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Multi-component Volterra and Toda type integrable equations

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Abstract

Multi-component integrable analogs related to the Jordan triple systems (JTS) are constructed for the Volterra equation. Differential difference substitutions lead to multi-component Toda type lattices. Associated equations generalize the derivative nonlinear Schrödinger equation. Multi-component master symmetries (both partial differential and differential difference ones) and zero curvature representations for lattice equations written in terms of the superstructure Lie algebra of the JTS arise for the first time. © 1999 Published by Elsevier Science B.V.

1. Introduction

In this paper, we construct multi-component analogs for the Volterra equation and some integrable equations closely connected to it and study their integrability properties. Many results demonstrate that very often multi-component partial differential and differential difference equations are integrable if connected to various Jordan structures [2,3,7,8]. The first papers on multi-component generalisations of integrable equations related to the Jordan and left-symmetric algebraic structures are due to S.I. Svinolupov, and main results in the field also have been obtained by him (see e.g. Refs. [1,2] and the review article [3] dedicated to his memory).

It is more convenient for our purpose to consider the following slightly changed form of the Volterra equation,

$$u_{n,x} = u_n^2(u_{n+1} - u_{n-1}), \quad n \in \mathbb{Z}. \tag{1}$$

This is a very well studied integrable model. It is bi-Hamiltonian, possesses an L–A pair, recursion operator, local master-symmetry, infinite hierarchy of higher symmetries and conservation laws, etc. Much of this stuff will be displayed in the next section. Now we only notice that higher symmetries of (1) can be rewritten as evolutionary partial differential systems for the variables $u = u_n, v = u_{n-1}$ and constitute a hierarchy of integrable systems, the first representative of which is of the form

$$u_t = u_{xx} + 2(u^2 v)_x, \qquad v_t = -v_{xx} + 2(v^2 u)_x \tag{2}$$

(see e.g. Refs. [4,5]). This is the well-known Kaup–Newell equation [6] which often is called the derivative nonlinear Schrödinger equation (DNLS).

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The main object of the present paper will be a class of multi-component generalizations of (1) related to the Jordan triple systems (JTS). We recall that a ternary algebra J with a multiplication $\{\} : J^3 \to J$ is called JTS if the following identities hold for arbitrary elements,

$$\{abc\} = \{cba\}, \tag{3}$$

$$\{ab\{cde\}\} - \{cd\{abe\}\} = \{\{cba\}de\} - \{c\{bad\}e\}. \tag{4}$$

All necessary information concerning JTS will be given later step-by-step. Details and bibliography can be found for example in Refs. [9–11].

It turns out that for arbitrary JTS the lattice

$$u_{n,x} = \{u_n(u_{n+1} - u_{n-1})u_n\}, \tag{5}$$

which will be called the Jordan Volterra equation, is integrable. This lattice can be written more explicitly, using the expansion $u_n = u_n^m e_m$ over a basis in J. The multiplication $\{\ \}$ is uniquely defined by the formula $\{e_i e_j e_k\} = a_{ijk}^m e_m$, and (5) takes the form

$$u_{n,x}^m = a_{ijk}^m u_n^i u_n^k (u_{n+1}^j - u_{n-1}^j).$$

However, we always will use coordinate-free notations.

Our aim is to transfer, if possible, properties of (1) to the multi-component case. Sometimes it can be attained just by the proper ordering of cofactors. For example, a correct Jordan analog of the symmetry (2) (i.e., of the DNLS equation) is of the form

$$u_t = u_{xx} + 2\{uvu\}_x, \qquad v_t = -v_{xx} + 2\{vuv\}_x. \tag{6}$$

In other cases, one should use rather complicated constructions. For instance, in order to obtain the zero curvature representations for (5) and (6), we use the superstructure Lie algebras.

In some points we are unable to achieve the full analogy. The Hamiltonian properties in the general Jordan case seem to be more poor. Moreover, there are some important transformations of scalar equations which are lacking in the multi-component case. An example of such a transformation is given by the formula $w_n = u_{n+1}u_n$. This transformation[1] brings (1) into the standard form of the Volterra equation

$$w_{n,x} = w_n(w_{n+1} - w_{n-1}), \tag{7}$$

which, unfortunately, does not have natural multi-component analogs corresponding to the Jordan algebraic structures. Nevertheless, some transformations of this kind are well defined in the Jordan case as well, give rise to integrable modifications of (5), (6), and lead, in particular, to multi-component lattice equations of the Toda type (see below).

As is known, there is an alternative and older approach for the construction of integrable multi-component equations connected to algebraic structures [12,13]. Some of examples obtained by these two approaches coincide, but the correspondence in the other cases remains an open problem. Multi-component integrable analogs of the DLNS equation (2) have been obtained for the first time in Ref. [14], using that alternative approach.

