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MODIFIED SERIES OF INTEGRABLE DISCRETE EQUATIONS ON A QUADRATIC LATTICE WITH A NONSTANDARD SYMMETRY STRUCTURE

R. N. Garifullin[*†] **and R. I. Yamilov**[*‡]

We recently constructed a series of integrable discrete autonomous equations on a quadratic lattice with a nonstandard structure of higher symmetries. Here, we construct a modified series using discrete nonpoint transformations. We use both noninvertible linearizable transformations and nonpoint transformations that are invertible on the solutions of the discrete equation. As a result, we obtain several new examples of discrete equations together with their higher symmetries and master symmetries. The constructed higher symmetries give new integrable examples of five- and seven-point differential–difference equations together with their master symmetries. The method for constructing noninvertible linearizable transformations using conservation laws is considered for the first time in the case of discrete equations.

Keywords: discrete equation, higher symmetry, master symmetry, differential–difference equation, noninvertible transformation

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1. Introduction

In [1], we presented an infinite series of discrete equations of the form

$$(u_{n,m+1}+1)(u_{n,m}-1)=\beta_N(u_{n+1,m+1}-1)(u_{n+1,m}+1), \tag{1}$$

where $n,m\in\mathbb{Z}$ are discrete independent variables and $u_{n,m}$ is an unknown function of two discrete variables. The equations of the series are labeled by natural numbers N, and the coefficients β_N are roots of unity: $\beta_N^N=1$, $N\geq 1$.

To distinguish equations with different N, we consider primitive roots of unity. Clearly, $\beta_1=1$. For $N>1$, primitive roots are defined as

$$\beta_N^N=1,\qquad \beta_N^j\neq 1,\quad 1\leq j<N. \tag{2}$$

[*]Mathematics Institute with Computation Center, Ufa Federal Research Center, RAS, Ufa, Russia, e-mail: rustem@matem.anrb.ru.

[†]Bashkir State University, Ufa, Russia.

[‡]Deceased.

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In particular,

$$\beta_1 = 1, \qquad \beta_2 = -1, \qquad \beta_3 = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \qquad \beta_4 = \pm i, \tag{3}$$

i.e., in the last two cases, there are two primitive roots corresponding to the plus and minus signs. For any $N > 2$, there exist at least two primitive roots $\beta_N = e^{\pm 2i\pi/N}$. Therefore, we consider the series of equations of form (1) where the coefficients β_N are Nth-order primitive roots of unity.

Equations of series (1) are integrable in the sense that they have infinite hierarchies of higher symmetries and conservation laws the same as L–A pairs. Infinite hierarchies of autonomous equations can be distinguished among these symmetries and conservation laws.

Analogous series of autonomous discrete equations were considered in [2]. Series of discrete Darboux-integrable equations and Burgers-type equations were studied in [3], [4].

Using nonpoint transformations, we here obtain new series of discrete equations together with higher symmetries. For simplicity, we restrict ourself to several simplest higher symmetries in each direction.

As shown in [1], the first- and second-order symmetries in the direction m for any equation of series (1) have the forms

$$\partial_{t_1} u_{n,m} = \beta_N^n (u_{n,m}^2 - 1)(u_{n,m+1} - u_{n,m-1}), \tag{4}$$

$$\begin{aligned} \partial_{t_2} u_{n,m} = \beta_N^{2n} (u_{n,m}^2 - 1)[&(u_{n,m+1}^2 - 1)(u_{n,m+2} + u_{n,m}) - \\ &- (u_{n,m-1}^2 - 1)(u_{n,m} + u_{n,m-2}) - 4(u_{n,m+1} - u_{n,m-1})]. \end{aligned} \tag{5}$$

There is an analogous higher symmetry in the direction m in each order. The structure of these symmetries is such that the Nth Eq. (1) has autonomous higher symmetries of orders kN, $k \in \mathbb{N}$.

The structure of higher symmetries is nonstandard in the direction n. The form and order of the simplest symmetry in this direction depends on N. The simplest higher symmetry of the equation of series (1) with $N = 1$ has the order one:

$$\partial_{\theta_1} u_{n,m} = (u_{n,m}^2 - 1)\left(\frac{a_{n+1}}{u_{n+1,m} + u_{n,m}} - \frac{a_n}{u_{n,m} + u_{n-1,m}}\right), \tag{6}$$

where $a_n = b + cn$ and b and c are arbitrary constants.

For $N = 2$, the simplest higher symmetry has the order two:

$$\begin{aligned} &\partial_{\theta_2} u_{n,m} = (u_{n,m}^2 - 1)(T_n - 1)\left(\frac{a_{n+1}(u_{n+1,m} + u_{n,m})}{U_{n,m}} + \frac{a_n(u_{n-1,m} + u_{n-2,m})}{U_{n-1,m}}\right), \\ &U_{n,m} = (u_{n+1,m} + u_{n,m})(u_{n,m} + u_{n-1,m}) - 2(u_{n,m}^2 - 1). \end{aligned} \tag{7}$$

Here, the function $a_n = b_n + cn$, where c is a constant and $b_n \equiv b_{n+2}$ is an arbitrary two-periodic function of n. It can be represented in the form $b_n = b^{(1)} + b^{(2)}(-1)^n$ with two arbitrary constants $b^{(1)}$ and $b^{(2)}$. Here, the shift operator in the direction n is denoted by T_n: $T_n h_{n,m} = h_{n+1,m}$.

