v_n of (19) or (24), the formula (23) yields a solution u_n of (2) or (5), respectively.

Following [**17**] and using (23), we can obtain one more key scalar integrable equation together with its corresponding master symmetry. Let us consider a system consisting of the equation (19) and the following ones:

$$\partial_t(V_n) = (\mu - V_n)V_n(V_{n+1} - V_{n-1}), \tag{25}$$

$$(\lambda - v_{n+1})v_n = (\mu - V_{n+1})V_n, \tag{26}$$

where $\mu(\tau)$ satisfies (21). We rewrite (26) in the form $v_n/V_n = (V_{n+1} - \mu)/(v_{n+1} - \lambda)$, and introduce a new variable $\tilde{w}_n = (V_n - \mu)/(v_n - \lambda)$ (then $\tilde{w}_{n+1} = v_n/V_n$). A new chain will be obtained in terms of $\tilde{w}_n$. Using the additional point transformation $w_n = (\tilde{w}_n - 1)/(\tilde{w}_n + 1)$ to simplify the new chain, we are led to the formulas

$$w_n = \frac{V_n - v_n + b}{V_n + v_n - a}, \quad w_{n+1} = \frac{v_n - V_n}{v_n + V_n} \qquad (a = \lambda + \mu, \quad b = \lambda - \mu). \tag{27}$$

If we exclude the case $\lambda = \mu = 0$, the change of variables (27) will be invertible:

$$2v_n = \frac{(1 + w_{n+1})(b + aw_n)}{w_{n+1} + w_n}, \quad 2V_n = \frac{(1 - w_{n+1})(b + aw_n)}{w_{n+1} + w_n}. \tag{28}$$

Let us differentiate some of the formulas (27) w.r.t. t and use (19), (25), (28). As a result, the following beautiful chain is constructed:

$$4\partial_t(w_n) = (1 - w_n^2)(b^2 - a^2w_n^2)\left(\frac{1}{w_{n+1} + w_n} - \frac{1}{w_n + w_{n-1}}\right). \tag{29}$$

The master symmetry of this chain

$$4\partial_\tau(w_n) = (1 - w_n^2)(b^2 - a^2w_n^2)\left(\frac{\varepsilon + n}{w_{n+1} + w_n} - \frac{\varepsilon + n - 1}{w_n + w_{n-1}}\right) + (1 - w_n^2)a^2w_n \tag{30}$$

is derived in the same way (one should use the same constraint (26) and eq. (24) instead of eq. (19). The discrete Miura transformations (28) reduce (29) and (30) to (19) and (24), respectively. The chain (29) obtained in **[15]** by classification is a difference analogue of the Calogero-Degasperis equation $u_t = u_{xxx} - u_x^3/8 + (c_1 e^u + c_2 e^{-u})u_x$ [**4**]. Eq. (30) is a new example of the master symmetry and the integrable chain.

Using the transformations (28), we can obtain a difference analogue of another Calogero-Degasperis equation [**4**]. In the complete list of scalar integrable chains of the form (1) [**15**], there are generalizations of both the difference Calogero-Degasperis equations. They can be constructed in the same way by applying the approach presented in [**16, 17**] to the Toda chain (3). Let us discuss briefly this case.

As in the case of (19) and (29), chains and their master symmetries explicitly depend on τ (master symmetries also depend on n). We shall express chains below in terms of functions $\lambda(\tau)$, $\mu(\tau)$, and $\nu(\tau)$, satisfying the equation $y' = y^2$ (compare with (21)). On the first step, we obtain the Volterra equation together with its master symmetry written in a different form:

$$\partial_t(u_n) = u_n(v_{n+1} - v_n), \quad \partial_t(v_n) = v_n(u_n - u_{n-1}). \tag{31}$$

$$\partial_\tau(u_n) = u_n[(\varepsilon + 2n + 1)v_{n+1} - (\varepsilon + 2n - 2)v_n + u_n + 2\lambda], \quad \partial_\tau(v_n) = v_n[(\varepsilon + 2n)u_n - (\varepsilon + 2n - 3)u_{n-1} + v_n + 2\lambda]. \tag{32}$$

