ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2021, Vol. 42, No. 6, pp. 1175–1182. © Pleiades Publishing, Ltd., 2021.

Integrals with a Meromorphic Function or the Difference of Subharmonic Functions over Discs and Planar Small Sets

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Received December 12, 2020; revised January 16, 2021; accepted January 30, 2021

Abstract—The maximum of the modulus of a meromorphic function cannot be restricted from above by the Nevanlinna characteristic of this meromorphic function. But integrals from the logarithm of the module of a meromorphic function allow similar restrictions from above. This is illustrated by one of the important theorems of Rolf Nevanlinna in the classical monograph by A. A. Goldberg and I. V. Ostrovskii on meromorphic functions, as well as by the Edrei–Fuchs Lemma on small arcs and its versions for small intervals in articles by A. F. Grishin, M. L. Sodin, T. I. Malyutina. Similar results for integrals of differences of subharmonic functions even with weights were recently obtained by B. N. Khabiblullin, L. A. Gabdrakhmanova. All these results are on integrals over subsets on a ray. In this article, we establish such results for integrals of the logarithm of the modulus of a meromorphic function and the difference of subharmonic functions over discs and planar small sets. Our estimates are uniform in the sense that the constants in these estimates are explicitly written out and do not depend on meromorphic functions and the difference of subharmonic functions provided that these functions has an integral normalization near zero.

DOI: 10.1134/S1995080221060111

Keywords and phrases: *meromorphic function, Nevanlinna characteristic, subharmonic function, δ-subharmonic function, Riesz measure.*

1. INTRODUCTION. DEFINITIONS AND NOTATION

Upper estimates of integrals for meromorphic functions in the *complex plane* $\mathbb{C}$ over segments or small sets on a ray via the Nevanlinna characteristic are considered in works [1, pp. 24–27], [2, Notes, Ch. 1], [2, Ch. 1, Theorem 7.2], [3, Lemma 3.1], [4, Theorem 8], [5, Theorem 1 (on small intervals), Remark 1.1, Conclusion of the Grishin–Malyutina Theorem], [6, Main Theorem]. Our article [6] contains all these listed results as special cases (see Theorem 1 below). We give in [5, Inrtoduction] and [6, 1.1–1.2] a detailed history of the issue with full formulations of all previous results. In this paper, we obtain similar upper estimates for integrals already over discs or small planar sets. As in [1], we establish the main theorem of the paper in a more general subharmonic version. Let's move on to precise definitions.

As usual, $\mathbb{R}$ is the *real line,* or the *real axis* of the complex plane $\mathbb{C}$, and $\mathbb{R}^+ := \{r \in \mathbb{R} \colon 0 \leq r\}$ is the *positive closed semiaxis*. We denote singleton sets by a symbol without curly brackets. So, $\mathbb{R}^+ \setminus 0$ is the *positive open semiaxis,* $\overline{\mathbb{R}}$ is the *extended real axis*. Besides, $D(z,r) := \{z' \in \mathbb{C} \colon |z'-z| < r\}$ is an *open disc,* $\overline{D}(z,r) := \{z' \in \mathbb{C} \colon |z'-z| \leq r\}$ is a *closed disc,* $\partial\overline{D}(z,r) := \{z' \in \mathbb{C} \colon |z'-z| = r\}$ is a *circle with center* $z \in \mathbb{C}$ *of radius* $r \in \overline{\mathbb{R}}^+$*;* $D(z,0) = \varnothing$, $\overline{D}(z,0) = \partial\overline{D}(z,0) = z$, $D(z,+\infty) = \mathbb{C}$, $D(r) := D(0,r)$, $\overline{D}(r) := \overline{D}(0,r)$, $\partial\overline{D}(r) := \partial\overline{D}(0,r)$.

Given a function $f \colon X \to \overline{\mathbb{R}}$, $f^+ := \sup\{0, f\}$ and $f^- := (-f)^+$ are *positive* and *negative parts* of function f, respectively; $|f| := f^+ + f^-$. Given $S \subset \mathbb{C}$, $\mathsf{sbh}(S)$ is the class of all *subharmonic* on an open neighbourhood of S. We set $\mathsf{sbh}_*(S) := \big\{v \in \mathsf{sbh}(S) \colon v \not\equiv -\infty\big\}$.

