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Necessary and Sufficient Conditions for Zero Subsets of Holomorphic Functions with Upper Constraints in Planar Domains

B. N. Khabibullin[1*] and F. B. Khabibullin[2]**

(Submitted by F. G. Avkhadiev)

[1] *Bashkir State University, Bashkortostan, 420076 Russian Federation*

[2] *Ufa State Aviation Technical University, Bashkortostan, 420008 Russian Federation*

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Abstract—Let D be a domain in the complex plane and M be an extended real-valued function on D. If f is a non-zero holomorphic function on D such that $|f| \leq \exp M$, then it is natural to expect that there should be some upper boundedness for the distribution of the zeros of f expressed exclusively in terms of the function M and the geometry of the domain D. We have investigated this question in detail in our previous works in the case when M is a subharmonic function and the domain D either is arbitrary or has a non-polar boundary. The answer was given in terms of constraints to the distribution of zeros of f from above via the Riesz measure of the subharmonic function M. In this article, the function M is the difference of subharmonic functions, or a δ-subharmonic function, and the upper constraints are given in terms of the Riesz charge of this δ-subharmonic function M. These results are also new to a certain extent for the subharmonic function M. The case when the domain D coincides with the whole complex plane is considered separately. For the complex plane, it is possible to reach the criterion level of our results.

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1. INTRODUCTION

As usual, $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{R}$, $\mathbb{R}^+ := \{x \in \mathbb{R} \bigm| x \geq 0\}$, $\mathbb{C}$ are the sets of all *natural, real, positive,* and *complex* numbers, respectively. We often denote singleton sets $\{x\}$ simply by x, without the use of curly braces. Let $\mathbb{N}_0 := 0 \cup \mathbb{N}$, $\mathbb{R}^+ \setminus 0$ be the set of all *strictly positive numbers,* $\overline{\mathbb{R}} := -\infty \cup \mathbb{R} \cup +\infty$ be the *extended real line,* $\overline{\mathbb{R}}^+ := \mathbb{R}^+ \cup +\infty$, and $\mathbb{C}_\infty := \mathbb{C} \cup \infty$ be the *extended complex plane.* Each of this sets is endowed with its natural order, algebraic, geometric, and topological structure. For $z \in \mathbb{C}$, we denote by $|z|$ its modulus and by $\bar{z}$ its conjugate. For the *empty set* $\varnothing$ we put $\inf \varnothing := +\infty := \sup \mathbb{R}$ and $\sup \varnothing := -\infty := \inf \mathbb{R}$. For a subset $S \subset \mathbb{C}_\infty$, let $\mathsf{Hol}(S)$ denote the algebra over $\mathbb{C}$ of all *holomorphic functions* f on open subsets $O_f \supset S$ in $\mathbb{C}_\infty$. Therefore, $\mathsf{Hol}(\mathbb{C})$ is the algebra of all *entire functions.*

Let S be a set, J be an *index set,* and $\mathsf{Z} := \{\mathsf{z}_j\}_{j \in J}$ be an *indexed set* of points $\mathsf{z}_j \in S$. Such indexed sets will be called *distributions of points* in S. We write $z \in \mathsf{Z}$ if there is z_j such that $\mathsf{z}_j = z$. For a subset $S' \subset S$ we write $\mathsf{Z} \subset S'$ if $\mathsf{z}_j \in S'$ for each $j \in J$. The *counting function* $n_{\mathsf{Z}} \colon S \to \overline{\mathbb{N}}$ of Z is defined by

$$n_{\mathsf{Z}}(z) := \sum_{\mathsf{z}_j = z} 1 \in \overline{\mathbb{N}}_0 := \mathbb{N}_0 \cup +\infty \quad \text{at each } z \in S. \tag{1z}$$

*E-mail: khabib-bulat@mail.ru

**E-mail: khabibullinfb@mail.ru

We denote by the same symbol $n_{\mathsf Z}$ the *counting measure* $n_{\mathsf Z}\colon 2^S\to\overline{\mathbb N}_0$ of $\mathsf Z$ defined by

$$n_{\mathsf Z}(S')\overset{(1z)}{:=}\sum_{\mathsf z_j\in S'}1=\sum_{z\in S'}n_{\mathsf Z}(z)\in\overline{\mathbb N}\quad\text{for each } S'\subset S. \tag{1n}$$

We say that distributions $\mathsf Z$ and $\mathsf Z'$ of points in S are equal and write $\mathsf Z=\mathsf Z'$ if $n_{\mathsf Z}=n_{\mathsf Z'}$ on S. Thus, both the counting function and the counting measure uniquely determine distributions of points in S. We write $\mathsf Z\subset\mathsf Z'$ if $n_{\mathsf Z}\leq n_{\mathsf Z'}$ on S.

Let $O\neq\varnothing$ be an open subset in $\mathbb C$. The *zero set* of $f\in\mathsf{Hol}(O)$ is a distribution Zero_f of points in O with the *counting function* of the *multiplicity of zeros* of f defined by

$$n_{\mathsf{Zero}_f}(z)\underset{z\,\in\,O}{:=}\sup\left\{p\in\mathbb N_0\,\middle|\,\limsup_{z\neq z'\to z}\frac{|f(z)|}{|z'-z|^p}<+\infty\right\}.$$

Therefore, for *the zero function* $0\in\mathsf{Hol}(O)$, we have $n_{\mathsf{Zero}_0}(z)\overset{(1z)}{=}+\infty$ for each $z\in O$ and $n_{\mathsf{Zero}_0}(S)\overset{(1n)}{=}+\infty$ for each $S\subset O$. The counting function n_{Zero_f} of $f\in\mathsf{Hol}(O)$ is often called the *divisor of zeros* of f. If $\mathsf Z\subset\mathsf{Zero}_f$, then we write $f(\mathsf Z)=0$ and say that $\mathsf Z$ is a *zero subset* of f and f *vanishes on* $\mathsf Z$.

For a subset $S\subset\mathbb C_\infty$, we denote by $\complement S:=\mathbb C_\infty\setminus S$, $\mathsf{clos}\,S$, $\mathsf{int}\,S:=\complement(\mathsf{clos}\,\complement S)$, and $\partial S:=\mathsf{clos}\,S\setminus\mathsf{int}\,S$ its *complement, closure, interior,* and *boundary* always considered in the topology of $\mathbb C_\infty$. We write $S\Subset O$ if $\mathsf{clos}\,S\subset O$.

A distribution of points $\mathsf Z\subset O$ is *locally finite* if $n_{\mathsf Z}(S)<+\infty$ for each $S\Subset O$.

Weierstrass Theorem. *Let Z be a distribution of points in an open set $O\subset\mathbb C$. The following three statements are equivalent:*

- *There is $f\in\mathsf{Hol}(O)$ such that $f\neq 0$ and $\mathsf Z=\mathsf{Zero}_f$.*
- *There is $f\in\mathsf{Hol}(O)$ such that $f\neq 0$ and $f(\mathsf Z)=0$.*
- *$\mathsf Z$ is locally finite.*

Throughout this article, we only consider locally finite distributions $\mathsf Z$ of points in an open connected subset $D\subset\mathbb C$, i.e. D is a *domain* in the complex plane $\mathbb C$.