These chains are reduced to (3), (6) by the following transformation: $\tilde{u}_n = u_n v_{n+1}$, $\tilde{v}_n = u_n + v_n + \lambda$. On the second step, we are led to

$$\partial_t(u_n) = (u_n^2 - a^2)(v_{n+1} - v_n), \quad \partial_t(v_n) = (v_n^2 - b^2)(u_n - u_{n-1}), \tag{33}$$

$$\partial_\tau(u_n) = (u_n^2 - a^2)[(\varepsilon + 2n)v_{n+1} - (\varepsilon + 2n - 2)v_n], \quad \partial_\tau(v_n) = (v_n^2 - b^2)[(\varepsilon + 2n - 1)u_n - (\varepsilon + 2n - 3)u_{n-1}] + cv_n, \tag{34}$$

where a is a constant, $b = (\mu - \lambda)/4a$, $c = \mu + \lambda$. Eq. (33) is a generalization of the modified Volterra equation (19). The transformation $\tilde{u}_n = (u_n + a)(v_{n+1} + b)$, $\tilde{v}_n = (u_n - a)(v_n - b)$ reduces (33), (34) to (31), (32). At last, the following chains arise:

$$\frac{1}{4}\partial_t(u_n) = u_n(bu_n + \alpha)(\beta u_n + a)\left(\frac{1}{v_{n+1} - u_n} + \frac{1}{u_n - v_n}\right),$$

$$\frac{1}{4}\partial_t(v_n) = v_n(bv_n + \alpha)(\beta v_n + a)\left(\frac{1}{u_n - v_n} + \frac{1}{v_n - u_{n-1}}\right),$$

$$\frac{1}{4}\partial_\tau(u_n) = u_n(bu_n + \alpha)(\beta u_n + a)\left(\frac{\varepsilon + 2n - 1}{v_{n+1} - u_n} + \frac{\varepsilon + 2n - 2}{u_n - v_n}\right) + u_n(b\beta u_n - \gamma),$$

$$\frac{1}{4}\partial_\tau(v_n) = v_n(bv_n + \alpha)(\beta v_n + a)\left(\frac{\varepsilon + 2n - 2}{u_n - v_n} + \frac{\varepsilon + 2n - 3}{v_n - u_{n-1}}\right) + v_n(b\beta v_n - \gamma),$$

where α is a constant, $\beta = (\mu - \nu)/4\alpha$, $\gamma = (\lambda + \nu)/4$. A transformation into (33), (34) has the form

$$\tilde{u}_n = \frac{2\beta u_n v_{n+1} + a(u_n + v_{n+1})}{u_n - v_{n+1}}, \quad \tilde{v}_n = \frac{b(u_n + v_n) + 2\alpha}{u_n - v_n}.$$

By introducing w_n ($u_n = w_{2n-1}$ and $v_n = w_{2n-2}$), we can write these systems in a scalar form:

$$(35)\quad \begin{aligned} \frac{1}{4}\partial_t(w_n) &= w_n(bw_n+\alpha)(\beta w_n+a)\left(\frac{1}{w_{n+1}-w_n}+\frac{1}{w_n-w_{n-1}}\right),\\ \frac{1}{4}\partial_\tau(w_n) &= w_n(bw_n+\alpha)(\beta w_n+a)\left(\frac{\varepsilon+n}{w_{n+1}-w_n}+\frac{\varepsilon+n-1}{w_n-w_{n-1}}\right)\\ &\quad + w_n(b\beta w_n-\gamma). \end{aligned}$$

The chain (35) generalizes (29).