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By λ we denote the *planar Lebesgue measure* on $\mathbb{C}$. We also use the notation mes for the *linear Lebesgue measure* on $\mathbb{R}$. For $r \in \mathbb{R}^+$ and a function $v\colon \partial\overline{D}(r) \to \overline{\mathbb{R}}$, we define

$$\mathsf{M}_v(r) := \sup_{|z|=r} v(z), \quad r \in \mathbb{R}^+, \tag{1M}$$

$$\mathsf{C}_v(r) := \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\varphi})\mathrm{d}\varphi, \quad r \in \mathbb{R}^+ \setminus 0, \tag{1C}$$

where $\mathsf{C}_v(r)$ is the *average over* the *circle* $\partial\overline{D}(0,r)$, if this integral exists. For a λ-measurable function $v\colon \overline{D}(r) \to \overline{\mathbb{R}}$, we use also the *average over* the *disc* $\overline{D}(r)$ defined as

$$\mathsf{B}_v(r) := \frac{1}{\pi r^2}\int_{\overline{D}(r)} v\,\mathrm{d}\lambda, \quad \mathsf{B}_v(0) := \mathsf{M}_v(0) \overset{(1M)}{=} v(0) =: \mathsf{C}_v(0). \tag{2}$$

See [7, 2.6], [8, 2.7], [9, 3] on properties of M_v, C_v, B_v in the case of a subharmonic function v.

For a Borel subset $S \subset \mathbb{C}$, the set of all Borel, or Radon, positive measures $\mu \geq 0$ on S is denoted by $\mathsf{Meas}^+(S)$, and $\mathsf{Meas}(S) := \mathsf{Meas}^+(S) - \mathsf{Meas}^+(S)$ is the set of all *charges,* or signed measures, on S. For a charge $\nu \in \mathsf{Meas}(S)$, we denote by

$$\nu^+ \in \mathsf{Meas}^+(S), \quad \nu^- := (-\nu)^+ \in \mathsf{Meas}^+(S), \quad |\nu| := \nu^+ + \nu^- \in \mathsf{Meas}^+(S)$$

the *upper, lower, total variations* of this charge ν, respectively. We set

$$\nu(z,r) := \nu\bigl(\overline{D}(z,r)\bigr) \in \overline{\mathbb{R}} \quad \text{if } \overline{D}(z,r) \subset S, \quad 0 \leq r \leq R, \tag{3z}$$

$$\nu^{\text{rad}}(r) := \nu(0,r) = \nu\bigl(\overline{D}(r)\bigr) \in \overline{\mathbb{R}} \quad \text{if } \overline{D}(r) \subset S, \quad 0 \leq r \leq R, \tag{3r}$$

$$\mathsf{N}_\nu(r,R) := \int_r^R \frac{\nu^{\text{rad}}(t)}{t}\mathrm{d}t \in \overline{\mathbb{R}}^+ \quad \text{if } \overline{D}(R) \subset S, \quad 0 \leq r \leq R, \tag{3N}$$

provided that the last integral is well defined. For $0 \leq r \leq R \in \mathbb{R}^+$ and functions $v\colon \partial\overline{D}(r) \cup \partial\overline{D}(R) \to \overline{\mathbb{R}}$, we define

$$\mathsf{C}_v(r,R) \overset{(1C)}{:=} \mathsf{C}_v(R) - \mathsf{C}_v(r) = \frac{1}{2\pi}\int_0^{2\pi} \bigl(v(Re^{i\varphi}) - v(re^{i\varphi})\bigr)\mathrm{d}\varphi \tag{4}$$

provided that $\mathsf{C}_v(R)$ and $\mathsf{C}_v(r)$ are well defined.

If $D \subset \mathbb{C}$ is a domain and $u \in \mathsf{sbh}_*(D)$, then there is its *Riesz measure*

$$\varDelta_u := \frac{1}{2\pi}\triangle u \in \mathsf{Meas}^+(D), \tag{5}$$

where $\triangle$ is the *Laplace operator* acting in the sense of the theory of distribution or generalized functions. This definition of the Riesz measures carries over naturally to $u \in \mathsf{sbh}_*(S)$ for connected subsets $S \subset \mathbb{C}$. By the Poisson–Jensen–Privalov formula [7, 8], we have

$$\mathsf{C}_v(r,R) = \mathsf{N}_{\varDelta_v}(r,R) \quad \text{for all } 0 < r < R < +\infty \text{ if } v \in \mathsf{sbh}_*\bigl(\overline{D}(R)\bigr). \tag{6}$$