[1] After the point transformations $w_n = \exp \widetilde{w}_n$, $u_n = \exp \widetilde{u}_n$, we have the transformation $\widetilde{w}_n = \widetilde{u}_{n+1} + \widetilde{u}_n$ and see that this is a discrete analog of the potentiation $w = u_x$. That is why (1) and (7) are almost the same from our point of view.

2. Scalar case

Here we introduce some integrable equations related to (1) and list those of their attributes which will be discussed in the multi-component case: master symmetries, zero curvature representations and Hamiltonian structures. We start from master symmetries because, as one can see later, those give us the shortest way to multi-component generalizations.

The master symmetry of (1) which has the form

$$\partial_y(u_n) = u_n^2((c+n+1)u_{n+1} - (c+n-1)u_{n-1}) \tag{8}$$

(c is an arbitrary constant) not only exemplifies an interesting integrable lattice with an essential dependence on the spatial variable n but also gives an easy way for the construction of higher symmetries and conservation laws. An infinite hierarchy of higher symmetries,

$$\partial_{t_k}(u_n) = f_k(u_{n+k},\dots,u_{n-k}), \quad k = 1,2,3,\dots,$$

of (1) can be obtained recursively in the following way: $\partial_{t_{k+1}} = [\partial_y, \partial_{t_k}]$, where $t_1 = x$. As the first step, we are led to the symmetry (denoting t_2 by t)

$$\partial_t(u_n) = u_n^2(u_{n+1}^2(u_{n+2}+u_n) - u_{n-1}^2(u_n+u_{n-2})). \tag{9}$$

Any higher symmetry of a lattice of the form $u_{n,x} = f(u_{n+1}, u_n, u_{n-1})$ is equivalent to a system of evolution partial differential equations [4,5]. In order to obtain such a system, one should eliminate the variables $u_{n+1}, u_{n\pm2}, u_{n\pm3},\dots$ in virtue of the lattice itself (more precisely, express these variables in terms of $u = u_n$, $v = u_{n-1}$ and their x-derivatives). In the case of (1) and (9), this procedure brings to the DNLS equation (2). Eq. (9) is not the only symmetry of the second order. Obviously, the differentiation $\partial_\tau = x\partial_t + \partial_y$ commutes with ∂_x and defines an x-dependent higher symmetry. That symmetry can be expressed in the form

$$u_\tau = (xu_x + 2xu^2v + (c+n)u)_x, \qquad v_\tau = (-xv_x + 2xv^2u + (c+n-1)v)_x. \tag{10}$$

For any constant c and integer n (one can put, for example, $c = -n$), this is nothing but the master symmetry of the DNLS equation because $[\partial_\tau, \partial_{t_k}] = \partial_{t_{k+1}}$ for all $k \geqslant 2$.

The concept of the master symmetry has been introduced in Ref. [15] (details can be found in the review articles [16,17]). Before that, master symmetries arose and were investigated as integrable equations with the spectral problem in which the spectral parameter depended on the time (see, e.g., Ref. [18]). For example, as an integrable equation, (8) has been found in Ref. [19]. It has arisen as the master symmetry of (1) in Ref. [20]. Eq. (10) has been found in Ref. [21].

As has been said in the introduction, equations under consideration can be rewritten in many forms by differential and discrete substitutions. We consider only one example of such a transformation. Introducing $u_{2n} = p_n$ and $u_{2n-1} = 1/(q_{n-1} - q_n)$, one can express (1), (8) as two Toda type equations,

$$q_{n,x} = p_n, \qquad p_{n,x} = p_n^2\left(\frac{1}{q_n - q_{n+1}} - \frac{1}{q_{n-1} - q_n}\right), \tag{11}$$

$$q_{n,y} = (c+2n)p_n, \qquad p_{n,y} = p_n^2\left(\frac{c+2n+1}{q_n - q_{n+1}} - \frac{c+2n-1}{q_{n-1} - q_n}\right), \tag{12}$$

found in Refs. [22,23], respectively. Eq. (12) remains the master symmetry of (11).

For all Eqs. (1), (2), (11) and their master symmetries (8), (10), (12), multi-component generalizations will be presented in the next section. Let us write down L–A pairs for the scalar equations (1), (2), (8), (10) (some of them are known, see e.g. Refs. [6,19]). For the lattice equation (1), its higher symmetry (9), and the associated system (2), we have, respectively, representations of the form

$$W_x = T(U)W - WU, \qquad W_t = T(V)W - WV, \qquad U_t = V_x + [V, U], \tag{13}$$

where T is the shift operator $n \mapsto n+1$, and we omit the index n for short, so that U denotes U_n and $T(U)$ denotes U_{n+1}. The representations mean that (1) defines the auto-Bäcklund transformation for the DNLS equation (2). For the master symmetry (8), the x-dependent higher symmetry corresponding to the differentiation ∂_τ, and the master symmetry (10) of (2), we have representations of the form