If an additional constraint $\ln|f|\leq M$ on D is imposed on a holomorphic function $f\neq 0$, where $M\colon D\to\overline{\mathbb R}$ is an *extended real-valued function,* then the problem of describing zero sets and zero subsets becomes much more complicated. In particular, zero sets are very often not the same as zero subsets. In our paper, we consider only zero subsets under a restriction from above of the form $\ln|f|\leq M$, when M is the difference of subharmonic functions with the Riesz charge $\varDelta_M$. For subharmonic functions M [1–3], this question was considered earlier in our series of works [4–7]. In this article, we consider the difference $M=M_{\rm up}-M_{\rm low}$ of subharmonic functions $M_{\rm up}$ and $M_{\rm low}$ with the Riesz measures $\varDelta_{M_{\rm up}}$ of $M_{\rm up}$ and $\varDelta_{M_{\rm low}}$ of $M_{\rm low}$ and with the Riesz charge $\varDelta_M:=\varDelta_{M_{\rm up}}-\varDelta_{M_{\rm low}}$ of M. It is shown that if $\ln|f|\leq M$, $f\neq 0$ and $f(\mathsf Z)=0$, then there is a number $C\in\mathbb R^+$ such that

$$\sum_{\mathsf z_j}v(\mathsf z_j)\leq\int v\,\mathrm d\varDelta_M+C \tag{2}$$

for v from a very wide class of positive test functions. There are the necessary conditions, see Sec. 2, Theorem 1, for (2) to be valid. Conversely, if (2) holds for much more narrow subclasses of positive smooth subharmonic test functions v, then we have almost converse statements in the form $\ln|f|\leq M^{\odot}+R$, where $M^{\odot}$ is a certain averaging of M over small disks, and R is a small addition related to the distance to the boundary ∂D of D. In Sec. 3, Theorem 2, we also give some sufficient conditions. In Sec. 3, we consider domains D with *non-polar* boundaries ∂D or, equivalently, with non-polar complements $\complement D$. This very broad class of domains D includes most of the considered in function theory and its applications. If the complement $\complement D$ contains a connected subset with more than one point, then the boundary ∂D is non-polar [1, Corollary 3.6.4], [3, Theorem 5.12]. But the boundary $\partial\mathbb C=\infty\in\mathbb C_\infty$

of $\mathbb{C}$ is polar. For the case of $D=\mathbb{C}$ in Sec. 4, Theorem 3, we obtain a criterion for a wide class of functions $M=M_{\rm up}-M_{\rm low}$ under a single condition: *There exist numbers* $P,C\in\mathbb{R}^+$ *such that*

$$\int\limits_0^{2\pi} M_{\rm up}\Bigl(z+\frac{1}{(1+|z|)^P}e^{i\theta}\Bigr)\,{\rm d}\theta\leq M_{\rm up}(z)+C\quad\text{for each }\ z\in\mathbb{C}. \tag{3}$$

2. NECESSARY CONDITIONS FOR ZERO SUBSETS OF HOLOMORPHIC FUNCTIONS WITH UPPER CONSTRAINTS IN DOMAIN

2.1. Subharmonic Functions and Measures

Let $\mathsf{sbh}(S)$ be the cone over $\mathbb{R}^+$ of all *subharmonic functions* u on open neighborhoods of $S\subset\mathbb{C}_\infty$ including the $(-\infty)$-function $\boldsymbol{-\infty}\colon z\underset{z\in S}{\longmapsto}-\infty$. Let $\mathsf{har}(S):=\mathsf{sbh}(S)\cap\bigl(-\mathsf{sbh}(S)\bigr)$ be the space over $\mathbb{R}$ of all *harmonic functions* u on S. We denote by $\mathsf{sbh}_*(D):=\bigl\{u\in\mathsf{sbh}(D)\bigm| u\neq\boldsymbol{-\infty}\bigr\}$ the class of all *nontrivial* subharmonic functions on a domain $D\subset\mathbb{C}$.

If $u\in\mathsf{sbh}_*(D)$, then its *Riesz measure* is denoted by

$$\varDelta_u:=\frac{1}{2\pi}\triangle u\in\mathsf{Meas}^+(D),$$

where $\triangle$ is the *Laplace operator* acting in the sense of the theory of distributions or generalized functions, and $\mathsf{Meas}^+(D)$ is the cone over $\mathbb{R}^+$ of *positive Radon measures* on D. But by definition, the Riesz measure of the $(-\infty)$-function on D is such that $\varDelta_{\boldsymbol{-\infty}}(S)=+\infty$ for each $S\subset D$.

If $f\in\mathsf{Hol}(D)$, then $\ln|f|\in\mathsf{sbh}(D)$ and $n_{\mathsf{Zero}_f}=\varDelta_{\ln|f|}$ [1, 3.7.8].

Given $f\in F\subset\overline{\mathbb{R}}^X$, we set $f^+\colon x\mapsto\max\{0,f(x)\}$, and $F^+:=\{f\in F\colon f=f^+\}$. For a sequence $(f_k)_{k\in\mathbb{N}}\subset\overline{\mathbb{R}}^X$, we write $f_k\underset{k\to\infty}{\nearrow}f$ if this sequence $(f_k)_{k\in\mathbb{N}}$ is increasing and $f=\lim\limits_{k\to\infty}f_k$;

$$F^{\uparrow}:=\Bigl\{f\in\overline{\mathbb{R}}^X\Bigm|\exists(f_k)\subset F,\ f_k\underset{k\to\infty}{\nearrow}f\Bigr\}. \tag{4}$$

Let $\mathsf{sbh}^+(S):=\bigl(\mathsf{sbh}(S)\bigr)^+$ be the class of all positive subharmonic functions on S. For a closed subset $S\Subset D$ we define the class [4, Definition 1], [6]

$$\mathsf{sbh}_0^+(D\setminus S;\leq b):=\Bigl\{v\in\mathsf{sbh}^+(D\setminus S)\Bigm| v\leq b\text{ on }D\setminus S,\quad \limsup_{D\ni z'\to z}v(z')\underset{z\,\in\,\partial D}{=}0\Bigr\} \tag{5}$$

of *positive subharmonic test functions* v *in* $D\setminus S$ *with an upper bound* $b\in\mathbb{R}^+$, and the class

$$\mathsf{sbh}_0^{+\uparrow}(D\setminus S;\leq b)\overset{(4)}{:=}\bigl(\mathsf{sbh}_0^+(D\setminus S;\leq b)\bigr)^{\uparrow}$$

of *upper positive test functions* v *in* $D\setminus S$ *with an upper bound* $b\in\mathbb{R}^+$.