Let $U = u - v$ be a difference of subharmonic functions $u, v \in \mathsf{sbh}_*\bigl(\overline{D}(0,R)\bigr)$, i. e., a δ*-subharmonic non-trivial* ($\not\equiv \pm\infty$) *function* [10], [11], [12], [13, 3.1] on $\overline{D}(R)$ with the *Riesz charge*

$$\varDelta_U \overset{(5)}{:=} \varDelta_u - \varDelta_v \overset{(5)}{:=} \frac{1}{2\pi}\triangle u - \frac{1}{2\pi}\triangle v \in \mathsf{Meas}\bigl(\overline{D}(0,R)\bigr), \text{ and } \varDelta_U^- := (\varDelta_U)^-$$

is the lower variation of the Riesz charge $\varDelta_U$ of U. Now we can determine the *difference Nevanlinna characteristic* T of δ-subharmonic non-trivial ($\not\equiv \pm\infty$) function U as a function of two variables

$$\mathsf{T}_U(r,R) :\overset{(6)}{=} \mathsf{C}_{U^+}(r,R) + \mathsf{N}_{\varDelta_U^-}(r,R), \quad 0 < r \leq R \in \mathbb{R}^+. \tag{7}$$

A representation $U = u_U - v_U$ with $u_U, v_U \in \mathsf{sbh}_*\bigl(\overline{D}(0,R)\bigr)$ is *canonical* if the Riesz measure $\varDelta_{u_U}$ of u_U is the *upper variation* $\varDelta_U^+$ of $\varDelta_U$ and the Riesz measure $\varDelta_{v_U}$ of v_U is the *lower variation* $\varDelta_U^- := (\varDelta_U)^-$ of $\varDelta_U$. The canonical representation for U is defined up to the harmonic function added simultaneously to each of the representing subharmonic functions u_U and v_U, and

$$\mathsf{T}_U(r,R) \overset{(6)}{=} \mathsf{C}_{U^+}(r,R) + \mathsf{C}_{v_U}(r,R) = \mathsf{C}_{\sup\{u_U,v_U\}}(r,R), \quad 0 < r \le R \in \mathbb{R}^+. \tag{8}$$

where $0 < r \le R \in \mathbb{R}^+$. By (8), the difference Nevanlinna characteristic T_U is already uniquely defined for all values $0 < r \le R < +\infty$ by positive values in $\mathbb{R}^+$, and is also increasing and convex with respect to the logarithmic function ln in the second variable R, but is decreasing in the first variable $r \le R$. Recall that the following notation is used for the meromorphic function $F \not\equiv 0$ on $\mathbb{C}$ in the classic monograph by A. A. Goldberg and I. V. Ostrovskii [2] for the maximum of module

$$M(r,F) \underset{r\,\in\,\mathbb{R}^+}{:=} \sup\bigl\{\bigl|F(z)\bigr| \colon |z| = r\bigr\}, \tag{9M}$$

and for the *Nevanlinna characteristic*

$$T(r,F) \underset{r\,\in\,\mathbb{R}^+}{:=} m(r,F) + N(r,F), \tag{9T}$$

$$m(r,F) \underset{r\,\in\,\mathbb{R}^+}{:=} \frac{1}{2\pi}\int_0^{2\pi} \ln^+\bigl|F(re^{i\varphi})\bigr|\,\mathrm{d}\varphi, \tag{9m}$$

$$N(r,F) \underset{r\,\in\,\mathbb{R}^+}{:=} \int_0^r \frac{n(t,F) - n(0,F)}{t}\,\mathrm{d}t + n(0,F)\ln r, \tag{9N}$$

where $n(r,F)$ is the number of poles of F in the closed disc $\overline{D}(r) := \{z \in \mathbb{C} \colon |z| \le r\}$, taking into account the multiplicity. The function $\ln|F|$ is non-trivial δ-subharmonic on $\mathbb{C}$, and

$$\ln M(r,F) \overset{(9M),(1M)}{=} \mathsf{M}_{\ln|F|}(r), \quad r \in \mathbb{R}^+, \tag{10M}$$

$$m(r,f) \overset{(9m),(1C)}{=} \mathsf{C}_{\ln^+|F|}(r), \quad r \in \mathbb{R}^+, \tag{10m}$$

$$N(R,F) - N(r,F) \overset{(9N),(3N)}{=} \mathsf{N}_{\varDelta^-_{\ln|F|}}(r,R), \quad 0 < r < R \in \mathbb{R}^+, \tag{10N}$$

$$T(R,F) - T(r,F) \overset{(9T),(7)}{=} \mathsf{T}_{\ln|F|}(r,R). \quad 0 < r < R \in \mathbb{R}^+. \tag{10T}$$