$$W_y + \mu\lambda^3 W_\lambda = T(Y)W - WY, \quad W_\tau + \mu\lambda^3 W_\lambda = T(xV+Y)W - W(xV+Y),$$
$$U_\tau + \mu\lambda^3 U_\lambda = (xV+Y)_x + [xV+Y, U], \tag{14}$$

where $\mu = 2$ and the matrices U, V, W, Y are

$$U = 2\lambda \begin{pmatrix} \lambda & u \\ -v & -\lambda \end{pmatrix}, \qquad Y = cU + \lambda \begin{pmatrix} (2n-1)\lambda & 2nu \\ -(2n-2)v & -(2n-1)\lambda \end{pmatrix},$$
$$V = 4\lambda^2 U + 2\lambda \begin{pmatrix} 2\lambda uv & u_x + 2u^2 v \\ v_x - 2uv^2 & -2\lambda uv \end{pmatrix}, \qquad W = \begin{pmatrix} 2\lambda/u & 1 \\ -1 & 0 \end{pmatrix}.$$

The representations (14) show that (1) also defines the auto-Bäcklund transformation for (10) and enable one to trace the connection between the discrete and continuous master symmetries at the level of the zero curvature representations.

All the equations under consideration are Hamiltonian. Let us briefly discuss the Hamiltonian structures of the lattice equations. In the scalar case of (1) and (8), the Poisson bracket is defined for formal functionals of the form $F = \sum_n f_n(u_n, u_{n+1}, \dots, u_{n+r})$ and is given by the following formula (we use square brackets to avoid a confusion with the multiplication in J),

$$[F, G] = \sum_n \frac{\delta f_n}{\delta u_n} K_n \frac{\delta g_n}{\delta u_n}.$$

Here $\delta f_n / \delta u_n$ denotes the formal variational derivative,

$$\frac{\delta f_n}{\delta u_n} = \frac{\partial}{\partial u_n}(1 + T^{-1} + \dots + T^{-r})(f_n),$$

and K_n is the Hamiltonian operator,

$$K_n = \sum_{m=-M}^{m=M} K_{nm} T^m,$$

which, of course, must be skew symmetrical and satisfy the Jacobi identity. The Hamiltonian lattice generated by the functional F is of the form

$$u_{n,t} = K_n \frac{\delta f_n}{\delta u_n}.$$

The Hamiltonian operator corresponding to (1) is

$$K_n = u_n^2 (T - T^{-1}) u_n^2. \tag{15}$$

It is easy to prove that in the variables $z_n = u_n^{-1}$ this operator becomes constant, and therefore the Jacobi identity is fulfilled automatically. The lattice (1) possesses an infinite set of local conservation laws,

$$\partial_x(\rho_n^k) = (T-1)(\sigma_n^k), \quad k=-1,0,1,\dots,$$

with densities

$$\rho_n^{-1} = \frac{c_n}{u_n}, \qquad \rho_n^0 = \log u_n, \qquad \rho_n^1 = u_n u_{n+1}, \qquad \rho_n^2 = 2u_n u_{n+1}^2 u_{n+2} + u_n^2 u_{n+1}^2, \quad \dots,$$

where c_n is defined by the relation $c_{n+2} = c_n$. The local master symmetry (8) enables one to obtain the conserved densities ρ_n^k ($k \geqslant 1$) recursively, using the formula

$$\partial_y(\rho_n^k) = \rho_n^{k+1} + (T-1)(h_n^k), \tag{16}$$

which means that $\rho_{n,y}^k$ and ρ_n^{k+1} are the same up to a total difference. The densities ρ_n^k ($k \geqslant 0$) define Hamiltonians $H_k = \sum_n \rho_n^k$ of (1) and its higher symmetries. The Hamiltonian H_{-1} is the annihilator of the Poisson structure (15). The master symmetry (8) also is Hamiltonian,

$$u_{n,y} = K_n \frac{\delta \widetilde{\rho}_n^0}{\delta u_n}, \qquad \widetilde{\rho}_n^0 = (c+n)\log u_n,$$

and ρ_n^{-1} is its conserved density.

Obviously, the lattice systems (11), (12) are Hamiltonian as well, and the Poisson bracket is defined in a similar way. The system (11), the master symmetry (12), and higher symmetries are expressed in the form

$$p_{n,z} = p_n^2 \frac{\delta \rho_n}{\delta q_n}, \qquad q_{n,z} = -p_n^2 \frac{\delta \rho_n}{\delta p_n}.$$

The Hamiltonians of (11) and (12) are $H = \sum \rho_n^0$ and $\widetilde{H} = \sum \widetilde{\rho}_n^0$, where

$$\rho_n^0 = \log \frac{q_n - q_{n+1}}{p_n}, \qquad \widetilde{\rho}_n^0 = (c+2n+1)\log(q_n - q_{n+1}) - (c+2n)\log p_n.$$

