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Master symmetries for differential-difference equations of the Volterra type

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Abstract

It is demonstrated that in the case of integrable differential-difference equations similar to the well-known Volterra equation (unlike the Korteweg-de Vries and nonlinear Schrödinger equations) there are many instances in which local master symmetries can be found. Those master symmetries are new interesting examples of local evolution chains explicitly depending on the time and discrete variable and integrable in a special sense. The examples are constructed by a direct and elementary approach which enables one to get new integrable equations, using Miura type transformations.

1. We consider integrable discrete-differential equations (chains) of the form

$$\partial_t(u_n) = f_n, \quad f_n = f(u_{n+1}, u_n, u_{n-1}), \tag{1}$$

where n is an arbitrary integer. The simplest example of this kind is the well-known Volterra equation

$$\partial_t(u_n) = u_n(u_{n+1} - u_{n-1}). \tag{2}$$

There is an exhaustive list of integrable chains of the form (1) in [1].

All chains of that list possess higher symmetries of the form

$$\partial_{t_i}(u_n) = f_n^{(i)}, \quad f_n^{(i)} = f^{(i)}(u_{n+i}, u_{n+i-1}, \ldots, u_{n-i}), \tag{3}$$

for any $i > 1$. Recall that, by definition, Eqs. (3) satisfy the following condition:

$$[f_n, f_n^{(i)}] = \partial_t(f_n^{(i)}) - \partial_{t_i}(f_n) = \sum_k (f_k \partial f_n^{(i)}/\partial u_k - f_k^{(i)} \partial f_n/\partial u_k) = 0. \tag{4}$$

In general, for two higher symmetries, $[f_n^{(i)}, f_n^{(j)}] = 0$. However, if a chain $\partial_\tau(u_n) = g_n$ is a master symmetry, the commutator $[g_n, f_n^{(i)}]$ is a higher symmetry.

We shall be interested in master symmetries of the form

$$\partial_\tau(u_n) = g_n, \quad g_n = g(n, \tau, u_{n+1}, u_n, u_{n-1}) \tag{5}$$

which enable us to construct all the higher symmetries (3) in the following way: $[g_n, f_n^{(i)}] = a_i f_n^{(i+1)}$, where $i \geq 1$, $f_n^{(1)} = f_n$ (see (1)), and a_i are constants. Master symmetry of such a kind for the Volterra equation (2) is the chain

$$\partial_\tau(u_n) = u_n[(\varepsilon + n + 2)u_{n+1} + u_n - (\varepsilon + n - 1)u_{n-1}], \tag{6}$$

where ε is an arbitrary constant. It is easy to check that in this case

$$[g_n, f_n] = f_n^{(2)} = u_n[u_{n+1}(u_{n+2} + u_{n+1} + u_n) - u_{n-1}(u_n + u_{n-1} + u_{n-2})], \tag{7}$$

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and this is the right hand side of the simplest higher symmetry of Eq. (2).

Sometimes, Eq. (2) is called the difference Korteweg-de Vries equation, for in the continuous limit it becomes the Korteweg-de Vries equation $u_t = u_{xxx} + uu_x$. Master symmetry of the KdV equation is

$$u_\tau = x(u_{xxx} + uu_x) + 4(u_{xx} + u^2/3) + (1/3)u_x\partial_x^{-1}(u) \quad (8)$$

[2]. In many cases, as in this example, known master symmetries are not local evolution equations. The differential-difference master symmetries we shall discuss here will be evolution chains without any difference analogues of ∂_x^{-1} (compare (6) and (8)).

2. Let us recall some properties of the known chains (2) and (6) (Eq. (6) and the corresponding $L-A$ pair which we write down below, are described in [3]) and explain why one can construct higher symmetries of (2) by (6).

Master symmetry (6) has the representation

$$\partial_\tau(L_n) = [A_n, L_n] + (1/2)L_n^3,$$
$$L_n = u_n^{1/2}D + u_{n-1}^{1/2}D^{-1},$$
$$2A_n = (\varepsilon + n + 1)u_{n+1}^{1/2}u_n^{1/2}D^2 - (\varepsilon + n - 1)u_{n-1}^{1/2}u_{n-2}^{1/2}D^{-2}, \quad (9)$$

where D is the shift operator, and A_n, L_n are multiplied as follows: $(p_nD^i)(q_nD^j) = p_nq_{n+i}D^{i+j}$. And that is why Eq. (6) is integrable (see, e.g., [2] and [3]). All other master symmetries in this paper are examples of integrable chains too, for they are reduced to integrable ones by Miura type transformations.