2. RECENT AND NEW RESULTS

Let us formulate the main result from [6] for the "one-dimensional" subset $E \subset [0,r] \subset \mathbb{R}^+$ in the special case without a weight function-multiplier $g \in L^p(E)$, $1 < p \le +\infty$, in the integrand.

Theorem 1 ([6, Main Theorem]). *Let $0 < r_0 < r < +\infty$, $1 < k \in \mathbb{R}^+$, $E \subset [0,r]$ be mes-measurable, $g \in L^p(E)$, where $1 < p \le \infty$ and $q \in [1,+\infty)$ is defined by $\frac{1}{p} + \frac{1}{q} = 1$. If $U \not\equiv \pm\infty$ is a non-trivial δ-subharmonic functions on $\mathbb{C}$, and $u \not\equiv -\infty$ is a subharmonic function on $\mathbb{C}$, then*

$$\int_E \mathsf{M}_{U^+}(t)g(t)\mathrm{d}t \le \frac{4qk}{k-1}\bigl(\mathsf{T}_U(r_0,kr) + \mathsf{C}_{U^+}(r_0)\bigr)||g||_{L^p(E)}\sqrt[q]{\operatorname{mes} E}\,\ln\frac{4kr}{\operatorname{mes} E}, \tag{11T}$$

$$\int_E \mathsf{M}_{|u|}(t)g(t)\mathrm{d}t \le \frac{5qk}{k-1}\bigl(\mathsf{M}_{u^+}(kr) + \mathsf{C}_{u^-}(r_0)\bigr)\sqrt[q]{\operatorname{mes} E}\,\ln\frac{4kr}{\operatorname{mes} E}. \tag{11M}$$

In particular, by (10M)–(10T) we have for meromorphic functions the following

Corollary 1. *Let, under the conditions of Theorem 1, $F\not\equiv 0$ be a meromorphic function, and $f\not\equiv 0$ be an entire function. Then, in the traditional notation (9M)–(9N), we have*

$$\int_E \bigl(\ln^+ M(t,F)\bigr)g(t)\mathrm{d}t \leq \frac{4qk}{k-1}\bigl(T(kr,F)-N(r_0,F)\bigr)||g||_{L^p(E)}\sqrt[q]{\operatorname{mes}E}\ln\frac{4kr}{\operatorname{mes}E}, \tag{12T}$$

$$\int_E \bigl|\ln M(t,f)\bigr|g(t)\mathrm{d}t \leq \frac{5qk}{k-1}\bigl(\ln^+ M(kr,f)+m(r_0,1/f)\bigr)\sqrt[q]{\operatorname{mes}E}\ln\frac{4kr}{\operatorname{mes}E}. \tag{12M}$$

Proof. By (10M)–(10T) we have for $U:=\ln|F|$

$$\mathsf{T}_U(r_0,kr)+\mathsf{C}_{U^+}(r_0)\overset{(10T),(10m)}{=}T(kr,F)-T(r_0,F)+m(r_0,f)\overset{(9T)}{=}T(kr,F)-N(r_0,F). \tag{13}$$

We can replace the bracket on the right-hand side of (11T) with the right side of these equalities. Then we obtain (12T) by (10T) for the integrand in the left-hand side of (11T).

For $u:=\ln|f|$, we have

$$\mathsf{M}_{u^+}(kr)+\mathsf{C}_{u^-}(r_0)\overset{(10M)}{=}\ln^+ M(kr,f)+\mathsf{C}_{\ln^-|f|}(r_0)\overset{(10m)}{=}\ln^+ M(kr,f)+m(r_0,1/f). \tag{14}$$

We can replace the bracket on the right-hand side of (11M) with the right side of these equalities. Then we obtain (12M) by (10M) for the integrand in the left-hand side of (11M). □