The function $\rho_n^{-1} = 1/p_n$ is the common conserved density of (11) and (12). Starting from ρ_n^0 and using the same formula (16), one can construct the other densities for (11), the first of which is

$$\rho_n^1 = \frac{p_{n+1} + p_n}{q_{n+1} - q_n}.$$

3. Main equations

Statements below can be proved by straightforward calculations, using the identities (3), (4), and their consequences,

$$\{ab\{aca\}\} = \{a\{bac\}a\} = \{\{aba\}ca\}, \qquad \{a\{bab\}c\} = \{\{aba\}bc\},$$
$$2\{\{abc\}bd\} = \{a\{bcb\}d\} + \{c\{bab\}d\}. \tag{17}$$

Also, it is convenient to use linear operators $L_{ab}, P_{ab}, P_a : J \to J$ which are defined for all elements $a, b \in J$ by

$$L_{ab}(c) = \{abc\}, \qquad P_{ab}(c) = \{acb\}, \qquad P_a(c) = P_{aa}(c) = \{aca\}. \tag{18}$$

Eq. (5) is expressed in terms of P_a as follows,

$$u_{n,x} = P_{u_n}(u_{n+1} - u_{n-1}) . \tag{19}$$

The following multi-component lattice equation,

$$u_{n,y} = P_{u_n}((c+n+1)u_{n+1} - (c+n-1)u_{n-1}) , \tag{20}$$

generalises the master symmetry (8).

Theorem 1. For any JTS, the lattice

$$u_{n,t} = P_{u_n}(P_{u_{n+1}}(u_{n+2} + u_n) - P_{u_{n-1}}(u_n + u_{n-2})) \tag{21}$$

obtained by the formula $\partial_t = [\partial_y, \partial_x]$ is the higher symmetry of the multi-component Volterra equation (19) (i.e. $[\partial_t, \partial_x] = 0$). It can be rewritten as the system (6) for $u = u_n$, $v = u_{n-1}$, and the symmetry of (19) corresponding to $\partial_\tau = x\partial_t + \partial_y$ is equivalent to the system

$$u_\tau = (xu_x + 2x\{uvu\} + (c+n)u)_x , \qquad v_\tau = (-xv_x + 2x\{vuv\} + (c+n-1)v)_x . \tag{22}$$

Theorem 1 proves, in a sense, that (20) is the master symmetry of the multi-component Volterra equation, i.e. generates higher symmetries and conservation laws in the standard way. Using the Jacobi identity for evolution differentiations and the fact that $[\partial_t, \partial_x] = 0$, one can easily prove that $[\partial_{\tilde{t}}, \partial_x] = 0$, where $\partial_{\tilde{t}} = [\partial_y, \partial_t]$, i.e. (20) generates a higher symmetry on the second step as well. A statement that (20) enables one to construct an infinite hierarchy of higher symmetries is more difficult to prove, and additional properties must be used (the homogeneity of equations, for instance). It should be remarked that in the other cases we also restrict ourselves to checking the determining equation for the master symmetries (for instance, in the case of (22), it has the form $[[\partial_\tau, \partial_t], \partial_t] = 0$), that turns out to be sufficient in all known examples.

Note that the identities (3), (4) are not only sufficient for the compatibility of (19) and (21) but also necessary, i.e. turn out to be the compatibility conditions for the lattices (19), (21) considered in an arbitrary ternary algebra.

Now we are going to present Jordan analogs of (11), (12) and transformations reducing them to (19), (20). In order to give the proper analog of the term $1/(q_{n-1} - q_n)$, we define the inverse element as $a^{-1} = P_a^{-1}(a)$. The following Lemma shows that this expression has many habitual features compared to the scalar case.

Lemma 1. Let the operator P_a be invertible, and $b = a^{-1} = P_a^{-1}(a)$. Then P_b is invertible as well, and

$$P_b^{-1} = P_a , \qquad a = P_b^{-1}(b) , \qquad L_{ab} = I , \qquad a_x = -P_a(b_x) .$$

Proof. First we prove that $P_bP_a = I$. Let c be an element of J, and $d = P_a(c)$, $e = P_b(d)$. Using (17), we obtain

$$P_a(e) = \{a\{bdb\}a\} = 2\{ab\{abd\}\} - \{a\{bab\}d\}$$
$$= 2\{ab\{ab\{aca\}\}\} - \{\{aba\}bd\} = \{ab\{aca\}\} = \{aca\} = P_a(c) ,$$

and therefore $e = c$. The formula $a = P_b^{-1}(b)$ is obvious. For arbitrary $c \in J$,

$$L_{ab}(c) = \{ab\{aP_a^{-1}(c)a\}\} = \{\{aba\}P_a^{-1}(c)a\} = \{aP_a^{-1}(c)a\} = c ,$$

i.e. $L_{ab} = I$. Taking this into account and differentiating the relation $a = \{aba\}$, we are led to the last formula.□