The difference of two nontrivial subharmonic functions is called a *nontrivial d-subharmonic function,* or a *nontrivial δ-subharmonic function.* A *d-subharmonic function* [8], [4, 3.1], [9]

$$M:=M_{\rm up}-M_{\rm low},\quad M_{\rm up}\in\mathsf{sbh}_*(D),\quad M_{\rm low}\in\mathsf{sbh}_*(D),\quad \varDelta_M:=\varDelta_{M_{\rm up}}-\varDelta_{M_{\rm low}}, \tag{6}$$

with the *Riesz charge* $\varDelta_M$ of M is defined at each point $z\in D$ where $M_{\rm low}(z)\neq-\infty$. Below we put $M(z):=+\infty$ if $M_{\rm low}(z)=-\infty$.

For a Borel subset $S\subset\mathbb{C}$, we denote by $\mathsf{Meas}(S)$ the class of all Borel signed measures, or *charges,* on S. $\mathsf{Meas}_{\rm cmp}(S)$ is the class of charges $\mu\in\mathsf{Meas}(S)$ with a compact support ${\rm supp}\mu\Subset S$;

$$\mathsf{Meas}^+(S):=(\mathsf{Meas}(S))^+,\quad \mathsf{Meas}^+_{\rm cmp}(S):=(\mathsf{Meas}_{\rm cmp}(S))^+.$$

For a charge $\mu\in\mathsf{Meas}(S)$, we let μ^+, $\mu^-:=(-\mu)^+$ and $|\mu|:=\mu^++\mu^-$ denote its *upper, lower,* and *total variations.* A function $f\colon S\to\overline{\mathbb{R}}$ is called *μ-integrable* if there are four integrals $\int f^{\pm}{\rm d}\mu^{\pm}\in\overline{\mathbb{R}}^+$ such that $\int f^+{\rm d}\mu^++\int f^-{\rm d}\mu^-<+\infty$ when $\int f^-{\rm d}\mu^++\int f^+{\rm d}\mu^-=+\infty$, and vice versa, $\int f^-{\rm d}\mu^++\int f^+{\rm d}\mu^-<+\infty$ when $\int f^+{\rm d}\mu^++\int f^-{\rm d}\mu^-=+\infty$. A μ-integrable function $f\colon S\to\overline{\mathbb{R}}$ is *μ-summable* if $\int|f|{\rm d}|\mu|\neq+\infty$.

2.2. Jensen Measures and its Potentials

A measure $\mu \in \mathsf{Meas}^+_{\mathrm{cmp}}(D)$ is a *Jensen measure for domain $D \subset D$ at $z_0 \in D$* if [10, 4]

$$u(z_0) \leq \int u \mathrm{d}\mu \quad \text{for each } z_0 \in \mathsf{sbh}(D). \tag{7}$$

We denote by $\mathsf{J}_{z_0}(D)$ the class of all these Jensen measures. Obviously, $\mu(D) = 1$ for every $\mu \in \mathsf{J}_{z_0}(D)$. For $\mu \in \mathsf{J}_{z_0}(D)$, the function

$$V_\mu \colon z \longmapsto \int \ln |z' - z| \mathrm{d}\mu - \ln |z|, \quad z \in \mathbb{C} \setminus z_0, \quad V(\infty) := 0, \tag{8}$$

is the *logarithmic potential of $\mu \in \mathsf{J}_{z_0}(D)$ with pole z_0*.

A positive subharmonic function $V \in \mathsf{sbh}^+(\mathbb{C}_\infty \setminus z_0)$ is a *Jensen potential for D with pole $z_0 \in D$* if there is a subset $S_V \Subset D$ such that $V(z) = 0$ for each $z \in D \setminus S_V$ and

$$\left(\limsup_{z_0 \neq z \to z_0} \frac{V(z)}{-\ln |z - z_0|} \leq 1\right) \overset{\text{or}}{\Longleftrightarrow} \left(V(z) \leq \ln \frac{1}{|z - z_0|} + O(1) \text{ as } z_0 \neq z \to z_0\right).$$

We denote by $\mathsf{PJ}_{z_0}(D)$ the class of all such Jensen potentials. If $D' \Subset D$ be a subdomain in domain $D \subset \mathbb{C}$, then its *Green's function* $\mathsf{g}_{D'}(\cdot, z_0)$ with pole $z_0 \in D'$ belongs to $\mathsf{PJ}_{z_0}(D)$.

Lemma 1 [[10, Propositions 1.2, 1.4]]. *Let $D \neq \varnothing$ be a domain in $\mathbb{C}$ and $z_0 \in D$.*

(i) *The mapping $\mathcal{P} \colon \mu \overset{(8)}{\longmapsto} V_\mu$ is an affine bijection from $\mathsf{J}_{z_0}(D)$ on $\mathsf{PJ}_{z_0}(D)$, and*

$$\mathcal{P}^{-1}(V) = \varDelta_V \big|_{\mathbb{C} \setminus z_0} + \left(1 - \limsup_{z_0 \neq z \to z_0} \frac{V(z)}{-\ln |z - z_0|}\right) \delta_{z_0}, \quad V \in \mathsf{PJ}_{z_0}(D). \tag{9}$$

where δ_{z_0} is the Dirac probability measure *with the support* $\operatorname{supp} \delta_{z_0} = z_0$.

(ii) *If $\mu \in \mathsf{J}_{z_0}(D)$, then the following Poisson–Jensen formula holds:*

$$u(z_0) = \int\limits_{D \setminus z_0} u \mathrm{d}\mu - \int\limits_{D} V_\mu \mathrm{d}\varDelta_u \quad \text{for each} \quad u \in \mathsf{sbh}(D) \quad \text{with} \quad u(z_0) \neq -\infty. \tag{10}$$

We denote by $\mathsf{PJ}^{\uparrow}_{z_0}(D) := \big(\mathsf{PJ}_{z_0}(D)\big)^{\uparrow}$ the class of all *test Jensen functions*.

If $D \subset \mathbb{C}$ is a domain with non-polar boundary ∂D, then there is its *Green's function* $\mathsf{g}_D(\cdot, z_0)$ with pole $z_0 \in D$. Moreover, $\mathsf{g}_D(\cdot, z_0)$ is the largest Jensen test function in $\mathsf{PJ}^{\uparrow}_{z_0}(D)$. If the boundary ∂D is polar, then the largest test Jensen function in $\mathsf{PJ}^{\uparrow}_{z_0}(D)$ is the $(+\infty)$-function $\boldsymbol{+\infty} \colon z \underset{z \in D}{\longmapsto} +\infty$.

2.3. Main Result on Necessary Conditions for Zero Subsets in Domains

The main aim of this section is to establish the largest possible range of necessary conditions for the distribution of zero subsets of holomorphic functions $f \in \mathsf{Hol}(D)$ satisfying the upper constraint $\ln |f| \leq M$ on D. We establish these conditions for arbitrary domains in $\mathbb{C}$ and arbitrary d-subharmonic majorants M from (6).