Remark 1. Our elementary example [6, 1.1, (3)] shows that it is impossible to discard the terms $-N(r_0,F)$ and $m(r_0,1/f)$ in parentheses on the right-hand sides of inequality (12T) and (12M), respectively. In particular, the classical Rolf Nevanlinna Theorem [2, Ch. 1, Theorem 7.2] can be formulated in the following correct form: *for each meromorphic function $F\not\equiv 0$ and $1<k\in\mathbb{R}^+$*

$$\frac{1}{r}\int_0^r \ln^+ M(t,F)\mathrm{d}t \leq \frac{4k\ln 4k}{k-1}\bigl(T(kr,F)-N(r_0,F)\bigr)\quad \textit{for all}\ \ 0<r_0\leq r\in\mathbb{R}^+. \tag{15}$$

Indeed, it is sufficient to choose $E:=[0,r]$, $g\equiv 1$ with $\operatorname{mes}E=r$, and $p:=\infty$ with $q=1$ in (12T).

Our main result is established for the case of a planar "two-dimensional" subset $E\subset\overline{D}(r)\subset\mathbb{C}$:

Theorem 2. *Let $0<r_0<r<+\infty$, $1<k\in\mathbb{R}^+$, $E\subset\overline{D}(r)$ be a λ-measurable subset. If $U\not\equiv\pm\infty$ is a δ-subharmonic functions on $\mathbb{C}$, and $u\not\equiv-\infty$ is a subharmonic function on $\mathbb{C}$, then*

$$\int_E U^+\mathrm{d}\lambda \leq \frac{2k}{k-1}\bigl(\mathsf{T}_U(r_0,kr)+\mathsf{C}_{U^+}(r_0)\bigr)\lambda(E)\ln\frac{100kr^2}{\lambda(E)}, \tag{16T}$$

$$\int_E |u|\mathrm{d}\lambda \leq \frac{3k}{k-1}\bigl(\mathsf{M}_{u^+}(kr)+\mathsf{C}_{u^-}(r_0)\bigr)\lambda(E)\ln\frac{100kr^2}{\lambda(E)}. \tag{16M}$$

Theorem 2 is proved at the end of Sec. 3 after some preparation.

We have not seen before these estimates (16T) and (16M) even for the case $E=\overline{D}(r)$:

Corollary 2. *If $U\not\equiv\pm\infty$ be a δ-subharmonic functions on $\mathbb{C}$, and $u\not\equiv-\infty$ be a subharmonic function on $\mathbb{C}$, then*

$$\mathsf{B}_{U^+}(r)\leq\frac{7k\ln(ek)}{k-1}\bigl(\mathsf{T}_U(r_0,kr)+\mathsf{C}_{U^+}(r_0)\bigr)\quad \textit{for all}\ \ 0<r_0<r\in\mathbb{R}^+\ \textit{and}\ \ 1<k\in\mathbb{R}^+, \tag{17T}$$

$$\mathsf{B}_{|u|}(r)\leq\frac{11k\ln(ek)}{k-1}\bigl(\mathsf{M}_{u^+}(kr)+\mathsf{C}_{u^-}(r_0)\bigr)\quad \textit{for all}\ \ 0<r_0<r\in\mathbb{R}^+\ \textit{and}\ \ 1<k\in\mathbb{R}^+. \tag{17M}$$

Proof. Let $E:=\overline{D}(r)$ in (16). Then $\lambda(E)=\pi r^2$,

$$\ln\frac{100kr^2}{\lambda(E)}=\ln\frac{100k}{\pi}\leq\frac{7}{2}\ln ek,\quad \mathsf{B}_{U^+}(r)\overset{(2)}{=}\frac{1}{\lambda(E)}\int_E U^+\mathrm{d}\lambda,\quad \mathsf{B}_{|u|}(r)\overset{(2)}{=}\frac{1}{\lambda(E)}\int_E |u|\mathrm{d}\lambda,$$

and by (16) we obtain (17). □

For a meromorphic function $F\not\equiv 0$ on $\mathbb{C}$, in the frame of traditional notation (9), we denote the average of $\ln^+|F|$ over a disc $\overline{D}(r)$ as

$$m^{[2]}(r,F)\underset{r\in\mathbb{R}^+}{:=}\frac{2}{r^2}\int_0^r\left(\frac{1}{2\pi}\int_0^{2\pi}\ln^+\bigl|F(te^{i\varphi})\bigr|\mathrm{d}\varphi\right)t\mathrm{d}t\overset{(1C)}{=}\frac{2}{r^2}\int_0^r m(t,F)\,t\mathrm{d}t\overset{(2)}{=}\mathsf{B}_{\ln^+|F|}(r). \tag{18}$$