Unfortunately, the operator P_a may be degenerate for some JTS. In such cases, the notion of the inverse element can be partly substituted by the notion of the deformation vector which is a solution of the system $\partial b/\partial a = -P_b$ [24]. We will restrict our consideration only to the case when $\det P_a \neq 0$ for almost all $a \in J$. The JTS with this property is called the JTS with invertible elements and admits the following generalization of the lattices (11), (12),

$$q_{n,x} = p_n\,, \qquad p_{n,x} = P_{p_n}((q_n - q_{n+1})^{-1} - (q_{n-1} - q_n)^{-1})\,, \tag{23}$$

$$q_{n,y} = (c+2n)p_n\,, \qquad p_{n,y} = P_{p_n}((c+2n+1)(q_n - q_{n+1})^{-1} - (c+2n-1)(q_{n-1} - q_n)^{-1})\,. \tag{24}$$

Using Lemma 1, one easily proves that the transformation $u_{2n} = p_n$, $u_{2n-1} = (q_{n-1} - q_n)^{-1}$ turns (23) and (24) into (19) and (20), respectively, in full analogy to the scalar case.

In conclusion, we present the most important examples of JTS and some of corresponding multi-component integrable equations. It should be remarked that those examples together with the symmetric (or skew-symmetric) reductions $u = \pm u^+$ of Examples 1 and 2 (with $M = N$) cover all the simple JTS aside from two exceptional ones [9].

Example 1. A linear space J of $N \times N$ matrices becomes the JTS if one defines the triple product with the help of the standard matrix multiplication as follows,

$$\{abc\} = \tfrac{1}{2}(abc + cba)\,.$$

The corresponding matrix Volterra equation reads

$$u_{n,x} = u_n(u_{n+1} - u_{n-1})u_n\,. \tag{25}$$

The operator P_a in this JTS is invertible iff $\det a \neq 0$, that is almost everywhere. The element a^{-1} coincides with the inverse matrix. The subspaces of symmetric and skewsymmetric matrices with the same triple product are the JTS as well. However, in the case of skewsymmetric matrices of the odd order, the operator P_a is degenerate for all a.

Example 2. The previous example can be generalised if one defines the triple product of $N \times M$ matrices by

$$\{abc\} = \tfrac{1}{2}(ab^+c + cb^+a)\,,$$

where $+$ denotes the transposition. The matrix Volterra equation in this case will have the form

$$u_{n,x} = u_n(u^+_{n+1} - u^+_{n-1})u_n\,. \tag{26}$$

If $N = M$, we easily can obtain (25): $u_{2n} \to u_{2n}$, $u_{2n+1} \to u^+_{2n+1}$. The operator P_a may be invertible only in the case $M = N$.

In the particular case $M = 1$, J turns into an N-dimensional vector space with the multiplication

$$\{abc\} = \tfrac{1}{2}(\langle a,b\rangle c + \langle c,b\rangle a)\,, \tag{27}$$

where $\langle\ \rangle$ denotes the standard scalar product, and (26) takes the form

$$u_{n,x} = \langle u_n, u_{n+1} - u_{n-1}\rangle u_n\,. \tag{28}$$

However, this lattice is not interesting, since it easily can be reduced to a scalar one. Indeed, all the coordinates of the vector u_n are proportional to each other, i.e. $u_n = \omega_n c_n$, where c_n is a constant vector, and $\omega_n(x)$ is a scalar function. So, the lattice is equivalent to

$$\omega_{n,x} = \omega_n^2(\gamma_n\omega_{n+1} - \gamma_{n-1}\omega_{n-1})\,,$$

and the constant factors $\gamma_n = \langle c_{n+1}, c_n \rangle$ can easily be removed by scaling. In spite of this degeneracy, higher symmetries of (28), the corresponding vector DNLS equation, its higher symmetries and master symmetry are nontrivial vector equations. The vector DNLS equation can be found in Ref. [14], and the multi-component master symmetry (22) turns into the following interesting example of an integrable vector equation

$$u_\tau = (xu_x + 2x\langle u, v\rangle u + (c+n)u)_x, \quad v_\tau = (-xv_x + 2x\langle v, u\rangle v + (c+n-1)v)_x.$$

Example 3. A nontrivial vector example generalizing the Volterra equation is related to the rule

$$\{abc\} = \langle a, b\rangle c + \langle c, b\rangle a - \langle a, c\rangle b, \tag{29}$$

which defines the structure of the JTS in an N-dimensional vector space as well as (27). The operator P_a, its inverse, and the vector a^{-1} are given by

$$P_a(b) = 2\langle a, b\rangle a - \langle a, a\rangle b, \qquad P_a^{-1} = \langle a, a\rangle^{-2} P_a, \qquad a^{-1} = \langle a, a\rangle^{-1} a. \tag{30}$$

The vector Volterra equation in this case reads

$$u_{n,x} = 2\langle u_n, u_{n+1} - u_{n-1}\rangle u_n - \langle u_n, u_n\rangle (u_{n+1} - u_{n-1}). \tag{31}$$

Vector analogs of the DNLS equation (2), the Toda type lattice equation (11), and all the three master symmetries also can be written down without any difficulties.