Theorem 1 (necessary conditions). *Let Z be a locally finite distribution of points in a domain $D \subset \mathbb{C}$ and let M be a d-subharmonic function (6). Suppose that there exists a function $f \in$ Hol(D) such that $f \neq 0$, $f(Z) = 0$ and*

$$\ln \big| f(z) \big| \leq M(z) \quad \text{at each } z \in D. \tag{11}$$

Then, for any closed set $S \Subset D$ with int $S \neq \varnothing$ *and for any $b \in \mathbb{R}^+$, there is a number $C \in \mathbb{R}^+$ such that, for each test upper positive function $v \overset{(5)}{\in} \mathsf{sbh}^{+\uparrow}_0(D \setminus S; \leq b)$, we have*

$$\sum_{\mathsf{z}_j \in D \setminus S} v(\mathsf{z}_j) \leq \int\limits_{D \setminus S} v \mathrm{d}\varDelta_M + C \quad \text{provided } v \text{ is } \varDelta_M\text{-summable on } D \setminus S. \tag{12}$$

If $z_0 \in D$, $f(z_0) \neq 0$ and $M_{up}(z_0) + M_{low}(z_0) \neq -\infty$, then there is $C \in \mathbb{R}$ such that (12) holds with the singleton $S := \{z_0\}$ for each test Jensen function $v \in \mathsf{PJ}^{\uparrow}_{z_0}(D)$.

Proof. In the case $v \overset{(5)}{\in} \mathsf{sbh}^+_0(D \setminus S; \leq b)$ without $\uparrow$ we use

Lemma 2 ([4, Main Theorem]). *For any point $z_0 \in \mathsf{int}\, S$ satisfying $M_{up}(z_0) + M_{low}(z_0) \neq -\infty$, any regular (for the Dirichlet problem) domain $\tilde{D} \subset \mathbb{C}$ with the Green function $\mathsf{g}_{\tilde{D}}(\cdot, z_0)$ with pole at $z_0 \in \tilde{D}$ which satisfies the conditions $S \Subset \tilde{D} \subset D$ and $\complement\, \mathsf{clos}\, \tilde{D} \neq \varnothing$, any function $u \in \mathsf{sbh}_*(D)$ satisfying the inequality $u \leq M$ on D, and any test function $v \in \mathsf{sbh}^+_0(D \setminus S; \leq b)$, the following inequality holds:*

$$Cu(z_0) + \int\limits_{D \setminus S} v \mathrm{d}\varDelta_u \leq \int\limits_{D \setminus S} v \mathrm{d}\varDelta_M + \int\limits_{\tilde{D} \setminus S} v \mathrm{d}\varDelta^-_{M_{\rm low}} + C\overline{C}_M, \tag{13}$$

where $C := b \bigm/ \inf_{z \in \partial S} \mathsf{g}_{\tilde{D}}(z, z_0) > 0$ and

$$\overline{C}_M := \int\limits_{\tilde{D} \setminus z_0} \mathsf{g}_{\tilde{D}}(\cdot, z_0) \mathrm{d}\varDelta_M + \int\limits_{\tilde{D} \setminus S} \mathsf{g}_{\tilde{D}}(\cdot, z_0) \mathrm{d}\varDelta^-_M + M^+(z_0) \in \overline{\mathbb{R}}, \tag{14}$$

but for $\tilde{D} \Subset D$, this constant $\overline{C}_M < +\infty$ in (13) is finite and independent of v and u.

We put $u \overset{(11)}{:=} \ln|f|$ and fix a point $z_0 \in \mathsf{int}\, S$ and a regular domain $\tilde{D} \Subset D$ such that $f(z_0) \neq 0$ and the assumptions of Lemma 2 hold. Then, by (13)–(14), $C \in \mathbb{R}$, $\overline{C}_M \in \mathbb{R}$, and

$$\int\limits_{\tilde{D} \setminus S} v \mathrm{d}\varDelta^-_{M_{\rm low}} \leq b\varDelta^-_{M_{\rm low}}\bigl(\tilde{D}\bigr) < +\infty$$

are independent of f and v. Thus, there is a number $\tilde{C}$ such that

$$\sum_{\mathsf{z}_j \in D \setminus S} v(\mathsf{z}_j) = \int\limits_{D \setminus S} v \mathrm{d}n_{\mathsf{Z}} \leq \int\limits_{D \setminus S} v \mathrm{d}n_{\mathsf{Zero}_f} = \int\limits_{D \setminus S} v \mathrm{d}\varDelta_{\ln|f|}$$

$$= \int\limits_{D \setminus S} v \mathrm{d}\varDelta_u \leq \int\limits_{D \setminus S} v \mathrm{d}\varDelta_M + \tilde{C} - C \ln\bigl|f(z_0)\bigr|$$

for each test function $v \in \mathsf{sbh}^+_0(D \setminus S; \leq b)$. Hence, for $C' := \tilde{C} - C \ln\bigl|f(z_0)\bigr| \in \mathbb{R}$, we obtain

$$\sum_{\mathsf{z}_j \in D \setminus S} v(\mathsf{z}_j) + \int\limits_{D \setminus S} v \mathrm{d}\varDelta_{M_{\rm low}} \leq \int\limits_{D \setminus S} v \mathrm{d}\varDelta_{M_{\rm up}} + C' \tag{15}$$

for each test function $v \in \mathsf{sbh}^+_0(D \setminus S; \leq b)$. Let $(v_k)_{k \in \mathbb{N}} \subset \mathsf{sbh}^+_0(D \setminus S; \leq b)$ be an increasing sequence and $v := \lim\limits_{k \to \infty} v_k \in \mathsf{sbh}^{+\uparrow}_0(D \setminus S; \leq b)$ is $\varDelta_M$-summable. Then

$$\int\limits_{D \setminus S} v_k \mathrm{d}(n_{\mathsf{Z}} + \varDelta_{M_{\rm low}}) = \sum_{\mathsf{z}_j \in D \setminus S} v_k(\mathsf{z}_j) + \int\limits_{D \setminus S} v_k \mathrm{d}\varDelta_{M_{\rm low}}$$

$$\overset{(15)}{\leq} \int\limits_{D \setminus S} v_k \mathrm{d}\varDelta_{M_{\rm up}} + C' \leq \int\limits_{D \setminus S} v \mathrm{d}\varDelta_{M_{\rm up}} + C' < +\infty \quad \text{for each } k \in \mathbb{N}.$$

Applying the monotone convergence theorem for integrals to the left-hand side, we obtain

$$\sum_{\mathsf{z}_j \in D \setminus S} v(\mathsf{z}_j) + \int\limits_{D \setminus S} v \mathrm{d}\varDelta_{M_{\rm low}} = \int\limits_{D \setminus S} v \mathrm{d}(n_{\mathsf{Z}} + \varDelta_{M_{\rm low}}) \leq \int\limits_{D \setminus S} v \mathrm{d}\varDelta_{M_{\rm up}} + C'$$

for each test upper function $v \in \mathsf{sbh}_0^{+\uparrow}(D \setminus S; \leq b)$.

It remains to consider the case of test Jensen functions.