Corollary 3. *Let, under the conditions of Theorem 1, $F\not\equiv 0$ be a meromorphic function, and $f\not\equiv 0$ be an entire function. Then, in the traditional notation (9M)–(9N) and (18), we have*

$$\int_E\ln^+\bigl|F(z)\bigr|\mathrm{d}\lambda(z)\leq\frac{2k}{k-1}\bigl(T(kr,F)-N(r_0,F)\bigr)\lambda(E)\ln\frac{100kr^2}{\lambda(E)}, \tag{19T}$$

$$\int_E\Bigl|\ln\bigl|f(z)\bigr|\Bigr|\mathrm{d}\lambda(z)\leq\frac{3k}{k-1}\bigl(\ln^+M(kr,f)+m(r_0,1/f)\bigr)\ln\frac{100kr^2}{\lambda(E)}. \tag{19M}$$

$$m^{[2]}(r,F)\leq\frac{7k\ln(ek)}{k-1}\bigl(T(kr,F)-N(r_0,F)\bigr), \tag{19F}$$

$$m^{[2]}(r,f)+m^{[2]}(r,1/f)\leq\frac{11k\ln(ek)}{k-1}\bigl(\mathsf{M}_{u^+}(kr)+\mathsf{C}_{u^-}(r_0)\bigr) \tag{19f}$$

for all $0<r_0<r\in\mathbb{R}^+$ and $1<k\in\mathbb{R}^+$.

Proof. For $U:=\ln|F|$, we obtain (19T) by (16T), (13), (10), and also (19F) by (17T), (13), (18).

For $u:=\ln|f|$ we obtain (19M) by (16M), (14), (10), and also (19f) by (17M), (14), (18) since $m^{[2]}(r,f)+m^{[2]}(r,1/f)\overset{(18)}{=}\mathsf{B}_{|\ln|f||}(r)$ for all $r\in\mathbb{R}^+$. □

3. LEMMATA AND PROOF OF THEOREM 1

Lemma 1. *Let $0\leq r<R<+\infty$, $E\subset\overline{D}(r)$ be λ-measurable, $U=u-v$ be a difference of subharmonic functions $u,v\in\mathsf{sbh}_*\bigl(\overline{D}(R)\bigr)$, $\varDelta_v$ be the Riesz measure of v. Then*

$$\int_E U^+\mathrm{d}\lambda\leq\frac{1}{2}\left(\frac{R+r}{R-r}\mathsf{C}_{U^+}(R)+\varDelta_v^{\text{rad}}(R)\right)\lambda(E)\ln\frac{(10R)^2}{\lambda(E)}. \tag{20}$$

*Proof.*For $w\in E\subset\overline{D}(r)$, by the Poisson–Jensen formula [7, 4.5], we have

$$U(w)=\frac{1}{2\pi}\int_0^{2\pi}U(Re^{i\varphi})\mathrm{Re}\frac{Re^{i\varphi}+w}{Re^{i\varphi}-w}\mathrm{d}\varphi-\int_{D(R)}\ln\Bigl|\frac{R^2-z\bar{w}}{R(w-z)}\Bigr|\mathrm{d}\varDelta_u(z)$$

$$+\int_{D(R)}\ln\Bigl|\frac{R^2-z\bar{w}}{R(w-z)}\Bigr|\mathrm{d}\varDelta_v(z)\leq\frac{R+r}{R-r}\mathsf{C}_{U^+}(R)+\int_{D(R)}\ln\frac{2R}{|w-z|}\mathrm{d}\varDelta_v(z)$$

where the right-hand side of the inequality is positive. Hence, by integrating, we get

$$\int_E U^+\mathrm{d}\lambda\leq\frac{R+r}{R-r}\mathsf{C}_{U^+}(R)\lambda(E)+\int_{D(R)}\int_E\ln\frac{2R}{|w-z|}\mathrm{d}\lambda(w)\mathrm{d}\varDelta_v(z)$$

$$\leq\frac{R+r}{R-r}\mathsf{C}_{U^+}(R)\lambda(E)+\varDelta_v\bigl(\overline{D}(R)\bigr)\int_E\ln\frac{2R}{|w-z|}\mathrm{d}\lambda(w). \tag{21}$$