There exist examples of a different kind. The following chain and its master symmetry

$$\partial_t(w_n) = (w_{n+1}-w_{n-1})^{-1}, \quad \partial_\tau(w_n) = (\varepsilon+n)(w_{n+1}-w_{n-1})^{-1}$$

are related to (19) and (24) with $\lambda = 0$ by the transformation $v_n = (w_{n+1}-w_{n-1})^{-1}$ which is simpler than the Miura transformation. Such an example can be obtained if a chain and its corresponding master symmetry possess a common conserved density p_n of the form $p_n = p(\tau, n, u_n)$ (we consider here just scalar chains; see, e.g., (2)). In this case, there are conservation laws $\partial_t(p_n) = (D-1)(q_n)$ and $\partial_\tau(p_n) = (D-1)(r_n)$. We can introduce a new variable $\tilde{u}_n$, so that $(D-1)(\tilde{u}_n) = p_n$, and new chains $\partial_t(\tilde{u}_n) = q_n$ and $\partial_\tau(\tilde{u}_n) = r_n$. We obtain a chain together with a master symmetry which are reduced to the given ones by the transformation $p(\tau, n, u_n) = (D-1)(\tilde{u}_n)$. These new equations can be simplified, using point transformations of the form $\tilde{u}_n = \tilde{u}(\tau, n, v_n)$.

The Volterra equation (2) and the master symmetry (5) possess the following common conserved density: $p_n = (-1)^n \log u_n$. The functions q_n and r_n are of the form:

$$\begin{aligned} q_n &= (-1)^{n-1}u_n + (-1)^n u_{n-1},\\ r_n &= (\varepsilon+n+1)(-1)^{n-1}u_n + (\varepsilon+n-1)(-1)^n u_{n-1}. \end{aligned}$$

Using the point transformation $\tilde{u}_n = (-1)^{n-1}\log(iv_n)$, we are led to (19), (23), (24) with $\lambda = 0$. There are many instances in which common conserved densities can be found. In the case of the modified Volterra equation (19) with $\lambda = 0$, such densities are v_n^{-1} and $(-1)^n v_n^{-1}$. If $\lambda \neq 0$ (recall that $\lambda = \lambda(\tau)$), we find the density $\log[(\lambda - v_n)/v_n]$. For the difference Calogero-Degasperis equation (29) and the master symmetry (30), the common density has the form: $\int(1-w_n^2)^{-1}\,dw_n$. There are examples of this kind in the case of chains similar to the Toda model (3), too. For instance, if we start from (33) with $b = 0$ (a is a constant), we are led to a chain and its master symmetry which can be expressed in the form:

$$\begin{aligned} u_{ntt} &= (a^2 - u_{nt}^2)(g_n - g_{n-1}), \quad g_n = (u_{n+1} - u_n)^{-1},\\ u_{n\tau\tau} &= [(\varepsilon+2n)a^2 - (\varepsilon+2n)^{-1}u_{n\tau}^2][(\varepsilon+2n+1)g_n - (\varepsilon+2n-1)g_{n-1}]. \end{aligned}$$

It is very important that the integrable chains, master symmetries, higher symmetries, conservation laws, and transformations under consideration are local. This allows one to prove, in particular, that the new master symmetries act on higher symmetries and conservation laws in the proper way. If $p_n^{(i)}[U_n]$ are conserved densities of a given chain (1) (we denote by $p_n^{(i)}[U_n]$ functions of the form $p^{(i)}(\tau, n, U_n, U_{n\pm1}, U_{n\pm2}, \dots)$), and a new chain $\partial_t(\overline{U}_n) = \overline{F}_n$ is reduced to the given one by a transformation $U_n = S_n[\overline{U}_n]$, then the functions

$\bar{p}_n^{(i)}[\bar{U}_n] = p_n^{(i)}\big[S_n[\bar{U}_n]\big]$ will be conserved densities of the new chain. Conserved densities obtained in this way also satisfy the relation (13), i.e. a new master symmetry generates conservation laws as well. Here, the main point is that the new master symmetry $\partial_\tau(\bar{U}_n) = \bar{G}_n$ is reduced to the master symmetry of the given chain by the same transformation $U_n = S_n[\bar{U}_n]$. Moreover, the approach proposed in [**16, 17**] allows us to construct higher symmetries $\partial_{t_i}(\bar{U}_n) = \bar{F}_n^{(i)}$ of the new chain, so that those are reduced to corresponding higher symmetries of the given chain by the same transformation. That is the reason why the formula (9) remains valid for the new higher symmetries.