By the Weierstrass Theorem there are a function $f_{\mathsf{Z}} \in \mathsf{Hol}(D)$ with zero set $\mathsf{Zero}_{f_{\mathsf{Z}}} = \mathsf{Z}$ and a function $g \in \mathsf{Hol}(D)$ such that $f_{\mathsf{Z}}(z_0) \neq 0$, $g(z_0) \neq 0$ and $\ln|f_{\mathsf{Z}}| + M_{\text{low}} \overset{(11)}{\leq} M_{\text{up}} - \ln|g|$ on D outside some polar set, and hence everywhere. Integrating with respect to a Jensen measure $\mu \overset{(7)}{\in} J_{z_0}(D)$, we obtain

$$\int_D \ln|f_{\mathsf{Z}}| \mathrm{d}\mu + \int_D M_{\text{low}} \mathrm{d}\mu \leq \int_D M_{\text{up}} \mathrm{d}\mu - \int_D \ln|g| \mathrm{d}\mu \leq \int_D M_{\text{up}} \mathrm{d}\mu - \ln\bigl|g(0)\bigr|.$$

By the Poisson–Jensen formula (10), for $\ln|f|$, M_{up}, and M_{low} we have

$$\int_D V_\mu \mathrm{d}n_{\mathsf{Z}} + \ln\bigl|f(z_0)\bigr| + \int_D V_\mu \mathrm{d}\varDelta_{M_{\text{low}}} + M_{\text{low}}(z_0) \leq \int_D V_\mu \mathrm{d}\varDelta_{M_{\text{up}}} + M_{\text{up}}(z_0) - \ln\bigl|g(0)\bigr|$$

Hence

$$\int_D V_\mu \mathrm{d}n_{\mathsf{Z}} + \int_D V_\mu \mathrm{d}\varDelta_{M_{\text{low}}} \leq \int_D V_\mu \mathrm{d}\varDelta_{M_{\text{up}}} + \underbrace{\Bigl(M_{\text{up}}(z_0) - \ln\bigl|f(z_0)\bigr| - M_{\text{low}}(z_0) - \ln\bigl|g(0)\bigr|\Bigr)}_{C}$$

for logarithmic potentials V_μ of all Jensen measures $\mu \in \mathsf{J}_{z_0}(D)$. By Lemma 1(i), if μ runs through $\mathsf{J}_{z_0}(D)$, then V_m runs through the whole class PJ_{z_0}. Thus,

$$\int_D V \mathrm{d}\bigl(n_{\mathsf{Z}} + \varDelta_{M_{\text{low}}}\bigr) = \int_D V \mathrm{d}n_{\mathsf{Z}} + \int_D V \mathrm{d}\varDelta_{M_{\text{low}}} \leq \int_D V \mathrm{d}\varDelta_{M_{\text{up}}} + C \quad \text{for each } V \in \mathsf{PJ}_{z_0}(D).$$

Let $(V_k)_{k\in\mathbb{N}} \subset \mathsf{PJ}_{z_0}(D)$ be increasing and $V := \lim\limits_{k\to\infty} V_k \in \mathsf{PJ}_{z_0}^{\uparrow}(D)$ be $\varDelta_M$-summable. Then

$$\int_D V_k \mathrm{d}\bigl(n_{\mathsf{Z}} + \varDelta_{M_{\text{low}}}\bigr) \leq \int_D V_k \mathrm{d}\varDelta_{M_{\text{up}}} + C \leq \int_D V \mathrm{d}\varDelta_{M_{\text{up}}} + C < +\infty.$$

Applying the monotone convergence theorem for integrals to the left-hand side, we obtain

$$\sum_{\mathsf{z}_j \in D\setminus S} V(\mathsf{z}_j) + \int_{D\setminus S} V \mathrm{d}\varDelta_{M_{\text{low}}} = \int_{D\setminus S} V \mathrm{d}(n_{\mathsf{Z}} + \varDelta_{M_{\text{low}}}) \leq \int_{D\setminus S} V \mathrm{d}\varDelta_{M_{\text{up}}} + C'$$

for each test Jensen function $V \in \mathsf{PJ}_{z_0}^{\uparrow}(D)$. □

3. SUFFICIENT CONDITIONS FOR ZERO SUBSETS OF HOLOMORPHIC FUNCTIONS WITH UPPER CONSTRAINTS IN DOMAINS

3.1. Integral Means of Subharmonic and d-subharmonic Functions

We denote by $D(z,t) := \bigl\{z' \in \mathbb{C} \colon |z'-z| < t\bigr\}$, $\overline{D}(z,t) := \bigl\{z' \in \mathbb{C} \colon |z'-z| \leq t\bigr\}$, $\partial\overline{D}(z,t) := \overline{D}(z,t) \setminus D(z,t)$ an *open disk,* a *closed disk,* a *circle of radius* $t \in \overline{\mathbb{R}}^+$ *centered at* $z \in \mathbb{C}$, respectively. If $D \neq \varnothing$ be a proper domain in $\mathbb{C}$, i.e. $D \neq \mathbb{C}$, then we use a function $r \colon D \to \mathbb{R}$ on D such that

$$\begin{cases} 0 \leq r(z) < \mathsf{dist}(z, \partial D) := \inf\limits_{z' \in \partial D} |z - z'| \quad \text{for each } z \in D, \\ \inf\limits_{z \in K} r(z) > 0 \quad \text{for each } K \Subset D, \end{cases} \tag{16D}$$

but if $D = \mathbb{C}$, then we use another function

$$r(z) \underset{z\in\mathbb{C}}{:=} \frac{1}{(1+|z|)^P} \quad \text{with a number } P \in \mathbb{R}^+. \tag{16ℂ}$$

Let $u\colon D\to\overline{\mathbb R}$ be a function. The *integral means* of u *over circles* $\partial\overline{D}\bigl(z,r(z)\bigr)$ are

$$u^{\odot r}(z):=\frac{1}{2\pi}\int_0^{2\pi}u\bigl(z+r(z)e^{i\theta}\bigr)\mathrm{d}\theta,\quad z\in D,\quad \overline{D}(z,r(z))\subset D, \tag{17$\odot$}$$

the *integral means* of u *over disks* $D\bigl(z,r(z)\bigr)$ are

$$u^{\bullet r}(z):=\frac{1}{\pi r^2(z)}\int_0^{r(z)}\int_0^{2\pi}u(z+te^{i\theta})\mathrm{d}\theta\, t\mathrm{d}t,\quad z\in D,\quad D(z,r(z))\subset D,. \tag{17$\bullet$}$$

Naturally, we assume that the above integrals are well defined and [1, 2.6], [11], [12, Theorem 3]

$$u\le u^{\bullet r}\le u^{\odot r}\le u^{\bullet(\sqrt{e}r)}\quad\text{on } D \text{ for each } u\in\mathsf{sbh}_*(D), \tag{18}$$

where the last inequality holds under the assumption that $\sqrt{e}r<\mathsf{dist}(\cdot,\partial D)$ on D.