For $N = 3$, the simplest higher symmetry has the order three:

$$\begin{aligned} &\partial_{\theta_3} u_{n,m} = (u_{n,m}^2 - 1)(T_n - 1)\left(\frac{a_{n+2}V_{n,m}}{U_{n,m}} + \frac{a_n W_{n,m}}{U_{n-2,m}} + (T_n + 1)\frac{a_{n+1}Z_{n,m}}{U_{n-1,m}}\right), \\ &V_{n,m} = \beta_3^2(u_{n+1,m}^2 - 1) + u_{n+1,m}(u_{n+2,m} - u_{n-1,m}) - u_{n+2,m}u_{n-1,m} + 1, \\ &W_{n,m} = \beta_3(u_{n-2,m}^2 - 1) + u_{n-2,m}(u_{n-1,m} + u_{n-3,m}) + u_{n-1,m}u_{n-3,m} + 1, \\ &Z_{n,m} = (u_{n+1,m} + u_{n,m})(u_{n-1,m} + u_{n-2,m}), \\ &U_{n,m} = \beta_3^2(u_{n+1,m}^2 - 1)(u_{n,m} + u_{n-1,m}) + \beta_3(u_{n,m}^2 - 1)(u_{n+2,m} + u_{n+1,m}) + \\ &\qquad + (u_{n+1,m}u_{n,m} + 1)(u_{n+2,m} + u_{n-1,m}) + (u_{n+1,m} + u_{n,m})(u_{n+2,m}u_{n-1,m} + 1). \end{aligned} \tag{8}$$

Here, β_3 is one of the two primitive roots of unity in (3). The function a_n is given by $a_n = b_n + cn$, where c is a constant and $b_n \equiv b_{n+3}$ is an arbitrary three-periodic function, which can be represented in the form $b_n = b^{(1)} + b^{(2)}\beta_3^n + b^{(3)}\beta_3^{2n}$, where $b^{(1)}$, $b^{(2)}$, and $b^{(3)}$ are arbitrary constants.

Each of formulas (6)–(8) contains an autonomous symmetry corresponding to $a_n \equiv 1$. The case where $a_n = n$ plays the role of the master symmetry for the corresponding discrete equation (1). For any fixed m, formulas (6)–(8) determine integrable differential–difference equations for $c = 0$ and their master symmetries for $a_n = n$.

Using nonpoint transformations, we obtain new examples of autonomous five- and seven-point differential–difference equations together with their master symmetries from formulas (6)–(8).

The nonpoint transformations that we use to construct modified series and higher symmetries are divided into two types. The first type comprises the discrete noninvertible linearizable transformations introduced in [5], which are constructed using special conservation laws. The method we use for constructing linearizable transformations from conservation laws in the case of discrete equations is apparently new.

The second type contains nonpoint transformations that are invertible on solutions of the discrete equation [6], [7]. In the case of higher symmetries and the corresponding differential–difference equations, such transformations lead to noninvertible Miura-type transformations or compound-linearizable transformations.

In Sec. 2, we consider discrete noninvertible linearizable transformations. We first describe the procedure for constructing such transformations using special conservation laws and the scheme for constructing modified discrete equations and the corresponding higher symmetries. As a result of applying this method, we then obtain four modified series of discrete equations together with the simplest higher symmetries. In Sec. 3, we use nonpoint transformations that are invertible on solutions of the discrete equation. This allows constructing two more modifications together with higher symmetries and master symmetries.

2. Linearizable transformations

Following [5], where differential–difference equations were discussed, we introduce the notion of a linearizable transformation. In this section, we consider first-order transformations that are linearized by point transformations. Such transformation in the direction n can be written in the form

$$U_{n,m} = F_{n,m}[(T_n - 1)G_{n,m}(V_{n,m})], \tag{9}$$

where $F'_{n,m}(x) \neq 0$ and $G'_{n,m}(x) \neq 0$ for any n and m. Using obvious nonautonomous point variable replacements $U_{n,m}$ and $V_{n,m}$, we can bring such a transformation to the form

$$\widehat{U}_{n,m} = (T_n - 1)\widehat{V}_{n,m},$$

which is the simplest linear transformation. Compositions of such transformations are also said to be linearizable [5]. In the case where the composition consists of transformations in different directions, we obtain a compound-linearizable transformation (see example (65) in Sec. 3.1). We note that Miura-type transformations are not linearizable [5]. An example of such a transformation is given in Sec. 3.2 (see (75)).

In the direction m, the first-order linearizable transformations studied in this section have an analogous form,

$$U_{n,m} = F_{n,m}[(T_m - 1)G_{n,m}(V_{n,m})],$$

where T_m denotes the shift operator in the direction m: $T_m h_{n,m} = h_{n,m+1}$.

2.1. Construction of the simplest linearizable transformations. We assume that Eq. (1) has a conservation law of the form

$$(T_m - 1)p_{n,m}(u_{n,m}) = (T_n - 1)q_{n,m}(u_{n,m+1}, u_{n,m}), \tag{10}$$

where $p'_{n,m}(x) \neq 0$ for any n and m, i.e., $p_{n,m}(x)$ is invertible for all n and m. We can introduce a new unknown function $v_{n,m}$:

$$p_{n,m}(u_{n,m}) = (T_n - 1)r_{n,m}(v_{n,m}), \tag{11}$$

where $r'_{n,m}(x) \neq 0$ for any n and m. This relation gives an explicit formula for $u_{n,m}$:

$$u_{n,m} = p^{-1}_{n,m}[(T_n - 1)r_{n,m}(v_{n,m})]. \tag{12}$$

Substituting function (12) in (10) and integrating over the discrete variable n, i.e., applying the inverse operator $(T_n - 1)^{-1}$, we obtain a discrete equation for the unknown function $v_{n,m}$:

$$(T_m - 1)r_{n,m}(v_{n,m}) = q_{n,m}\big(p^{-1}_{n,m+1}[(T_n - 1)r_{n,m+1}(v_{n,m+1})], p^{-1}_{n,m}[(T_n - 1)r_{n,m}(v_{n,m})]\big). \tag{13}$$

Discrete equation (13) is transformed into Eq. (1) by linearizable replacement (12).

The choice of the function $r_{n,m}$ in replacement (11) corresponds to a nonautonomous point transformation of the unknown function $v_{n,m}$. Because of integration over n in relation (13), an integration function ω_m must appear; it is eliminated by the choice of $r_{n,m}$. Moreover, we choose the function $r_{n,m}$ to simplify replacement (12) and Eq. (13).