The Volterra equation (2) admits Lax representation

$$\partial_t(L_n) = [B_n, L_n],$$
$$2B_n = u_{n+1}^{1/2}u_n^{1/2}D^2 - u_{n-1}^{1/2}u_{n-2}^{1/2}D^{-2} \quad (10)$$

(L_n is the operator of (9)). Therefore the functions $p_n^{(i)} = \mathrm{res}(L_n^{2i})$ (i.e. coefficients of L_n^{2i} at D^0) are conserved densities of Eq. (2) for any $i \geq 1$. For example $p_n^{(1)} = u_n + u_{n-1} \sim 2u_n$, $p_n^{(2)} \sim 4(u_{n+1}u_n + u_n^2/2)$. Note that the local conservation laws $\partial_t(p_n) = (D-1)(q_n)$ and $\partial_t(\tilde{p}_n) = (D-1)(\tilde{q}_n)$ are the same if $\tilde{p}_n = p_n + (D-1)(r_n)$. We call such conserved densities p_n and $\tilde{p}_n$ equivalent ($p_n \sim \tilde{p}_n$).

It should be remarked that the representation (9) implies the following relationships: $\partial_\tau(p_n^{(i)}) \sim ip_n^{(i+1)}$. So we can see that, differentiating a conserved density with respect to τ, one easily obtains the next one. This is another important property of master symmetries which we consider in this paper, that they allow one to construct not only higher symmetries but also local conservation laws.

In order to show that master symmetry (6) enables us to get higher symmetries, we shall use the bi-Hamiltonian structure of the Volterra equation (2). The operators $H_n^{(1)} = u_n(D - D^{-1})u_n$ and

$$H_n^{(2)} = u_n[u_{n+1}D^2 + (u_{n+1} + u_n)D - (u_n + u_{n-1})D^{-1} - u_{n-1}D^{-2}]u_n$$

are Hamiltonian. Let f_n^+ (now f_n is the right hand side of (2)) act on them as follows:

$$f_n^+(H_n^{(i)}) = H_n^{(i)}(\partial_t + f_n^{*T}) - (\partial_t - f_n^*)H_n^{(i)},$$

$$f_n^* = \partial f_n/\partial u_{n+1}D + \partial f_n/\partial u_n + \partial f_n/\partial u_{n-1}D^{-1},$$

$$f_n^{*T} = \partial f_{n+1}/\partial u_nD + \partial f_n/\partial u_n + \partial f_{n-1}/\partial u_nD^{-1}.$$

It is easy to see that $f_n^+(H_n^{(1)}) = 0$, hence $H_n^{(1)}$ takes formal variational derivatives $\delta p_n^{(i)}/\delta u_n$ of conserved densities $p_n^{(i)}$ into right hand sides of higher symmetries:

$$\partial_{t_i}(u_n) = F_n^{(i)} = H_n^{(1)}(G_n^{(i)}),$$
$$G_n^{(i)} = \delta p_n^{(i)}/\delta u_n = \sum_k \partial p_k^{(i)}/\partial u_n. \quad (11)$$

For example $F_n^{(1)} = 2f_n$, $F_n^{(2)} = 4f_n^{(2)}$ (see (7)). Indeed, the following identity

$$(\delta/\delta u_n)\partial_t(p_n^{(i)}) = (\partial_t + f_n^{*T})(G_n^{(i)}) \quad (12)$$

holds, and $(\delta/\delta u_n)\partial_t(p_n^{(i)}) = 0$ because $\partial_t(p_n^{(i)}) \sim 0$. Therefore the functions $F_n^{(i)}$ satisfy the conditions $[f_n, F_n^{(i)}] = (\partial_t - f_n^*)(F_n^{(i)}) = 0$.

For the second Hamiltonian operator we also have $f_n^+(H_n^{(2)}) = 0$, but in this case

$$H_n^{(2)}(G_n^{(i)}) = (i/(i+1))F_n^{(i+1)}. \quad (13)$$

It is interesting that

$$g_n^{*}(H_n^{(1)}) = H_n^{(2)}, \quad g_n^{+}(H_n^{(2)}) = 0, \quad (14)$$

where g_n is the right hand side of (6). The second of the equalities (14) means that we can try to represent master symmetry (6) in Hamiltonian form. Indeed, Eq. (6) has conserved density $p_n = (1/2)(\varepsilon + n + 1/2)\log(u_n)$, and we get $g_n = H_n^{(2)}(\delta p_n/\delta u_n)$.