We impose one very weak requirement on function (16D). For the function

$$\widehat{r}(z):=\inf\Bigl\{R\in\mathbb R^+\Bigm| \bigcup_{z'\in D(z,r(z))}D\bigl(z',r(z')\bigr)\subset D(z,R)\Bigr\},\quad z\in D, \tag{19r}$$

we require

$$\overline{D}\bigl(z,\widehat{r}(z)\bigr)\subset D\quad\text{for each } z\in D. \tag{19$\widehat{\mathrm{r}}$}$$

We define the class [6, (1.12)]

$$\mathsf{sbh}^+_{00}(D\setminus S;\le b):=\Bigl\{v\in\mathsf{sbh}^+_0(D\setminus S;\le b)\Bigm| \tag{20}$$

$$\text{there is a subset } S_v\Subset D \text{ such that } v(z)=0 \text{ at each } z\in D\setminus S_v\Bigr\}. \tag{21}$$

of *test subharmonic positive compactly supported functions for D outside of $S\Subset D$.*

3.2. Main Result on Sufficient Conditions for Zero Subsets in Domains

The order of formulating sufficient conditions in Theorem 2 differs from the order of formulating necessary conditions in Theorem 1. First, we give sufficient conditions for arbitrary domains D and d-subharmonic majorants M in terms of smooth Jensen potentials from $\mathsf{PJ}_{z_0}(D)$, and then we state sufficient conditions for arbitrary domains D with non-polar boundary ∂D and arbitrary d-subharmonic majorants M from (6) in terms of smooth test subharmonic functions from $\mathsf{sbh}_0(D\setminus S;\le 1)$. The main task of this section is to establish the smallest possible set of sufficient conditions for the distribution of zero subsets of holomorphic functions $f\in\mathsf{Hol}(D)$ satisfying the upper constraint $\ln|f|\le M$ on D.

Theorem 2 [sufficient conditions]**.** *Let Z be a locally finite distribution of points in a domain $D\subset\mathbb C$ containing $z_0\notin Z$ and M be a d-subharmonic function (6) with $M_{\rm up}(z_0)+M_{\rm low}(z_0)\ne-\infty$.*

Let there be a subdomain $U_{z_0}\Subset D$ containing $z_0\in U_{z_0}$ such that the inequality (12) with $S:=\{z_0\}$ is fulfilled for each smooth Jensen potential

$$v\in\mathsf{PJ}_{z_0}(D)\bigcap\mathsf{har}(U_{z_0}\setminus z_0)\bigcap C^\infty(D\setminus z_0) \tag{21P}$$

satisfying

$$v(z)=-\ln|z-z_0|+O(1)\quad\text{as } z_0\ne z\to z_0. \tag{21_0}$$

Then, for each function (16), satisfying (19), and for any number $a>0$ there exists a function $f\in\mathit{Hol}(D)$ such that $f\ne0$, $f(Z)=0$ and

$$\ln|f|\overset{17_\odot}{\le}M_{\rm up}^{\odot\widehat{r}}-M_{\rm low}+R\quad\text{on } D, \text{ where } \widehat{r} \text{ is defined in (19), and} \tag{22M}$$

$$R(z) := \begin{cases} \ln \dfrac{1}{r(z)} + (1+a)\ln\bigl(1+|z|\bigr), & \text{if } D \neq \mathbb{C}, \\ \ln \dfrac{1}{r(z)}, & \text{if } D \neq \mathbb{C} \text{ is simply connected or } \complement\, \mathsf{clos}\, D \neq \varnothing, \\ 0, & \text{if } D = \mathbb{C}, \end{cases} \tag{22R}$$

at each $z \in D$.

In addition, if the boundary ∂D *of* D *is non-polar and there exist a closed subset* $S \Subset D$ *with* $\mathsf{int}\, S \neq \varnothing$ *and a number* $C \in \mathbb{R}^+$ *such that the inequality (12) is fulfilled for each smooth test function* $v \overset{(20)}{\in} \mathsf{sbh}^+_{00}(D \setminus S; \leq 1) \bigcap C^\infty(D \setminus S)$*, then, for any function (16D), there is a function* $f \in \mathsf{Hol}(D)$ *such that* $f \neq 0$*,* $f(\mathsf{Z}) = 0$ *and (22) is fulfilled.*

Proof of Theorem 2. We first prove the statement for Jensen potentials. We denote by $\mathsf{PJ}^1_{z_0}(D)$ the class of all Jensen potentials v satisfying (21_0) and put (cf. [7, (13V)])

$$\mathcal{V}(U_{z_0}) := \mathsf{PJ}^1_{z_0}(D) \bigcap \mathsf{har}(U_{z_0} \setminus z_0) \bigcap C^\infty(D \setminus z_0). \tag{23}$$

We denote by $\mathsf{Meas}^+_\infty(D)$ the subclass of all measures $\mu \in \mathsf{Meas}^+(D)$ with densities $m \in C^\infty(D)$, i.e. $\mathrm{d}\mu = m \mathrm{d}\lambda$, where λ is the Lebesgue measure on D, and put (cf. e[7, (13M)])

$$\mathcal{M}(U_{z_0}) := \mathsf{J}_{z_0}(D) \bigcap \mathsf{Meas}_{\mathrm{cmp}}(D \setminus U_{z_0}) \bigcap \mathsf{Meas}^+_\infty(D). \tag{24}$$

By Lemma 1(i), it is easy to see that the mapping $\mathcal{P}^{-1}$ in (9) defines a bijection from the subclass $\mathcal{V}(U_{z_0})$ to the subclass $\mathcal{M}(U_{z_0})$ [7, Theorem A].

By the Weierstrass Theorem there is a function $f_{\mathsf{Z}} \in \mathsf{Hol}(D)$ with zero set $\mathsf{Zero}_{f_{\mathsf{Z}}} = \mathsf{Z}$.

For all Jensen potentials $v \in \mathcal{V}(U_{z_0})$ we have the inequality (12) with $S := \{z_0\}$. Hence, by the Poisson–Jensen formula of Lemma 1(ii), (10), applied to $\ln|f|$, $M_{\rm up}$, $M_{\rm low}$, and by bijection $\mathcal{P}^{-1}(D)\colon \mathcal{V}(U_{z_0}) \overset{(24)}{\longrightarrow} \mathcal{M}(U_{z_0})$, we have (cf. [7, (15)])

$$\int_D \underbrace{\bigl(\ln|f_{\mathsf{Z}}| + M_{\rm low}\bigr)}_{u} \mathrm{d}\mu = \int_D \ln|f_{\mathsf{Z}}| \mathrm{d}\mu + \int_D M_{\rm low} \mathrm{d}\mu$$
$$\leq \int_D M_{\rm up} \mathrm{d}\mu + \underbrace{\Bigl(\ln\bigl|f_{\mathsf{Z}}(z_0)\bigr| + M_{\rm low}(z_0) - M_{\rm up}(z_0)\Bigr)}_{c} \quad \text{for each } \mu \in \mathcal{M}(U_{z_0}). \tag{25}$$