Higher symmetries of discrete equation (13) in the direction n can be found from (11) (see [5]). For example, as the diagonalization method for L–A pairs shows [8], $p_{n,m}(u_{n,m})$ becomes a conservation law density of not only discrete equation (1) but also its higher symmetries (6)–(8) with $c = 0$. By virtue of such higher symmetries, we have

$$D_{\theta_j} p_{n,m}(u_{n,m}) = (T_n - 1)h^{(j)}_{n,m}(u_{n+j-1,m}, u_{n+j-2,m}, \ldots, u_{n-j,m}), \quad j = 1, 2, 3. \tag{14}$$

Differentiating (11) with respect to θ_j and integrating over n, we obtain

$$r'_{n,m}(v_{n,m}) D_{\theta_j} v_{n,m} = h^{(j)}_{n,m} + \widetilde{\omega}^{(j)}_m, \tag{15}$$

where all functions of the form $u_{n+k,m}$ are replaced with $v_{n+k,m}$ by virtue of (12). Using the consistency condition for discrete equation (13) and Eq. (15), we specify the integration function $\widetilde{\omega}^{(j)}_m$.

For master symmetries (6)–(8) with $a_n = n$, we rewrite the function $D_{\theta_j} p_{n,m}(u_{n,m})$ in terms of $v_{n+k,m}$ using (12) and in examples obtain a representation of the form

$$D_{\theta_j} p_{n,m}(u_{n,m}) = (T_n - 1)\tilde{h}^{(j)}_{n,m}(v_{n+j,m}, v_{n+j-1,m}, \ldots, v_{n-j,m}). \tag{16}$$

This representation allows finding a new master symmetry as in case (14).

To obtain higher symmetries of discrete equation (13) in the direction m, we write (13) in the form

$$(T_m - 1)r_{n,m}(v_{n,m}) = q_{n,m}(u_{n,m+1}, u_{n,m}). \tag{17}$$

Differentiating with respect to t_j, $j = 1, 2$, by virtue of (4) and (5), we obtain

$$(T_m - 1)D_{t_j} r_{n,m}(v_{n,m}) = s^{(j)}_{n,m}(u_{n,m+j+1}, u_{n,m+j}, \ldots, u_{n,m-j}). \tag{18}$$

Eliminating $u_{n,m+k}$ in the right-hand side of this relation using (12), we obtain the dependence on $v_{n,m+k}$ and $v_{n+1,m+k}$. Using discrete equation (13), we bring this relation to the form

$$(T_m - 1)[r'_{n,m}(v_{n,m})D_{t_j} v_{n,m}] = \tilde{s}^{(j)}_{n,m}(v_{n+1,m}, v_{n,m+j+1}, v_{n,m+j}, \ldots, v_{n,m-j}). \tag{19}$$

In all examples considered below, we find that $\tilde{s}^{(j)}_{n,m}$ is independent of $v_{n+1,m}$ and this function belongs to the image of the operator $T_m - 1$. Integrating relation (19) over m, we obtain a differential–difference equation for $v_{n,m}$ with the integration function $\widehat{\omega}^{(j)}_n$. For this equation, we verify the compatibility with not only discrete equation (13) but also replacement (12). Moreover, we specify the function $\widehat{\omega}^{(j)}_n$.

In the case of a conservation law symmetric to (10) that has the form

$$(T_n - 1)p_{n,m}(u_{n,m}) = (T_m - 1)q_{n,m}(u_{n+1,m}, u_{n,m}), \tag{20}$$

where $p'_{n,m}(x) \neq 0$ for any n and m, we use the scheme described above up to the exchange $n \leftrightarrow m$.

2.2. First modification. All discrete equations of series (1) have a conservation law of form (10):

$$(T_m - 1)(\beta^n_N u_{n,m}) = (T_n - 1)\left[-\frac{1}{2}\beta^n_N (u_{n,m+1} - 1)(u_{n,m} + 1)\right]. \tag{21}$$

We change the variables $\beta^n_N u_{n,m} = (T_n - 1)(\beta^n_N v_{n,m})$, which are written in form (12) as

$$u_{n,m} = \beta_N v_{n+1,m} - v_{n,m}. \tag{22}$$

Using (13), we obtain the discrete equation for $v_{n,m}$:

$$(\beta_N v_{n+1,m+1} - v_{n,m+1} - 1)(\beta_N v_{n+1,m} - v_{n,m} + 1) + 2(v_{n,m+1} - v_{n,m}) = 0. \tag{23}$$

For Eq. (23), we write four higher symmetries, which reduce to (4)–(7) by transformation (22). In the direction m, we have the symmetries for any N

$$\partial_{t_1} v_{n,m} = -2\beta^n_N (v_{n,m+1} - v_{n,m})(v_{n,m} - v_{n,m-1}), \tag{24}$$

$$\partial_{t_2} v_{n,m} = 4\beta^{2n}_N (v_{n,m+2} - v_{n,m-2})(v_{n,m+1} - v_{n,m})(v_{n,m} - v_{n,m-1}) \tag{25}$$

corresponding to (4) and (5). Relation (17), which is used to construct these symmetries, has the form

$$2(v_{n,m} - v_{n,m+1}) = (u_{n,m+1} - 1)(u_{n,m} + 1). \tag{26}$$

In the direction n for $N = 1$, $\beta_N = 1$, we obtain the first-order symmetry corresponding to (6):

$$\partial_{\theta_1} v_{n,m} = -a_n \frac{(v_{n+1,m} - v_{n,m})(v_{n,m} - v_{n-1,m}) + 1}{v_{n+1,m} - v_{n-1,m}} + c v_{n,m}, \tag{27}$$

where $a_n = b + cn$ as in (6). For $N = 2$, $\beta_N = -1$, the higher second-order symmetry corresponding to (7) has the form

$$\begin{aligned}\partial_{\theta_2} v_{n,m} = {} & [(v_{n+1,m} + v_{n,m})^2 - 1] \times \\ & \times \left(a_{n+1}\frac{v_{n+2,m} + v_{n,m} + 2v_{n-1,m}}{V_{n,m}} - a_n \frac{v_{n,m} + 2v_{n-1,m} + v_{n-2,m}}{V_{n-1,m}}\right) + \\ & + (b_n - b_{n+1})(v_{n+1,m} + v_{n,m}) + c(v_{n,m} - v_{n+1,m}), \\ V_{n,m} = {} & (v_{n+2,m} + v_{n,m})(v_{n+1,m} + v_{n-1,m}) + 2(v_{n+2,m}v_{n,m} + v_{n+1,m}v_{n-1,m} + 1),\end{aligned} \tag{28}$$

where $a_n = b_n + cn$ as in (7).