Denote by λ_E the restriction of the Lebesgue measure λ to λ-measurable set $E\subset\overline{D}(r)$. Obviously,

$$\operatorname{supp}\lambda_E\subset\overline{D}(r)\subset D(R), \tag{22s}$$

$$\lambda_E(w,t) \overset{(3z)}{\leq} \pi t^2 \quad \text{for each } w\in\mathbb{C} \text{ and } t\in\mathbb{R}^+, \tag{22t}$$

$$\lambda_E(\mathbb{C})=\lambda(E)\leq \pi r^2\leq \pi R^2. \tag{22E}$$

Consider the last integral in (21):

$$\int\limits_E \ln\frac{2R}{|w-z|}\,\mathrm{d}\lambda(w)=\int\limits_{\mathbb{C}} \ln\frac{2R}{|w-z|}\,\mathrm{d}\lambda_E(w)\overset{(22s)}{=}\int\limits_0^{2R}\ln\frac{2R}{t}\,\mathrm{d}\lambda_E(z;t)\overset{(22t)}{=}\int\limits_0^{2R}\frac{\lambda_E(z;t)}{t}\,\mathrm{d}t. \tag{23}$$

By (22E) we have $\sqrt{\lambda(E)}\overset{(22E)}{\leq}\sqrt{\pi}r<2r\leq 2R$. From here we can split the last integral in (23) into the sum of two positive integrals:

$$\int\limits_0^{2R}\frac{\lambda_E(z;t)}{t}\,\mathrm{d}t=\int\limits_0^{\sqrt{\lambda(E)}}\frac{\lambda_E(z;t)}{t}\,\mathrm{d}t+\int\limits_{\sqrt{\lambda(E)}}^{2R}\frac{\lambda_E(z;t)}{t}\,\mathrm{d}t. \tag{24}$$

Using (22t), we have for first integral on the right-hand side of this equality (24) the estimate

$$\int\limits_0^{\sqrt{\lambda(E)}}\frac{\lambda_E(z;t)}{t}\,\mathrm{d}t\overset{(22t)}{\leq}\int\limits_0^{\sqrt{\lambda(E)}}\frac{\pi t^2}{t}\,\mathrm{d}t=\frac{\pi}{2}\lambda(E). \tag{25}$$

Using (22E), we have for second integral on the right-hand side of (24) the estimate

$$\int\limits_{\sqrt{\lambda(E)}}^{2R}\frac{\lambda_E(z;t)}{t}\,\mathrm{d}t\leq\lambda(\mathbb{C})\int\limits_{\sqrt{\lambda(E)}}^{2R}\frac{\lambda_E(z;t)}{t}\,\mathrm{d}t\overset{(22E)}{=}\frac{1}{2}\lambda(E)\ln\frac{4R^2}{\lambda(E)}. \tag{26}$$

Thus, it follows from (25), (26), and (24) that

$$\int\limits_0^{2R}\frac{\lambda_E(z;t)}{t}\,\mathrm{d}t\leq\frac{\pi}{2}\lambda(E)+\frac{1}{2}\lambda(E)\ln\frac{4R^2}{\lambda(E)}=\frac{1}{2}\lambda(E)\ln\frac{4e^{\pi}R^2}{\lambda(E)}\leq\frac{1}{2}\lambda(E)\ln\frac{(10R)^2}{\lambda(E)}, \tag{27}$$

where $\ln\frac{(10R)^2}{\lambda(E)}\geq 2$, since $\lambda(E)\overset{(22E)}{\leq}\pi R^2$, and estimate (27) together with (23) and (21) gives (20). □

Lemma 2 ([6, Lemma 1]). *Let* $\mu\in \mathit{Meas}^+\bigl(\overline{D}(R)\bigr)$. *Then*

$$\mu^{\mathrm{rad}}(r)\leq\frac{R}{R-r}\mathsf{N}_\mu(r,R)\quad \textit{for each}\ \ 0\leq r\leq R. \tag{28}$$