Lemma 3 (A very special case of [13, Corollary 8.1.II.1] with $H := \mathsf{sbh}_*(D)$, cf. [7, Theorem B]). *If, for some number* $c \in \mathbb{R}$*, assertion (25) holds, then, for any function* r*, satisfying (16D), there are a function* $h \in \mathsf{sbh}_*(D)$ *and a positive function* $\check{r} \leq r$ *from the class* $C^\infty(D)$ *such that*

$$u + h \leq M_{\rm up}^{\circledast \check{r}} \in C^\infty(D) \quad \text{on } D, \tag{26}$$

where, by the construction from [13, (8.3–6), (8.10)], [6, (2.18–19)], $M_{up}^{\circledast \check{r}}$ *are "moving contracting" smoothing averages over some probabilistic measures* $\alpha^{(\check{r}(z))} \in \mathsf{Meas}^+_\infty\bigl(\overline{D}(z, \check{r}(z))\bigr)$*, obtained by the shift, compression, and normalization of a single approximate unit* $a \in C^\infty(\mathbb{C})$*, depending on the modulus* $|\cdot|$ *only with support* $\operatorname{supp} a \subset \overline{D}(0,1)$.

By Lemma 3 we choose a subharmonic function $h \in \mathsf{sbh}_*(D)$ such that $\ln|f_{\mathsf{Z}}| + M_{\rm low} + h \leq M_{\rm up}^{\circledast \check{r}}$ on D. By [14, Proposition 3] or [15, Theorem 4], for the subharmonic function $M_{\rm up}$ we have $M_{\rm up}^{\circledast \check{r}} \leq M_{\rm up}^{\odot \check{r}} \leq M_{\rm up}^{\odot r}$ on D. Thus $\ln|f_{\mathsf{Z}}| + M_{\rm low} + h \leq M_{\rm up}^{\odot r}$ on D. Hence,

$$\ln|f_{\mathsf{Z}}| + M_{\rm low} + h^{\bullet r} \overset{17\bullet}{\leq} \bigl(\ln|f_{\mathsf{Z}}|\bigr)^{\bullet r} + M_{\rm low}^{\bullet r} + h^{\bullet r} \leq \bigl(M_{\rm up}^{\odot r}\bigr)^{\bullet r} \quad \text{on } D. \tag{27}$$

Lemma 4 ([16, Theorem 3, Corollary 3(i),(iii)]). *Let* $h \in \mathsf{sbh}_*(D)$ *be a subharmonic function on a domain* $D \subset \mathbb{C}$*. Then, for any number* $a > 0$*, there is a function* $g \in \mathsf{Hol}(D)$ *such that* $g \neq 0$ *and*

$$\ln|g| \leq h^{\bullet t} + R \quad \text{on } D, \tag{28}$$

where R is a function from (22R).

By Lemma 4 and (27), we get

$$\ln|f_{\mathsf Z}g| + M_{\text{low}} = \ln|f_{\mathsf Z}| + M_{\text{low}} + \ln|g| \le \bigl(M_{\text{up}}^{\odot r}\bigr)^{\bullet r} + R \quad \text{on } D, \tag{29}$$

where $f := f_{\mathsf Z}g \neq 0$ and $f(\mathsf Z) = 0$.

The following lemma is an elementary very special case of [15, Theorems 2, 4].

Lemma 5. *If r and $\widehat{r}$ are defined by either (16D) or (16$\mathbb{C}$) and (19), then $(u^{\odot r})^{\bullet r} \le u^{\odot \widehat{r}}$ on D.*

By Lemma 5, it follows from (29) that

$$\ln|f| + M_{\text{low}} = \ln|f_{\mathsf Z}g| + M_{\text{low}} \le M_{\text{up}}^{\odot \widehat{r}} + R \quad \text{on } D.$$

Thus, under (12) for smooth Jensen potentials, we have proved (22).

Now consider the case of a domain D with non-polar boundary ∂D.

Lemma 6 ([6, Theorem 3]). *Under the conditions of Theorem 2, there is a subharmonic function $u \in sbh_*(D)$ such that $n_Z \le \varDelta_u$ and $u \le M^{\bullet r}$ on D.*

By Lemma 6 there is a subharmonic function $u \in \mathsf{sbh}_*(D)$ such that

$$n_{\mathsf Z} \le \varDelta_u \quad \text{and} \quad u \le M^{\bullet r} = M_{\text{up}}^{\bullet r} - M_{\text{low}}^{\bullet r} \overset{(18)}{\le} M_{\text{up}}^{\bullet r} - M_{\text{low}} \quad \text{on } D. \tag{30}$$

By the Weierstrass Theorem, there is $f_{\mathsf Z} \in \mathsf{Hol}(D)$ with $\mathsf{Zero}_{f_{\mathsf Z}} = \mathsf Z$, and $f_{\mathsf Z} \neq 0$.

Consider a d-subharmonic function $h := u - \ln|f_{\mathsf Z}|$ with Riesz charge $\varDelta_h = \varDelta_u - n_{\mathsf Z} \overset{(30)}{\ge} 0$, i. e., $\varDelta_h \in \mathsf{Meas}^+(D)$ and $h \in \mathsf{sbh}_*(D)$. It follows from (30) that

$$\ln|f_{\mathsf Z}| + h = u \overset{(30)}{\le} M_{\text{up}}^{\bullet r} - M_{\text{low}} \quad \text{on } D. \tag{31}$$

Hence, for $\ln|f_{\mathsf Z}| \in \mathsf{sbh}_*(D)$ and $h \in \mathsf{sbh}_*(D)$, we obtain

$$\begin{aligned}\ln|f_{\mathsf Z}| + h^{\bullet r} &\le \bigl(\ln|f_{\mathsf Z}|\bigr)^{\bullet r} + h^{\bullet r} = \bigl(\ln|f_{\mathsf Z}| + h\bigr)^{\bullet r} = u^{\bullet r}\\ &\overset{(31)}{\le} \bigl(M_{\text{up}}^{\bullet r}\bigr)^{\bullet r} - M_{\text{low}}^{\bullet r} \overset{18}{\le} \bigl(M_{\text{up}}^{\bullet r}\bigr)^{\bullet r} - M_{\text{low}} \quad \text{on } D.\end{aligned} \tag{32}$$

Using Lemma 4, we put $f := f_{\mathsf Z}g \neq 0$. Then $f(\mathsf Z) = 0$ since $\mathsf Z = \mathsf{Zero}_{f_{\mathsf Z}}$, and

$$\ln|f| = \ln|f_{\mathsf Z}g| = \ln|f_{\mathsf Z}| + \ln|g| \overset{(28)}{\le} \ln|f_{\mathsf Z}| + h^{\bullet r} + R \overset{(32)}{\le} \bigl(M_{\text{up}}^{\bullet r}\bigr)^{\bullet r} - M_{\text{low}} + R \quad \text{on } D. \tag{33}$$