Fixing the integration functions $\widetilde{\omega}_m^{(j)}$ and $\widetilde{\omega}_n^{(j)}$ that arise in constructing these higher symmetries, in all cases, we obtain an additional term of the form $\nu\beta_N^{-n}$ with an arbitrary constant coefficient ν, which we omit because discrete equation (13) is obviously invariant under the one-parameter transformation group $v_{n,m} \to v_{n,m} + \tau\beta_N^{-n}$ corresponding to the point symmetry $\partial_\tau v_{n,m} = \beta_N^{-n}$. This point symmetry explains the presence of the abovementioned addition to the higher symmetries.

We note that for $N = 2$, autonomous discrete equation (23) has autonomous second-order higher symmetries in both directions. Symmetry (25) becomes autonomous because $\beta_2 = -1$, and among symmetries of (28), we have the autonomous particular case with $b_n \equiv 1$ and $c = 0$.

2.3. New examples constructed in this paper. Discrete equation (23) with $N = 1$, i.e., $\beta_N = 1$, coincides with Eq. (T4) in [9] (Table 1 on p. 17) up to the change of variables $v_{n,m} = -2u_{m,n} + m$. The other equations of series (23) are apparently new. The same holds for remaining modified series of discrete equations. All equations of the series are new except the case $N = 1$ in some of them.

For each fixed n, higher symmetries (24) and (25) in the direction m are one of the known modifications of the Volterra equation and its higher symmetry. Integrable equations of the Volterra type were completely classified in [10] (see review [11] for details). Symmetry (24) corresponds to a particular case of Eq. (V6) in the list of Volterra-type equations in [11]. All the remaining higher symmetries in the direction m listed below correspond to differential–difference equations of a similar nature; moreover, all three-point examples belong to the same list in [11].

For each fixed m, higher symmetry (27) in the direction n determines an integrable differential–difference equation with $b = 1$ and $c = 0$ and its master symmetry with $b = 0$ and $c = 1$. This differential–difference equation is a particular case of Eq. (V4) with $\nu = 0$ in [11]. For all equations of form (V4) with $\nu = 0$ in [12], master symmetries with an explicit dependence on its time have been constructed (see Sec. 4.3 in [13] for more details). In particular, in our case, the master symmetry has the form

$$\partial_\tau v_n = -\frac{n}{\tau + 1}\frac{(v_{n+1} - v_n)(v_n - v_{n-1}) + (\tau + 1)^2}{v_{n+1} - v_{n-1}}, \tag{29}$$

where we omit the index m for brevity. In contrast to (29), the master symmetry obtained from (27) is independent of its time and allows constructing higher symmetries not only for the differential–difference equation but also for discrete equation (23) with $N = 1$. Moreover, it naturally arises in constructing higher symmetries of the discrete equation.[1]

In the other examples, the higher symmetries in the direction n corresponding to $N = 1$ in the discrete equation in the particular case $b = 1$ and $c = 0$ determine known Volterra-type differential–difference equations in [11]. The master symmetries corresponding to the case $b = 0$ and $c = 1$ are mostly known.

With a fixed m, higher symmetry (28) determines an integrable five-point autonomous differential–difference equation with $b_n \equiv 1$ and $c = 0$, its nonautonomous symmetry with $b_n = (-1)^n$ and $c = 0$, and the master symmetry with $b_n \equiv 0$ and $c = 1$. The equation and master symmetry are apparently new.

The remaining higher symmetries in the direction n given below and corresponding to the cases $N = 2$ and $N = 3$ in the discrete equations generate examples of five- and seven-point differential–difference equations. The condition $b_n \equiv 1$, $c = 0$ distinguishes the autonomous integrable cases among them. For each of them, we automatically obtain one or two nonautonomous symmetries and the master symmetry. All such examples including the nonautonomous equations and master symmetries except two five-point autonomous equations mentioned below are new.

[1] Without going into the details, we note that there is a connection between these master symmetries, i.e., the example obtained from (27) is not essentially new.

2.4. Second modification. All discrete equations of series (1) have a conservation law of the form

$$(T_m - 1)\log\frac{u_{n,m}+1}{u_{n,m}-1} = (T_n - 1)\log\Big(\beta_N^n (u_{n,m+1}-1)(u_{n,m}+1)\Big). \tag{30}$$

To simplify new discrete equation (13), we choose $r_{n,m} = \log w_{n,m} + m\log 4$, where $w_{n,m}$ is a new unknown function, as the function $r_{n,m}$ in relation (11). Then (11) implies that

$$\frac{u_{n,m}+1}{u_{n,m}-1} = \frac{w_{n+1,m}}{w_{n,m}}, \tag{31}$$

i.e., in explicit form, we have the transformation

$$u_{n,m} = \frac{w_{n+1,m}+w_{n,m}}{w_{n+1,m}-w_{n,m}}. \tag{32}$$

Discrete equation (13) for $w_{n,m}$ can be written as

$$(w_{n+1,m+1} - w_{n,m+1})(w_{n+1,m} - w_{n,m}) = \beta_N^n w_{n+1,m} w_{n,m}. \tag{33}$$

We obtain a series of modified discrete equations with one nonautonomous coefficient β_N^n. The first equation of this series, corresponding to $N = 1$, in the variable $u_{n,m} = (-1)^n w_{m,n}$ coincides with (T6) in [9].