Finally, let us write down the following formulae:

$$\begin{aligned} iF_n^{(i+1)} &= iH_n^{(1)}(\delta p_n^{(i+1)}/\delta u_n) = H_n^{(1)}(\delta/\delta u_n)\partial_\tau(p_n^{(i)}) \\ &= H_n^{(1)}(\partial_\tau + g_n^{*T})(G_n^{(i)}) \\ &= [(\partial_\tau - g_n^*)H_n^{(1)} + H_n^{(2)}](G_n^{(i)}) \\ &= (\partial_\tau - g_n^*)(F_n^{(i)}) + (i/(i+1))F_n^{(i+1)}. \end{aligned}$$

Here we use (11), the relationship $\partial_\tau(p_n^{(i)}) \sim ip_n^{(i+1)}$, the identity (12) with τ and g_n instead of t and f_n, the first of the equalities (14), and then (11) and (13). Since $[g_n, F_n^{(i)}] = (\partial_\tau - g_n^*)(F_n^{(i)})$, we are led to the beautiful formula $[g_n, F_n^{(i)}] = (i^2/(i+1))F_n^{(i+1)}$, which shows that the chain (6) is master symmetry of the Volterra equation (2).

3. All integrable chains of the form (1) are reduced to the Volterra equation or Toda model but a difference analog of the Krichever-Novikov equation $u_t = u_{xxx} - (3/2)u_x^{-1}[u_{xx}^2 + P(u)]$ (here $P(u)$ is an arbitrary polynomial of an arbitrary degree $N \le 4$) [4]. There are several chains among them being reduced to the Volterra and Toda equations by discrete Miura type transformations (we shall call them key equations). All the other chains are reduced to the key ones by more simple transformations like $\tilde{u}_n = u_{n+1} - u_n$. We can construct local evolution master symmetries for all the key chains and for some of the other ones. We use a scheme which allows one to obtain new integrable equations, using Miura transformations. This scheme was presented in [5] (the case of partial differential equations) and [6] (the case of discrete-differential ones). In this Section we explain how to get local master symmetries for the modified Volterra equation and a difference analog of the Calogero-Degasperis equation $u_t = u_{xxx} - u_x^3/8 + (ae^u + be^{-u})u_x$ [7]. Besides, we discuss the problem of the construction of exact solutions.

In order to obtain master symmetry of the modified Volterra equation

$$\partial_t(v_n) = (\lambda - v_n)v_n(v_{n+1} - v_{n-1}), \quad (15)$$

let us consider the system

$$L_n(\psi_n) = \lambda\psi_n, \quad \partial_\tau(\psi_n) = (A_n + \alpha)(\psi_n), \quad (16)$$

where L_n and A_n are the operators of (9), $\alpha = \alpha(\tau)$, $\lambda = \lambda(\tau)$, and

$$\lambda' = \lambda^3/2. \quad (17)$$

It is easy to check that the function

$$v_n = u_n^{1/2}\psi_{n+1}/\psi_n \quad (18)$$

satisfies the following relationships:

$$u_n = (\lambda - v_{n+1})v_n, \quad (19)$$

$$\begin{aligned} \partial_\tau(v_n) = (\lambda - v_n)v_n[(\varepsilon + n + 1)v_{n+1} \\ -(\varepsilon + n - 1)v_{n-1}] + \lambda v_n(v_n - \lambda/2). \end{aligned} \quad (20)$$

Eq. (20) is master symmetry of (15). Note that (19) is a discrete Miura type transformation reducing (15) to the Volterra equation (2) and (20) to its master symmetry (6). For any solution v_n of (15) or (20), the formula (19) yields a solution u_n of (2) or (6), respectively.

Now, following [6], we shall obtain one more key integrable equation of the form (1) and corresponding master symmetry. Let us consider a system consisting of Eq. (15) and the following ones:

$$\partial_t(V_n) = (\mu - V_n)V_n(V_{n+1} - V_{n-1}), \quad (21)$$

$$(\lambda - v_{n-1})v_n = (\mu - V_{n+1})V_n, \quad (22)$$

where $\mu(\tau)$ satisfies (17). We rewrite (22), so that $v_n/V_n = (V_{n+1} - \mu)/(v_{n+1} - \lambda)$, and introduce a new variable $\tilde{w}_n = (V_n - \mu)/(v_n - \lambda)$ (then $\tilde{w}_{n+1} = v_n/V_n$). A new chain will be expressed in terms of $\tilde{w}_n$. Using the additional point transformation $w_n = (\tilde{w}_n - 1)/(\tilde{w}_n + 1)$ to make that new chain more simple, we are led to the formulae

$$w_n = \frac{V_n - v_n + b}{V_n + v_n - a}, \quad w_{n+1} = \frac{v_n - V_n}{v_n + V_n}$$
$$(a = \lambda + \mu, \quad b = \lambda - \mu). \qquad (23)$$