4. Exact solutions of master symmetries

As is known, master symmetries, which are examples of equations explicitly depending on the spatial variable (x in the continuous case and n in the differential-difference one) and on time, are integrable by the inverse scattering method (the integrability of (7) was discussed, e.g., in [**3, 5**]; about (5), (6), (24), (34) see [**1, 2, 10–12**]). The master symmetries (5), (6) admit the representations (12), (15) and that is why they are integrable. All the other differential-difference master symmetries above are reduced to (5), (6). In this section, we discuss the problem of the construction of exact soliton-like solutions for the new examples of master symmetries.

Using the approach of [**16, 17**], one can not only construct new equations but also obtain solutions for them [**17**]. Let us consider the case of the Volterra equation (the case of the Toda model is analogous to this one). Let us assume that we have a solution u_n of (5) and a corresponding solution ψ_n of (20) (for any λ satisfying (21)). Using (22), we can obtain a solution v_n of (24) and a solution V_n of the same chain (24) with μ in place of λ, such that those solutions satisfy the Bäcklund transformation (26) (this is true, since relation (23) holds). Then we can construct a solution w_n of the master symmetry (30) by the explicit formulas (27).

For example, if $u_n = \kappa^2/4$ ($\kappa(\tau)$ satisfies (21)), then the general solution of (20) with $\alpha = 0$ is $\psi_n = \lambda\kappa^{-1/2}\{c_1\exp[(\varepsilon+n)\nu]+c_2\exp[-(\varepsilon+n)\nu]\}$, where $\cosh\nu = \lambda/\kappa$, and c_1, c_2 are arbitrary constants. In the particular case $\psi_n = \lambda\kappa^{-1/2}\cosh(\varphi_n)$, $\varphi_n = (\varepsilon + n)\nu + \delta$, where δ is an arbitrary constant, we are led to the solution $v_n = (\kappa/2)[\sinh(\nu)\tanh(\varphi_n) + \lambda/\kappa]$ of (24). In a similar way, one easily can obtain the following trivial solution of (24) with μ instead of λ ($\mu(\tau)$ satisfies (21)): $V_n = e^\eta\kappa/2$, where $\cosh\eta = \mu/\kappa$. The second of the formulas (27) yields for the chain (30) the solution

$$w_{n+1} = \frac{\cosh\varphi_{n+1} - e^\eta\cosh\varphi_n}{\cosh\varphi_{n+1} + e^\eta\cosh\varphi_n}.$$

The dressing method enables one to construct more complicated solutions of the master symmetry (5) of the Volterra equation (2) and, therefore, solutions of the master symmetry (30). Note that the transformation $\tilde{v}_n = \lambda - v_n$ does not change (24). That is why (24) is reduced to (5) not only by (23) but also by the transformation $u_n = v_{n+1}(\lambda - v_n)$. Using this transformation together with (22), we obtain the following formula:

$$\tilde{u}_n = u_{n+1}^{1/2}u_{n-1}^{1/2}\frac{\psi_{n+2}\psi_{n-1}}{\psi_{n+1}\psi_n},$$

where $\tilde{u}_n$ is a new solution of (5). If $u_n = \kappa^2/4$ and $\psi_n = \lambda\kappa^{-1/2}\cosh(\varphi_n)$ (see formulas above), then

$$\tilde{u}_n = \frac{\kappa^2}{4} + \lambda \frac{\lambda^2 - \kappa^2}{\lambda + \kappa\cosh(2\varphi_n + \nu)}.$$

Assume that $\tilde{L}_n$ and $\tilde{A}_n$ are the operators L_n and A_n of (11), (12), (20) with $\tilde{u}_n$ in place of u_n (ε remains unchanged). There is an operator

$$R_n = u_n^{1/4} u_{n-1}^{1/4} \left(\psi_{n-1}^{1/2} \psi_{n+1}^{-1/2} D - \psi_{n+1}^{1/2} \psi_{n-1}^{-1/2} D^{-1}\right)$$

satisfying the following relationships:

$$\tilde{L}_n R_n = R_n L_n, \quad \partial_\tau(R_n) = \tilde{A}_n R_n - R_n A_n + \frac{1}{2} R_n (L_n^2 + \lambda^2). \tag{36}$$