Relation (17), which is used to construct higher symmetries in the direction m, has the form

$$4\frac{w_{n,m+1}}{w_{n,m}} = \beta_N^n (u_{n,m+1}-1)(u_{n,m}+1). \tag{34}$$

Both symmetries in this direction turn out to be autonomous:

$$\frac{1}{4}\partial_{t_1} w_{n,m} = w_{n,m+1} + \frac{w_{n,m}^2}{w_{n,m-1}}, \tag{35}$$

$$\frac{1}{16}\partial_{t_2} w_{n,m} = w_{n,m+2} + \frac{w_{n,m+1}^2}{w_{n,m}} + \frac{2w_{n,m+1}w_{n,m}}{w_{n,m-1}} + \frac{w_{n,m}^3}{w_{n,m-1}^2} + \frac{w_{n,m}^2}{w_{n,m-2}}. \tag{36}$$

In the direction n with $N = 1$ and $\beta_N = 1$, we obtain the first-order symmetry corresponding to (6):

$$\partial_{\theta_1} w_{n,m} = -a_n \frac{(w_{n+1,m}-w_{n,m})(w_{n,m}-w_{n-1,m})}{w_{n+1,m}-w_{n-1,m}} + cmw_{n,m}, \tag{37}$$

where $a_n = b + cn$ as in (6). The autonomous particular case $b = 1$, $c = 0$ of this symmetry is invariant under linear-fractional transformations $w_{n,m}$ (or Möbius invariant). This is just the well-known Schwarzian version of the Volterra chain.

It is convenient to introduce the notation from [14] (Sec. 4)

$$\begin{aligned} Y_{n,m} &= \frac{(w_{n+1,m}-w_{n,m})(w_{n,m}-w_{n-1,m})}{w_{n+1,m}-w_{n-1,m}}, \\ X_{n,m} &= \frac{(w_{n+1,m}-w_{n,m})(w_{n-1,m}-w_{n-2,m})}{(w_{n+1,m}-w_{n-1,m})(w_{n,m}-w_{n-2,m})}. \end{aligned} \tag{38}$$

Then Eq. (37) can be easily written in terms of $Y_{n,m}$. With $N = 2$ and $\beta_N = -1$, the second-order symmetry corresponding to (7) has the form

$$\partial_{\theta_2} w_{n,m} = Y_{n,m}\left(\frac{a_{n+1}}{2X_{n+1,m}-1} + \frac{a_n}{2X_{n,m}-1}\right) + 2cmw_{n,m}, \tag{39}$$

where $a_n = b_n + cn$ as in (7). For $c = 0$, it is obviously Möbius invariant under $w_{n,m}$ the same as (37). Fixing the variable m, we see that the autonomous case $b_n \equiv 1$, $c = 0$ coincides with the well-known Eq. (32) in [14] up to a shift of θ_2. Here, we find nonautonomous symmetry (39) with $b_n = (-1)^n$ and $c = 0$ and the master symmetry corresponding to $b_n \equiv 0$ and $c = 1$ for this equation. Moreover, discrete equation (33) with $N = 2$ compatible with symmetry (39) determines a Bäcklund autotransformation for this well-known equation (see Sec. 4.1 in [13] for details).

For $N = 3$, the third-order symmetry corresponding to (8) has the form

$$\partial_{\theta_3} w_{n,m} = Y_{n,m}\left(\frac{a_{n+2}(W_{n+2,m}+\beta_3)}{\beta_3 W_{n+1,m} - W_{n+2,m}} + \frac{a_{n+1}(\beta_3 - 1)}{\beta_3 W_{n,m} - W_{n+1,m}} - \frac{a_n(\beta_3 W_{n-1,m}+1)}{\beta_3 W_{n-1,m} - W_{n,m}}\right) + 3cmw_{n,m}, \quad (40)$$
$$W_{n,m} = 3X_{n,m} - 1,$$

where $X_{n,m}$ and $Y_{n,m}$ are functions given by (38) and β_3 and $a_n = b_n + cn$ are defined as in Eq. (8). For fixed m and $c = 0$, Eq. (40) is obviously Möbius invariant. It contains three commuting particular cases distinguished by the conditions $b_n \equiv 1$, $b_n = \beta_3^n$, and $b_n = \beta_3^{2n}$. The first of these is a new autonomous seven-point Möbius-invariant integrable equation. The case $b_n \equiv 0$, $c = 1$ determines the master symmetry for all three equations. Equation (33) with $N = 3$ determines a Bäcklund autotransformation for them.

2.5. Third modification. We start with Eq. (33), which for any N has the conservation law

$$(T_n - 1)[(-1)^{n+m}\log w_{n,m}] = (T_m - 1)[(-1)^{n+m}\log(\beta_N^{-n/2}(w_{n,m} - w_{n+1,m}))]. \quad (41)$$

This conservation law has form (20), and the formula symmetric to (11) is

$$(-1)^{n+m}\log w_{n,m} = (T_m - 1)[(-1)^{n+m+1}\log z_{n,m}]. \quad (42)$$

This yields the simple linearizable transformation

$$w_{n,m} = z_{n,m+1}z_{n,m}, \quad (43)$$

which allows obtaining the discrete equation

$$z_{n,m+1}z_{n,m} - z_{n+1,m+1}z_{n+1,m} = \beta_N^{n/2} z_{n+1,m}z_{n,m}. \quad (44)$$

The relation for constructing higher symmetries in the direction n has the form

$$\beta_N^{n/2} z_{n,m}z_{n+1,m} = w_{n,m} - w_{n+1,m}. \quad (45)$$

The presence of half-integer powers of β_N, which is inconvenient for studies, is a disadvantage of Eq. (44). Using the nonautonomous point transformation $z^{\text{new}}_{n,m} = \beta_N^{n(n-1)/4} z_{n,m}$, we transform it into a form analogous to (33):

$$z_{n,m+1}z_{n,m} - \beta_N^{-n} z_{n+1,m+1}z_{n+1,m} = z_{n+1,m}z_{n,m}. \quad (46)$$

This discrete equation with $\beta_N = 1$ is not new, because after the transformation $u_{n,m} = i^{n-m} z_{m,n}$, it coincides with (T7) with $c_2 = 0$ in [9].