If we eliminate the case $\lambda = \mu = 0$, the change of variables (23) will be invertible:

$$2v_n = \frac{(1 + w_{n+1})(b + aw_n)}{w_{n+1} + w_n},$$
$$2V_n = \frac{(1 - w_{n+1})(b + aw_n)}{w_{n+1} + w_n}. \qquad (24)$$

Let us differentiate some of the formulae (23) with respect to t and use Eqs. (15), (21) together with the substitutions (24). As a result, the following difference analog of the Calogero-Degasperis equation is constructed:

$$4\partial_t(w_n) = (1 - w_n^2)(b^2 - a^2 w_n^2) \times \left(\frac{1}{w_{n+1} + w_n} - \frac{1}{w_n + w_{n-1}}\right). \qquad (25)$$

Master symmetry of this chain

$$4\partial_\tau(w_n) = (1 - w_n^2)(b^2 - a^2 w_n^2) \times \left(\frac{\varepsilon + n}{w_{n+1} + w_n} - \frac{\varepsilon + n - 1}{w_n + w_{n-1}}\right) + (1 - w_n^2)a^2 w_n \qquad (26)$$

is derived in the same way (one should use the same constraint (22) and Eq. (20) instead of (15)). The discrete Miura transformations (24) reduce the chains (25) and (26) to the corresponding chains (15) and (20).

By this direct and elementary approach, one can not only construct master symmetries similar to (26) but also get exact soliton-like solutions. Let us assume we have a solution u_n of (6) and the corresponding solution ψ_n of (16) for any λ satisfying (17). Using (18), we can obtain a solution v_n of (20) and a solution V_n of the same chain (20) with μ instead of λ, such that they satisfy the Bäcklund transformation (22) (this is true, since the relationship (19) holds). Then we can construct a solution w_n of master symmetry (26) by the formulae (23).

For example, if $u_n = \kappa^2/4$ ($\kappa(\tau)$ satisfies (17)), then the general solution of (16) with $\alpha = 0$ is $\psi_n = \lambda\kappa^{-1/2}\{c_1 \exp[(\varepsilon + n)\nu] + c_2 \exp[-(\varepsilon + n)\nu]\}$, where $\cosh\nu = \lambda/\kappa$, and c_1, c_2 are arbitrary constants. In the particular case $\psi_n = \lambda\kappa^{-1/2}\cosh(\varphi_n)$, $\varphi_n = (\varepsilon + n)\nu + \delta$, where δ is an arbitrary constant, we are led to the solution $v_n = (\kappa/2)[\sinh(\nu)\tanh(\varphi_n) + \lambda/\kappa]$ of Eq. (20). By the similar way, one can easily get the following trivial solution of (20) with μ in place of λ ($\mu(\tau)$ satisfies (17)): $V_n = e^\eta\kappa/2$, where $\cosh\eta = \mu/\kappa$. The second of the formulae (23) yields for the chain (26) the solution

$$w_{n+1} = \frac{\cosh\varphi_{n+1} - e^\eta\cosh\varphi_n}{\cosh\varphi_{n+1} + e^\eta\cosh\varphi_n}.$$

The dressing method enables one to construct more complicated solutions of master symmetry (6) of the Volterra equation (2) and, therefore, solutions of master symmetry (26). Note that the transformation $\tilde{v}_n = \lambda - v_n$ does not change (20). That is why Eq. (20) is reduced to (6) not only by (19) but also by the transformation $u_n = v_{n+1}(\lambda - v_n)$. Using this transformation together with (18), we obtain the following formula:

$$\tilde{u}_n = u_{n+1}^{1/2} u_{n-1}^{1/2} (\psi_{n+2}\psi_{n-1})/(\psi_{n+1}\psi_n),$$

where $\tilde{u}_n$ is a new solution of Eq. (6). If $u_n = \kappa^2/4$ and $\psi_n = \lambda\kappa^{-1/2}\cosh(\varphi_n)$ (see formulae above), then

$$\tilde{u}_n = \kappa^2/4 + \lambda(\lambda^2 - \kappa^2)/[\lambda + \kappa\cosh(2\varphi_n + \nu)].$$