Let θ_n be a solution of the system (20) with $\tilde{\lambda}$ and β instead of λ and α. It follows from (36) that $\tilde{\psi}_n = R_n(\theta_n)$ is a new solution of (20) corresponding to $\tilde{u}_n$. Namely, $\tilde{L}_n(\tilde{\psi}_n) = \tilde{\lambda}\tilde{\psi}_n$ and $\partial_\tau(\tilde{\psi}_n) = (\tilde{A}_n + \tilde{\alpha})(\tilde{\psi}_n)$, where $\tilde{\alpha} = \beta + (\tilde{\lambda}^2 + \lambda^2)/2$.

References

1. O. I. Bogoyavlensky, *Breaking solitons. Nonlinear integrable equations*, Nauka, Moscow, 1991 (Russian).
2. M. Bruschi, D. Levi, and O. Ragnisco, *Discrete version of the modified Korteweg-de Vries equation with x-dependent coefficients*, Nuovo Cimento A **48** (1978), 213–226.
3. S. P. Burtsev, V. E. Zakharov, and A. V. Mikhailov, *Inverse scattering method with variable spectral parameter*, Teoret. Mat. Fiz. **70** (1987), no. 3, 323–341 (Russian); English transl. in Theoret. and Math. Phys. **70** (1987).
4. F. Calogero and A. Degasperis, *Reduction technique for matrix nonlinear evolution equations solvable by the spectral transform*, Preprint 151, Istituto di Fisica G. Marconi, Univ. di Roma "La Sapienza", 1979.
5. ______, *Spectral transform and solitons: tools to solve and investigate nonlinear evolution equations*, vol. 1, North-Holland, Amsterdam, New York, Oxford, 1982.
6. A. S. Fokas, *Symmetries and integrability*, Stud. Appl. Math. **77** (1987), 253–299.
7. A. S. Fokas and B. Fuchssteiner, *The hierarchy of the Benjamin-Ono equation*, Phys. Lett. A **86** (1981), 341–345.
8. I. M. Krichever and S. P. Novikov, *Holomorphic bundles over algebraic curves, and nonlinear equations*, Uspekhi Mat. Nauk **35** (1980), no. 6, 47–68 (Russian); English transl. in Russian Math. Surveys **35** (1980).
9. B. A. Kupershmidt, *Discrete Lax equations and differential-difference calculus*, Astérisque, vol. 123, Paris, 1985.
10. D. Levi and O. Ragnisco, *Extension of the spectral transform method for solving nonlinear differential-difference equations*, Lett. Nuovo Cimento **22** (1978), 691–696.
11. ______, *Non-linear differential-difference equations with N-dependent coefficients* I, J. Phys. A **12** (1979), 157–162.
12. ______, *Non-linear differential-difference equations with N-dependent coefficients* II, J. Phys. A **12** (1979), 163–167.
13. R. M. Miura, *Korteweg-de Vries equation and generalizations* I. *A remarkable explicit nonlinear transformation*, J. Math. Phys. **9** (1968), 1202–1204.
14. M. Toda, *Exact treatment of nonlinear lattice waves*, Prog. Theoret. Phys. Suppl. **59** (1976), 1–35.
15. R. I. Yamilov, *Classification of discrete evolution equations*, Uspekhi Mat. Nauk **38** (1983) no. 6, 155–156 (Russian).
16. ______, *On the construction of Miura type transformations by others of this kind*, Phys. Lett. A **173** (1993), 53–57.
17. ______, *Construction scheme for discrete Miura transformations*, J. Phys. A **27** (1994), 6839–6851.

Ufa Institute of Mathematics, Russian Academy of Sciences, Chernyshevsky str. 112, Ufa 450000, Russia

E-mail address: ken@nkc.bashkiria.su

Ufa Institute of Mathematics, Russian Academy of Sciences, Chernyshevsky str. 112, Ufa 450000, Russia

E-mail address: yamilov@nkc.bashkiria.su