4. Zero curvature representations

In this section we discuss the zero curvature representations (13), (14) in the Jordan case for the main of the equations presented here. Naturally, models corresponding to different JTS admit zero curvature representations in matrices of a different size. Nevertheless, it is possible to give uniform representations

$$U_t = V_x + [V, U], \qquad U_\tau - \lambda^3 U_\lambda = (xV + Y)_x + [xV + Y, U], \tag{32}$$

for all the systems (6), (22) in terms of the superstructure Lie algebra $K(J)$ of the JTS. Notice that the multiplier μ in (14) can be set to an arbitrary constant by rescaling λ, and sometimes it is convenient to use different values. Zero curvature representations

$$W_x = T(\widehat{U})W - W\widehat{U}, \qquad W_y - \lambda^3 W_\lambda = T(\widehat{Y})W - W\widehat{Y}, \tag{33}$$

for the lattice equations (19), (20) cannot be defined in Lie-algebraic terms. However, the transition $U \mapsto \operatorname{ad} U = \widehat{U}$, $Y \mapsto \operatorname{ad} Y = \widehat{Y}$ allows one to obtain the matrix realization of the representations (32) as well and then to find a matrix W which, of course, is not of $\operatorname{ad} K(J)$. The problem of the realization in matrices of a minimal size is reduced to the studying matrix representations of the Lie algebra $K(J)$.

Let us remind the definition of the superstructure Lie algebra $K(J)$ or the Tits–Cantor–Koecher construction (see e.g. Ref. [11] for details). At first, one should define the structure Lie algebra $\operatorname{strl} J$ of operators $J \to J$ spanned over the multiplication operators L_{ab} (18) and the identity operator I. The commutator in this algebra is the usual operator commutator $[A, B] = AB - BA$. Note that the identity (4) is equivalent to the relation

$$[L_{ab}, L_{cd}] = L_{\{cba\}d} - L_{c\{bad\}}, \tag{34}$$

which shows that the commutator does not lead out of $\operatorname{strl} J$. The superstructure Lie algebra is defined as the direct sum $K(J) = J \oplus \operatorname{strl} J \oplus J$ with the commutator

$$[(a, M, b), (c, N, d)] = (M(c) - N(a),\ [M, N] + L_{ad} - L_{cb},\ \sigma M(d) - \sigma N(b)),$$

where the operator $\sigma : \mathrm{strl}\, J \to \mathrm{strl}\, J$ acts as follows,

$$\sigma I = -I, \qquad \sigma L_{ab} = -L_{ba}.$$

The following statement is proved immediately.

Theorem 2. The Jordan DNLS equation (6) and its master-symmetry (22) admit the zero curvature representations (32) with $U, V, Y \in K(J)$ which are of the form

$$U = 2\lambda(u, -\lambda I, v), \qquad Y = cU + \lambda(2nu, -(2n-1)\lambda I, (2n-2)v),$$
$$V = -2\lambda^2 U + 2\lambda(u_x + 2\{uvu\}, -2\lambda L_{uv}, -v_x + 2\{vuv\}).$$

Remark. In Ref. [2], multi-component analogs of the NLS equation of the form

$$u_t = u_{xx} - \{uvu\}, \qquad v_t = -v_{xx} + \{vuv\}, \quad u, v \in J, \tag{35}$$

have been proved to be integrable if J is some JTS. These integrable systems also admit the representation of (32) which, as in the case of (6), can be realized in terms of $K(J)$,

$$U = (u, \lambda I, v), \qquad V = \lambda U + (u_x, \ -L_{uv}, \ -v_x).$$

Before Ref. [2], integrable systems of the form (35) were considered in Ref. [12] as systems possessing an L–A pair associated with Hermitian symmetric spaces.

In order to obtain the adjoint representation in $K(J)$, we have to write down the operator $\mathrm{ad}(a, M, b)$ for an arbitrary element of $K(J)$. Introduce the following operators,

$$i_a : \mathrm{strl}\, J \to J, \quad i_a(M) = M(a); \qquad l_a : J \to \mathrm{strl}\, J, \quad l_a(b) = L_{ab}.$$

It is easy to verify that

$$i_a l_b = P_{ab}, \qquad i_a \sigma l_b = -L_{ab}. \tag{36}$$

In terms of these operators, $\mathrm{ad}(a, M, b)$ is written as the block matrix

$$\mathrm{ad}(a, M, b) = \begin{pmatrix} M & -i_a & 0 \\ \sigma l_b & \mathrm{ad}\, M & l_a \\ 0 & -i_b\sigma & \sigma M \end{pmatrix},$$

where $\mathrm{ad}\, M$ denotes the adjoint representation in $\mathrm{strl}\, J$.