By Lemma 5 with $u := M_{\text{up}}$, we obtain $\ln|f| \overset{(33)}{\le} M_{\text{up}}^{\odot \widehat{r}} - M_{\text{low}} + R$ on D. □

4. ZERO SUBSETS IN THE COMPLEX PLANE

In this section, we give to the results of Theorems 1 and 2 a form related to subharmonic functions of polynomial growth and point out a very general case when the necessary and sufficient conditions coincide. We denote by

$$\mathsf{Pot} := \Bigl\{p \in \mathsf{sbh}_*(\mathbb{C}) \Bigm| \limsup_{z\to\infty} \frac{p(z)}{\ln|z|} < +\infty\Bigr\}$$

the convex cone over $\mathbb{R}^+$ of all *subharmonic functions of polynomial growth* [17, 6.7.2]. We use the convex subcone over $\mathbb{R}^+$

$$\mathsf{Pot}_0^{+1} := \Bigl\{p \in \mathsf{Pot} \Bigm| p(0) = 0, \quad p \ge 0 \text{ on } \mathbb{C}, \quad \limsup_{z\to\infty} \frac{p(z)}{\ln|z|} \le 1\Bigr\} \subset \mathsf{Pot} \tag{34}$$

of *positive subharmonic functions of polynomial growth with unit upper seminormization at ∞.*

Theorem 3. *Let Z be a locally finite distribution of points in $\mathbb{C}$ and $0 \notin Z$. Let M be a d-subharmonic function (6) on $D := \mathbb{C}$. Suppose that $M_{up}(0) + M_{low}(0) \neq -\infty$ and there are numbers $P \in \mathbb{R}^+$ and $C \in \mathbb{R}^+$ such that (3) holds. Then the following three statements are equivalent:*

I. *There exists an entire function $f \neq 0$ such that $f(Z) = 0$ and*

$$\ln\bigl|f(z)\bigr| \leq M(z) \quad \text{at each } z \in \mathbb{C}. \tag{35}$$

II. *There is a number $C \in \mathbb{R}^+$ such that, for each $p \in \mathsf{Pot}_0^{+1}$, we have*

$$\sum_j p\Bigl(\frac{1}{\bar{\mathsf{z}}_j}\Bigr) \leq \int\limits_{\mathbb{C}} p\Bigl(\frac{1}{\bar{z}}\Bigr) \mathrm{d}\varDelta_M(z) + C \quad \text{provided } p\Bigl(\frac{1}{\bar{z}}\Bigr) \text{ is } \varDelta_M\text{-summable on } \mathbb{C} \setminus 0. \tag{36}$$

III. *There are numbers $C \in \mathbb{R}^+$ and $R_0 > 0$ such that (36) is fulfilled for each*

$$p \in \mathsf{Pot}_0^{+1} \bigcap C^\infty(\mathbb{C}) \bigcap \mathsf{har}\bigl(\mathbb{C} \setminus \overline{D}(0, R_0)\bigr) \tag{37P}$$

such that $p = 0$ on some neighborhood of the origin and

$$p(z) = \ln|z| + O(1) \quad \text{as } z \to \infty. \tag{37$_0$}$$

Proof. Here (35) is a particular case of (11) for $D = \mathbb{C}$. Let $z_0 := 0$. The inversion transformation $z \underset{z\in\mathbb{C}}{\longmapsto} \frac{1}{\bar{z}}$, $\quad 0 \mapsto \infty \mapsto 0$ from $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$ gives a bijection from Pot_0^{+1} onto $\mathsf{PJ}_0^{\uparrow}(\mathbb{C})$ and a bijection from the class (37) onto the class (21), or onto the class $\mathcal{V}(U_{z_0})$ from (23). Thus, (36) is (12) with $v(z) \underset{z\in\mathbb{C}_\infty}{=} p(1/\bar{z})$. Hence, the implication I $\Longrightarrow$ II follows from Theorem 1, the implication II $\Longrightarrow$ III is obvious, and the implication III $\Longrightarrow$ I follows from Theorem 2 if we take into account condition (3). $\square$

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

REFERENCES

1. Th. Ransford, *Potential Theory in the Complex Plane* (Cambridge Univ. Press, Cambridge, 1995).
2. W. K. Hayman and P. B. Kennedy, *Subharmonic Functions*, Vol. 9 of *London Math. Soc. Monography* (Academic, London, New York, 1976), Vol. 1.
3. L. L. Helms, *Introduction to Potential Theory* (Wiley-Interscience, New York, London, Sydney, Toronto, 1969).
4. B. N. Khabibullin and A. P. Rozit, “On the distribution of zero sets of holomorphic functions,” Funct. Anal. Appl. **52**, 21–34 (2018).
5. E. B. Menshikova and B. N. Khabibullin, “On the distribution of zero sets of holomorphic functions. II,” Funct. Anal. Appl. **53**, 65–68 (2019).
6. B. N. Khabibullin and F. B. Khabibullin, “On the distribution of zero sets of holomorphic functions. III. Inversion theorems,” Funct. Anal. Appl. **53**, 110–123 (2019).
7. E. B. Menshikova and B. N. Khabibullin, “A criterion for the sequence of roots of holomorphic function with restrictions on its growth,” Russ. Math. **64** (5), 49–55 (2020).
8. M. G. Arsove, “Functions representable as differences of subharmonic functions,” Trans. Am. Math. Soc. **75**, 327–365 (1953).
9. V. Azarin, *Growth Theory of Subharmonic Functions* (Birkhäuser, Basel, Boston, Berlin, 2009).
10. B. N. Khabibullin, “Criteria for (sub-)harmonicity and continuation of (sub-)harmonic functions,” Sib. Math. J. **44**, 713–728 (2003).

11. A. F. Beardon, "Integral means of subharmonic functions," Proc. Cambridge Philos. Soc. **69**, 151–152 (1971).
12. P. Freitas and J. P. Matos, "On the characterization of harmonic and subharmonic functions via mean-value properties," Potential Anal. **32**, 189–200 (2010).
13. B. N. Khabibullin, A. P. Rozit, and E. B. Khabibullina, "Order versions of the Hahn–Banach theorem and envelopes. II. Applications to the function theory," Itogi Nauki Tekh., Ser.: Sovrem. Mat. Pril. Temat. Obz. **162**, 93–135 (2019).
14. T. Yu. Bayguskarov, G. R. Talipova, and B. N. Khabibullin, "Subsequences of zeros for classes of entire functions of exponential type, allocated by restrictions on their growth," SPb. Math. J. **28**, 127–151 (2017).
15. B. N. Khabibullin, "Balayage of measures with respect to (sub-)harmonic functions," Lobachevskii J. Math. **41** (11), 2179–2189 (2020).
16. B. N. Khabibullin and T. Yu. Baiguskarov, "The logarithm of the modulus of a holomorphic function as a minorant for a subharmonic function," Math. Notes **99**, 576–589 (2016).
17. W. K. Hayman, *Subharmonic Functions* (Academic, London, 1990), Vol. 2.