Lemma 3. *Let* $0<r<+\infty$, $0<b\in\mathbb{R}^+$, $E\subset\overline{D}(r)$ *be* λ*-measurable,* $U=u-v$ *be a difference of subharmonic functions* $u,v\in \mathit{sbh}_*\Bigl(\overline{D}\bigl((1+b)^2r\bigr)\Bigr)$. *Then*

$$\int\limits_E U^+\,\mathrm{d}\lambda\leq\frac{1+b}{b}\Bigl(\mathsf{C}_{U^+}\bigl((1+b)r\bigr)+\mathsf{N}_{\Delta_v}\bigl((1+b)r,(1+b)^2r\bigr)\Bigr)\lambda(E)\ln\frac{\bigl(10(1+b)r\bigr)^2}{\lambda(E)}. \tag{29}$$

Proof. By Lemma 1 with $R:=(1+b)r$ we have

$$\int\limits_E U^+\,\mathrm{d}\lambda\leq\frac{1}{2}\left(\frac{2+b}{b}\mathsf{C}_{U^+}(R)+\varDelta_v^{\mathrm{rad}}\bigl((1+b)r\bigr)\right)\lambda(E)\ln\frac{\bigl(10(1+b)r\bigr)^2}{\lambda(E)}.$$

Hence, by Lemma 2 with $(1+b)^2r$ instead of R and $(1+b)r$ instead of r, we obtain

$$\int_E U^+ \mathrm{d}\lambda \leq \frac{1}{2}\frac{2+b}{b}\Bigl(\mathsf{C}_{U^+}\bigl((1+b)r\bigr)+\mathsf{N}_{\Delta_v}\bigl((1+b)r,(1+b)^2r\bigr)\Bigr)\lambda(E)\ln\frac{\bigl(10(1+b)r\bigr)^2}{\lambda(E)},$$

which gives (29). □

Proof of Theorem 1. We can assume that $U = u - v$ is the canonical representation of U. Consider a number $b > 0$ such that $(1+b)^2 = k$. By Lemma 3, we have

$$\int_E U^+ \mathrm{d}\lambda \leq \frac{\sqrt{k}}{\sqrt{k}-1}\Bigl(\mathsf{C}_{U^+}(\sqrt{k}r)+\mathsf{N}_{\Delta_v}(\sqrt{k}r,kr)\Bigr)\lambda(E)\ln\frac{\bigl(10\sqrt{k}r\bigr)^2}{\lambda(E)}$$

$$\leq 2\frac{k}{k-1}\Bigl(\underbrace{\mathsf{C}_{U^+}(r_0,kr)+\mathsf{N}_{\Delta_v}(r_0,kr)}_{\mathsf{T}_U(r_0,kr)}+\mathsf{C}_{U^+}(r_0)\Bigr)\lambda(E)\ln\frac{\bigl(10r\bigr)^2k}{\lambda(E)},$$

and, by definition (7), obtain (16T). If $u \in \mathsf{sbh}_*(\mathbb{C})$, then the function M_u^+ is increasing, and

$$\int_E u^+ \mathrm{d}\lambda \overset{(1M)}{\leq} \mathsf{M}_{u^+}(r)\lambda(E) \quad \text{for } E \subset \overline{D}(r). \tag{30}$$

For $U_u := 0 - u$, the difference $0 - u$ is the canonical representation of δ-subharmonic non-trivial function U_u and we have

$$\mathsf{T}_{U_u}(r,R) \overset{(8)}{=} \mathsf{C}_{\sup\{0,u\}}(r,R) = \mathsf{C}_{u^+}(r,R) \leq \mathsf{C}_{u^+}(R) \leq \mathsf{M}_{u^+}(R). \tag{31}$$

Hence, by Theorem 2 in part (16T) for U_u in the role of U, we obtain

$$\int_E (-u)^+ \mathrm{d}\lambda \overset{(16T)}{\leq} \frac{2k}{k-1}\bigl(\mathsf{T}_{U_u}(r_0,kr)+\mathsf{C}_{U_u^+}(r_0)\bigr)\lambda(E)\ln\frac{100kr^2}{\lambda(E)}$$

$$\overset{(31)}{\leq} \frac{2k}{k-1}\bigl(\mathsf{M}_{u^+}(kr)+\mathsf{C}_{(-u)^+}(r_0)\bigr)\lambda(E)\ln\frac{100kr^2}{\lambda(E)}.$$

The latter together with (30) gives (16M). □

FUNDING

The research is funded in the framework of executing the development program of Scientific Educational Mathematical Center of Volga Federal District by additional agreement no. 075-02-2020-1421/1 to agreement no. 075-02-2020-1421.

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