Now we can find the matrix W of (33), assuming that $\widehat{U} = \mathrm{ad}\, U$, $\widehat{Y} = \mathrm{ad}\, Y$, i.e.

$$\widehat{U} = 2\lambda \begin{pmatrix} -\lambda I & -i_u & 0 \\ \sigma l_v & 0 & l_u \\ 0 & -i_v\sigma & \lambda I \end{pmatrix}, \qquad \widehat{Y} = c\widehat{U} + \lambda \begin{pmatrix} -(2n-1)\lambda I & -2n i_u & 0 \\ (2n-2)\sigma l_v & 0 & 2n l_u \\ 0 & -(2n-2) i_v \sigma & (2n-1)\lambda I \end{pmatrix}. \tag{37}$$

The answer is simple enough, however it only exists for the JTS with the invertible P_u.

Theorem 3. Let J be the JTS with the invertible operator P_u, then the multi-component Volterra equation (19) and its master symmetry (20) admit the representations (33) with matrices $\widehat{U}, \widehat{Y}$ given by the formulae (37) and matrix W of the form

$$W = \begin{pmatrix} -2\lambda^2 P_u^{-1} & -2\lambda P_u^{-1} i_u & I \\ 2\lambda\sigma l_u P_u^{-1} & \sigma & 0 \\ I & 0 & 0 \end{pmatrix}$$

(it is assumed that $u = u_n$, $v = u_{n-1}$).

The upper bound for the size of matrices in the zero curvature representations (33) is $(d+1)^2$ where $d = \dim J$. For a concrete JTS, it is possible to find more compact representations. For instance, the vector Volterra lattice (31) of Example 3 and its master symmetry admit representations (14) with $\mu = -1$ in $(d+2) \times (d+2)$ matrices,

$$U = 2\lambda \begin{pmatrix} \lambda & u^+ & 0 \\ -v & 0 & -u \\ 0 & v^+ & -\lambda \end{pmatrix}, \qquad W = \begin{pmatrix} -2\lambda^2/\langle u,u\rangle & -2\lambda u^+/\langle u,u\rangle & 1 \\ 2\lambda u/\langle u,u\rangle & I & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$Y = cU + \lambda \begin{pmatrix} (2n-1)\lambda & 2nu^+ & 0 \\ -(2n-2)v & 0 & -2nu \\ 0 & (2n-2)v^+ & -(2n-1)\lambda \end{pmatrix},$$

where u is the column vector, and I is the $d \times d$ unit matrix. For the corresponding vector DNLS equation and its master symmetry, the matrix V is of the form

$$V = 2\lambda^2 U + 2\lambda \begin{pmatrix} 2\lambda\langle u,v\rangle & u_x^+ + 2\{uvu\}^+ & 0 \\ v_x - 2\{vuv\} & 2\lambda(uv^+ - vu^+) & -u_x - 2\{uvu\} \\ 0 & -v_x^+ + 2\{vuv\}^+ & -2\lambda\langle u,v\rangle \end{pmatrix}.$$

The zero curvature representations (14) with $\mu = 2$ for the continuous vector equations of Example 2 are given by the matrices (cf. with the scalar case)

$$U = 2\lambda \begin{pmatrix} \lambda & u^+ \\ -v & -\lambda I \end{pmatrix}, \qquad V = 4\lambda^2 U + 2\lambda \begin{pmatrix} 2\lambda\langle u,v\rangle & u_x^+ + 2\langle u,v\rangle u^+ \\ v_x - 2\langle u,v\rangle v & -2\lambda v u^+ \end{pmatrix},$$

$$Y = cU + \lambda \begin{pmatrix} (2n-1)\lambda & nu^+ \\ -(2n-2)v & -(2n-1)\lambda I \end{pmatrix}.$$

5. Hamiltonian structures

In this section we assume that J is the JTS with invertible elements, and the bilinear form $\langle a,b\rangle = \operatorname{tr} L_{ab}$ is symmetric and nondegenerate. Note that for the vector examples (27), (29), the bilinear form coincides with the standard scalar product, and no confusion happens. For Examples 1 and 2, one correspondingly has $\langle a,b\rangle = \operatorname{tr} ab$ and $\langle a,b\rangle = \operatorname{tr} ab^+$. The bilinear form is symmetric and nondegenerate in all these examples. The invariance property

$$\langle a, \{bcd\}\rangle = \langle\{abc\}, d\rangle$$

is a consequence of formula (34).

The Poisson bracket is defined in the case of the lattices (19) and (20) in the following way,

$$[F,G] = \sum_n \left\langle \frac{\delta f_n}{\delta u_n}, K_n \frac{\delta g_n}{\delta u_n} \right\rangle,$$

the definition of the formal variational derivative and Hamiltonian lattice remains formally the same as in the scalar case, but the partial derivative $\partial f/\partial u$ now must be understood as the gradient. The Jordan analog of the operator (15) is quite natural,

$$K_n = P_{u_n}(T - T^{-1})P_{u_n}.$$

As in the scalar case, K_n becomes constant in the variables $z_n = u_n^{-1}$, and that is why it is easy to see that K_n is Hamiltonian. This Poisson structure is highly degenerate: using Lemma 1, one easily can prove that the density $c_n u_n^{-1}$ (with $c_{n+2} = c_n$) is its annihilator.