Assume that $\tilde{L}_n$ and $\tilde{A}_n$ are the operators L_n and A_n of (9), (16) with $\tilde{u}_n$ in place of u_n (ε remains unchanged). There is an operator

$$R_n = u_n^{1/4} u_{n-1}^{1/4} (\psi_{n-1}^{1/2}\psi_{n+1}^{-1/2} D - \psi_{n+1}^{1/2}\psi_{n-1}^{-1/2} D^{-1})$$

satisfying the following relationships:

$$\tilde{L}_n R_n = R_n L_n,$$
$$\partial_\tau(R_n) = \tilde{A}_n R_n - R_n A_n + (1/2)R_n(L_n^2 + \lambda^2). \qquad (27)$$

Let θ_n be a solution of the system (16) with $\tilde{\lambda}$ and β instead of λ and α. It follows from (27) that $\tilde{\psi}_n = R_n(\theta_n)$ is a new solution of (16) corresponding to $\tilde{u}_n$. Namely, $\tilde{L}_n(\tilde{\psi}_n) = \tilde{\lambda}\tilde{\psi}_n$ and $\partial_\tau(\tilde{\psi}_n) = (\tilde{A}_n + \tilde{\alpha})(\tilde{\psi}_n)$, where $\tilde{\alpha} = \beta + (\tilde{\lambda}^2 + \lambda^2)/2$.

4. So, we have cited three beautiful examples of local evolution chains (6), (20), (26) explicitly depending on the discrete parameter n and on the time τ

(as far as we know, the last of them is new). Chains constructed in this way are integrable, for they are reduced to integrable ones by Miura type transformations. At least, one easily can construct exact soliton-like solutions for such chains.

It is important that integrable chains, master and higher symmetries, conservation laws, and transformations under consideration are local. This allows one to prove, in particular, that new master symmetries act on higher symmetries and conserved densities in the proper way. If $p_n^{(i)}[u_n]$ are conserved densities of a given chain (1) (we denote by $p_n^{(i)}[u_n]$ functions of the form $p^{(i)}(n, \tau, u_n, u_{n\pm1}, u_{n\pm2}, \ldots)$), and a new chain $\partial_t(U_n) = F_n$ is reduced to the given one by a transformation $u_n = S_n[U_n]$, then the functions $P_n^{(i)}[U_n] = p_n^{(i)}[S_n[U_n]]$ will be conserved densities of this new chain. Conserved densities obtained in this way also satisfy the relationships $\partial_\tau(P_n^{(i)}) \sim iP_n^{(i+1)}$ (see Section 2), i.e. one can construct local conservation laws by master symmetry of the new chain as well. The main point is that master symmetry $\partial_\tau(U_n) = G_n$ of the new chain is reduced to master symmetry of the given one by the same transformation $u_n = S_n[U_n]$. Moreover, the approach presented in [5,6] allows us to construct higher symmetries $\partial_{t_i}(U_n) = F_n^{(i)}$ of the new chain, so that they are reduced to corresponding higher symmetries of the given chain by the same transformation. That is the reason why the formula $[G_n, F_n^{(i)}] = (i^2/(i+1))F_n^{(i+1)}$ (see Section 2) remains valid for new higher symmetries.

As has been said above, we can construct other examples of integrable local master symmetries in a similar way. However, we shall not discuss them in this paper. Let us show only one more example of a different kind. We can get master symmetries not only if the construction scheme we used in the previous Section can be applied, but also if both a chain and its master symmetry possess the same local conservation law of zeroth order (i.e. with a density of the form $p_n = p(u_n)$). In this case, as is known, we can construct a new chain and its master symmetry being reduced to the given ones by a transformation that is simpler than the Miura type one. For instance, Eqs. (15), (20) with $\lambda = 0$ possess the conserved density $p_n = v_n^{-1}$. Therefore there are the chains

$$\partial_t(w_n) = (w_{n+1} - w_{n-1})^{-1},$$
$$\partial_\tau(w_n) = (\varepsilon + n)(w_{n+1} - w_{n-1})^{-1} \quad (28)$$

related to Eqs. (15) and (20) with $\lambda = 0$, respectively, by the transformation $v_n = (w_{n+1} - w_{n-1})^{-1}$. The second of Eqs. (28) is master symmetry of the first one. This chain is integrable and acts on higher symmetries and conserved densities in the proper way.

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