We rewrite relations (43) and (45), which are needed for constructing higher symmetries, as

$$w_{n,m} = \beta_N^{-n(n-1)/2} z_{n,m+1}z_{n,m}, \qquad \beta_N^{-n(n-1)/2} z_{n,m}z_{n+1,m} = w_{n,m} - w_{n+1,m}. \quad (47)$$

We note that there are no half-integer powers of β_N here because $n(n-1)/2$ is always an integer.

Two higher symmetries of discrete equation (46) in the direction m independently of N have the forms

$$\frac{1}{4}\partial_{t_1} z_{n,m} = \frac{z_{n,m+1} z_{n,m}}{z_{n,m-1}}, \tag{48}$$

$$\frac{1}{16}\partial_{t_2} z_{n,m} = \frac{z_{n,m+2} z_{n,m+1}}{z_{n,m-1}} + \frac{z^2_{n,m+1} z_{n,m}}{z^2_{n,m-1}} + \frac{z_{n,m+1} z^2_{n,m}}{z_{n,m-1} z_{n,m-2}}. \tag{49}$$

For $N = 1$ and $\beta_N = 1$, Eq. (37) yields the first-order symmetry in the direction n

$$2\partial_{\theta_1} z_{n,m} = -a_n z_{n,m} \frac{z_{n+1,m} - z_{n-1,m}}{z_{n+1,m} + z_{n-1,m}} + c z_{n,m}\left(m - \frac{1}{2}\right), \tag{50}$$

where $a_n = b + cn$ as in (6). After the nonautonomous shift $z^{\text{new}}_{n,m} = i^n z_{n,m}$, this equation with any fixed m coincides with the well-known equation in [11], whose master symmetry is also known [12], [13].

For $N = 2$ and $\beta_N = -1$, the second-order symmetry in the direction n obtained from (39) has the form

$$\begin{aligned}\partial_{\theta_2} z_{n,m} = z_{n,m}\Bigg(&\frac{a_n}{1 + (-1)^n (z_{n+1,m}(z_{n-2,m} - (-1)^n z_{n,m}))/(z_{n-1,m}(z_{n-2,m} + (-1)^n z_{n,m}))} - \\ &- \frac{a_{n+1}}{1 + (-1)^n (z_{n-1,m}(z_{n+2,m} - (-1)^n z_{n,m}))/(z_{n+1,m}(z_{n+2,m} + (-1)^n z_{n,m}))} + \\ &+ cm + \frac{b_{n+1} - b_n}{2}\Bigg),\end{aligned} \tag{51}$$

where $a_n = b_n + cn$ as in (7). To eliminate the coefficient $(-1)^n$, we use the nonautonomous point transformation $z^{\text{new}}_{n,m} = i^{-(n-1)^2/2-1/4} z_{n,m}$ and obtain

$$\begin{aligned}\partial_{\theta_2} z_{n,m} = z_{n,m}\Bigg(&\frac{a_n}{1 - (z_{n+1,m}(z_{n-2,m} - z_{n,m}))/(z_{n-1,m}(z_{n-2,m} + z_{n,m}))} - \\ &- \frac{a_{n+1}}{1 - (z_{n-1,m}(z_{n+2,m} - z_{n,m}))/(z_{n+1,m}(z_{n+2,m} + z_{n,m}))} + cm + \frac{b_{n+1} - b_n}{2}\Bigg).\end{aligned} \tag{52}$$

Because c is a constant and b_n is an arbitrary two-periodic function here, we have three cases for any fixed m: an autonomous differential–difference equation for $a_n \equiv 1$, its symmetry for $a_n = (-1)^n$, and their common master symmetry for $a_n = n$.

Moreover, the second relation in (47) becomes

$$z_{n,m} z_{n+1,m} = w_{n,m} - w_{n+1,m}, \tag{53}$$

and for a fixed m, this relates differential–difference equations (39) and (52). With $a_n \equiv 1$, Eq. (52) becomes autonomous and is related to the well-known autonomous equation (39) with $a_n \equiv 1$ by autonomous transformation (53). Transformation (53) is linearizable, but it is not explicit in both directions [5].

2.6. Fourth modification. The right- and left-hand sides of relation (26) can be denoted by $y_{n,m}$. We thus introduce a new unknown function, which as follows from (4) and (24) satisfies the equation

$$\partial_{t_1} y_{n,m} = \beta_N^n y_{n,m}(y_{n,m+1} - y_{n,m-1}). \tag{54}$$

This is the Volterra equation for any fixed n, which is natural because one of the transformations defined here,

$$y_{n,m} = (u_{n,m+1} - 1)(u_{n,m} + 1), \tag{55}$$

is just the known transformation of the modified Volterra equation into the Volterra equation. The other transformation,

$$y_{n,m} = 2(v_{n,m} - v_{n,m+1}),$$

relating (24) and (54), is also known, as is (55) (see, e.g., review [11]). Letting $\beta_N^n y_{n,m}$ denote the right- and left-hand sides of relation (34), we again obtain relations between (35) and (4) with the same Eq. (54).

We can expect an analogous result from relation (53). Here, we show that the transformation

$$y_{n,m} = w_{n,m} - w_{n+1,m} \tag{56}$$

obtained from (53) allows deriving one more modification together with higher symmetries from discrete equation (33). We note that in contrast to (22), (32), and (43), transformation (56) has another direction in the sense that it allows expressing the new unknown function $y_{n,m}$ in terms of $w_{n,m}$ in an explicit form.

Transformation (56) allows easily rewriting higher symmetries in the direction n. Symmetries (37) and (39) become

$$\partial_{\theta_1} y_{n,m} = y_{n,m}^2\left(\frac{a_{n+1}}{y_{n+1,m}+y_{n,m}} - \frac{a_n}{y_{n,m}+y_{n-1,m}}\right) + c(m-1)y_{n,m}, \tag{57}$$

$$\begin{gathered}\partial_{\theta_2} y_{n,m} = -y_{n,m}^2\left(\frac{a_{n+2}(y_{n+2,m}-y_{n+1,m})}{\Upsilon_{n+1,m}} + \frac{a_{n+1}(y_{n+1,m}-y_{n-1,m})}{\Upsilon_{n,m}} + \right. \\ \left. + \frac{a_n(y_{n-1,m}-y_{n-2,m})}{\Upsilon_{n-1,m}}\right) + 2c(m-1)y_{n,m}, \\ \Upsilon_{n,m} = (y_{n+1,m}+y_{n,m})(y_{n,m}+y_{n-1,m}) - 2y_{n+1,m}y_{n-1,m},\end{gathered} \tag{58}$$

where the functions a_n are the same as in (6) and (7). We can also rewrite higher symmetry (40).