Theorem 4. The Jordan Volterra equation (19) and its master symmetry (20) are Hamiltonian, and corresponding Hamiltonian densities are $\rho_n = \frac{1}{2}\log\det(P_{u_n})$, $\widetilde{\rho}_n = [(c+n)/2]\log\det(P_{u_n})$.

Proof. It is sufficient to prove that $\partial\rho/\partial u = u^{-1}$ for $\rho = \frac{1}{2}\log\det(P_u)$, where ∂/∂_u denotes gradient

$$\left\langle \frac{\partial f}{\partial u}, v \right\rangle = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(u + \varepsilon v)|_{\varepsilon=0}.$$

We have for any a,

$$2\left\langle \frac{\partial \rho}{\partial u}, a \right\rangle = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\log\det(P_{u+\varepsilon a})|_{\varepsilon=0} = 2\,\mathrm{tr}(P_u^{-1}P_{au}).$$

For arbitrary b,

$$P_u^{-1}P_{au}(b) = P_u^{-1}(\{\{uP_u^{-1}(a)u\}bu\}) = P_u^{-1}(\{u\{P_u^{-1}(a)ub\}u\}) = \{P_u^{-1}(a)ub\} = L_{P_u^{-1}(a)u}(b),$$

and we obtain

$$\left\langle \frac{\partial \rho}{\partial u}, a \right\rangle = \langle P_u^{-1}(a), u\rangle = \langle u^{-1}, a\rangle.$$

The lattices (23) and (24) also can be written in the Hamiltonian form,

$$p_{n,z} = P_{p_n}\frac{\delta\rho_n}{\delta q_n}, \qquad q_{n,z} = -P_{p_n}\frac{\delta\rho_n}{\delta p_n}.$$

The corresponding Hamiltonian densities are

$$2\rho_n = \log\det P_{q_n - q_{n+1}} - \log\det P_{p_n}, \qquad 2\widetilde{\rho}_n = (c + 2n + 1)\log\det P_{q_n - q_{n+1}} - (c + 2n)\log\det P_{p_n}.$$

Using the master symmetries and the formula (16), we can construct higher conserved densities for the Jordan lattices (19) and (23) if we start from the above Hamiltonian densities ρ_n. The simplest higher conservation laws read

$$\langle u_n, u_{n+1}\rangle_x = (T-1)\langle\{u_n u_{n-1} u_n\}, u_{n+1}\rangle,$$

$$(\langle u_n, \{u_{n+1}u_{n+2}u_{n+1}\}\rangle + \tfrac{1}{2}\langle u_n, \{u_{n+1}u_n u_{n+1}\}\rangle)_x = (T-1)\langle\{u_n u_{n-1} u_n\}, \{u_{n+1}(u_{n+2} + u_n)u_{n+1}\}\rangle$$

in the case of (19), and

$$\langle p_n + p_{n+1}, (q_n - q_{n+1})^{-1}\rangle_x = (T-1)\langle\{p_n(q_{n-1} - q_n)^{-1}p_n\}, (q_n - q_{n+1})^{-1}\rangle$$

in the case of (23).

6. Conclusion

There are only a few papers devoted to the construction and investigation of multi-component integrable lattices, using the Jordan algebraic structures [7,8,25]. Multi-component lattices generalised in those papers the Toda model and an integrable approximation of the NLS equation. Here we have succeeded in finding the Jordan analogs of the Volterra equation (5). Their modifications (23) are lattice equations of the Toda type. Also, we have considered here the known multi-component DNLS equations (6) (see Ref. [14]).

The master symmetries (20), (22), (24) are, as far as we know, the first examples of multi-component master symmetries. They are local and, for this reason, give an easy way to construct higher symmetries and conservation laws. On the other hand, they exemplify local integrable multi-component equations with an essential dependence on the spatial variable.

An L–A pair has been written down for the main of equations we consider, their master symmetries, and even for the Jordan NLS equation. If L–A pairs for continuous equations in terms of the superstructure Lie algebra of the JTS arose before [24], L–A pairs for discrete ones in terms of the adjoint representation of the superstructure Lie algebra have appeared for the first time.

Hamiltonian structures, higher symmetries, and conservation laws have been given in the general algebraic form as well. Nevertheless, one always can write down vector and matrix examples of equations, higher and master symmetries, etc., and obtain matrix realization of L–A pairs, and we have done this in most interesting cases.

Lattices of the Toda type similar to (11) can be useful for studying integrable many-body models [26]. We hope the multi-component lattice equations (23), (24) will be of interest from this point of view as well.

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