The new discrete equation for $y_{n,m}$ and its higher symmetries in the direction m contain square roots, but it is important that transformation (56) allows constructing them.

We rewrite discrete equation (33) using transformation (56) in the form of a quadratic equation for the unknown function $w_{n,m}$:

$$y_{n,m+1}y_{n,m} = \beta_N^n(w_{n,m} - y_{n,m})w_{n,m}. \tag{59}$$

We introduce the notation

$$\Theta_{n,m} = 1 + 4\beta_N^{-n}\frac{y_{n,m+1}}{y_{n,m}} \tag{60}$$

and then write solution (59) as

$$w_{n,m} = \frac{y_{n,m}}{2}\big(1 + \sqrt{\Theta_{n,m}}\,\big), \tag{61}$$

where $\sqrt{\Theta_{n,m}}$ is any branch of the square root. Applying the operator $1 - T_n$ to relation (61), we obtain the new discrete equation

$$y_{n+1,m}\big(1 + \sqrt{\Theta_{n+1,m}}\,\big) + y_{n,m}\big(1 - \sqrt{\Theta_{n,m}}\,\big) = 0. \tag{62}$$

To find the simplest higher symmetry in the direction m, we differentiate transformation (56) with respect to t_1 by virtue of symmetry (35). In the right-hand side of the obtained expression, we eliminate $w_{n+1,m+1}$, $w_{n+1,m}$, and $w_{n+1,m-1}$ by virtue of transformation (56) and then $w_{n,m}$ and $w_{n,m-1}$ by virtue of (61). We obtain the higher symmetry

$$\partial_{t_1} y_{n,m} = 2\beta_N^n y_{n,m}\big(\sqrt{\Theta_{n,m}}\,\sqrt{\Theta_{n,m-1}} - 1\big). \tag{63}$$

The point transformation $y_{n,m} = e^{\hat{y}_{n,m}}$ allows obtaining a symmetry in terms of $\hat{y}_{n,m}$ that for each fixed n has the form of the known Eq. (V6) in [11].

3. Transformations invertible on solutions of discrete equations

In this section, for the discrete equations, we construct nonpoint transformations that are invertible on solutions of these discrete equations. For higher symmetries of the discrete equations, these transformations generate Miura-type transformations or compound-linearizable transformations [5]. In this case, we use the method developed in [6], [7] for discrete and semidiscrete equations. Another version of this method intended for constructing Miura-type transformations was presented in [15], [16].

As a result, we obtain two more modifications of series (1) of discrete equations together with their higher symmetries.

3.1. First modification. We rewrite discrete equation (1) as

$$\frac{u_{n,m}-1}{u_{n+1,m}+1}=\beta_N\frac{u_{n+1,m+1}-1}{u_{n,m+1}+1}. \tag{64}$$

This representation allows introducing the new unknown function $v_{n,m}$ as

$$v_{n,m+1}=\frac{u_{n,m}-1}{u_{n+1,m}+1},\qquad v_{n,m}=\beta_N\frac{u_{n+1,m}-1}{u_{n,m}+1}. \tag{65}$$

We can solve the obtained expressions for the old unknowns $u_{n,m}$ and $u_{n+1,m}$:

$$\begin{aligned} u_{n,m}&=\frac{v_{n,m+1}v_{n,m}+2\beta_N v_{n,m+1}+\beta_N}{\beta_N-v_{n,m+1}v_{n,m}},\\ u_{n+1,m}&=\frac{v_{n,m+1}v_{n,m}+2v_{n,m}+\beta_N}{\beta_N-v_{n,m+1}v_{n,m}}. \end{aligned} \tag{66}$$

Writing these expressions at the one point $u_{n+1,m}$, we obtain a new discrete equation. It is essentially simplified if we rewrite it in terms of $(u_{n+1,m}-1)/(u_{n+1,m}+1)$:

$$\frac{v_{n,m}(v_{n,m+1}+1)}{v_{n,m}+\beta_N}=\frac{v_{n+1,m+1}(v_{n+1,m}+\beta_N)}{\beta_N(v_{n+1,m+1}+1)}.$$

One more equivalent form is

$$(v_{n+1,m}+\beta_N)(1+\beta_N v_{n,m}^{-1})=\beta_N(v_{n,m+1}+1)(1+v_{n+1,m+1}^{-1}). \tag{67}$$

We see that modified equation (67) is obtained from (1) using transformation (65), which is invertible on solutions of discrete equation (1). The inverse transformation has form (66). We note that for $\beta_N=1$, nothing new is obtained because the point transformation

$$\hat v_{n,m}=\frac{1-v_{n,m}}{1+v_{n,m}} \tag{68}$$

leads to a discrete equation that after the exchange $n\leftrightarrow m$ becomes Eq. (1) with $\beta_N=1$.

Invertible transformation (65) allows rewriting higher symmetries in a regular way [7]. From symmetries (4) and (5) of Eq. (1), we thus obtain the two simplest higher symmetries in the direction m for the new discrete equation (67):

$$\begin{aligned} &\partial_{t_1}v_{n,m}=4\beta_N^{n+1}v_{n,m}(v_{n,m}+1)(v_{n,m}+\beta_N)\frac{v_{n,m+1}-v_{n,m-1}}{V_{n,m+1}V_{n,m}},\\ &V_{n,m}=v_{n,m}v_{n,m-1}-\beta_N, \end{aligned} \tag{69}$$

$$\begin{aligned} \partial_{t_2}v_{n,m}=16\beta_N^{2n+2}(v_{n,m}+1)(v_{n,m}+\beta_N)\biggl[&-\frac{v_{n,m}(v_{n,m+1}+1)(v_{n,m+1}+\beta_N)}{V_{n,m+2}V_{n,m+1}^2}+\\ &+\frac{v_{n,m}(v_{n,m-1}+1)(v_{n,m-1}+\beta_N)}{V_{n,m}^2V_{n,m-1}}-\frac{v_{n,m+1}+v_{n,m}+\beta_N+1}{V_{n,m+1}^2}+\\ &+\frac{v_{n,m}+v_{n,m-1}+\beta_N+1}{V_{n,m}^2}+\frac{\beta_N(v_{n,m}+1)(v_{n,m}+\beta_N)(v_{n,m+1}-v_{n,m-1})}{V_{n,m+1}^2V_{n,m}^2}\biggr]. \end{aligned} \tag{70}$$

We note that for any fixed n, the point transformation $v^{\mathrm{new}}_{n,m} = (\sqrt{\beta_N} - v_{n,m})/(\sqrt{\beta_N} + v_{n,m})$ transforms symmetry (69) into the known equation of form (V2) in [11].

We turn to symmetries in the direction n and consider the case $N = 1$. The higher symmetry obtained from (6) and then rewritten in terms of $\hat{v}_{n,m}$ in (68) for any fixed m becomes the modified Volterra equation with the known master symmetry [17]:

$$4\partial_{\theta_1}\hat{v}_{n,m} = (\hat{v}^2_{n,m} - 1)(a_{n+2}\hat{v}_{n+1,m} - a_n\hat{v}_{n-1,m}), \qquad a_n = b + cn. \tag{71}$$

The simplest higher symmetry for $N = 2$ obtained from (7) has the form

$$\begin{aligned} &\partial_{\theta_2} v_{n,m} = v_{n,m}(1 - T_n^2)\left(a_{n+1}\frac{v_{n,m} - 1}{\widehat{V}_{n,m}} - a_n\frac{v_{n-2,m} + 1}{\widehat{V}_{n-1,m}}\right) - 2cv_{n,m},\\ &\widehat{V}_{n,m} = v_{n,m}v_{n-1,m} - v_{n,m} + v_{n-1,m} + 1, \end{aligned} \tag{72}$$

where a_n is defined as in (7). For any fixed m with $a_n \equiv 1$, it generates a five-point analogue of the modified Volterra equation. On the other hand, this case $a_n \equiv 1$ is a new example of an autonomous differential–difference equation with the nonautonomous symmetry $a_n = (-1)^n$ and the master symmetry $a_n = n$.

Each of transformations (65) transforms higher symmetry (7) directly into (72) because they are completely defined on line n in the same way as the symmetries. This is how they differ from transformations (66), which relate the same symmetries but on solutions of discrete equation (67). In contrast to the transformations studied in Sec. 2, transformations (65) are linearized in a more complicated way. For example, the first of them is a composition of transformations:

$$v_{n,m+1} = \frac{y_{n+1,m}}{y_{n,m}}, \qquad y_{n,m} = z_{n,m} - z_{n-1,m}, \qquad u_{n,m} = \frac{z_{n-1,m} + z_{n,m}}{z_{n-1,m} - z_{n,m}}. \tag{73}$$

They are linearizable because they can be easily written in form (9) up to the shift T_n.

The same compound-linearizable transformations (65) relate higher symmetry (8) to the symmetry

$$\begin{aligned} \partial_{\theta_3} v_{n,m} = v_{n,m}(T_n^2 - 1)\bigg[&a_{n+2}\beta_3\frac{v_{n+1,m}v_{n,m} + \beta_3 v_{n+1,m} + v_{n,m} + \beta_3^2}{\widetilde{V}_{n,m}} + \\ &+ a_{n+1}\beta_3^2\frac{(v_{n,m} + \beta_3)(v_{n-2,m} + 1)}{\widetilde{V}_{n-1,m}} + \\ &+ a_n\frac{v_{n-2,m}v_{n-3,m} + \beta_3 v_{n-2,m} + v_{n-3,m} + 1}{\widetilde{V}_{n-2,m}}\bigg] - 3cv_{n,m},\\ \widetilde{V}_{n,m} = (v_{n+1,m} + 1)&(v_{n,m} + \beta_3^2)(v_{n-1,m} + \beta_3) + (\beta_3 + 2)(v_{n-1,m} - v_{n+1,m}), \end{aligned} \tag{74}$$

where the coefficients a_n, c, and β_3 are the same as in (8). Here, for any fixed m, we have one autonomous seven-point analogue of the modified Volterra equation, its two nonautonomous commuting symmetries, and one master symmetry.

Just like Eq. (23), Eq. (67) is an autonomous modification of autonomous discrete equation (1). Just as in the case of Eq. (1), we can show that it is an example of an autonomous discrete equation with two hierarchies of autonomous higher symmetries. For example, for $N = 2$ and $\beta_N = -1$, the simplest autonomous higher symmetries are symmetries (70) and (72) with $a_n \equiv 1$, which turn out to be five-point.

3.2. Second modification. We let $3\beta_N/w_{n,m+1}$ denote the right- and left-hand sides of discrete equation (67), and these two equalities yield the formulas

$$w_{n,m} = \frac{3v_{n+1,m}}{(v_{n+1,m}+1)(v_{n,m}+1)}, \qquad w_{n,m+1} = \frac{3\beta_N v_{n,m}}{(v_{n+1,m}+\beta_N)(v_{n,m}+\beta_N)}. \tag{75}$$

They determine a transformation of $v_{n,m}$ and $v_{n+1,m}$ into $w_{n,m}$ and $w_{n,m+1}$ that is invertible on solutions of discrete equation (67). As in case (65), we can construct a new discrete equation and its higher symmetries in both directions in terms of $w_{n,m}$. But the inverse transformation contains square roots, and the discrete equation and the symmetries in the direction m must therefore also contain square roots. Formulas (75) provide explicit rational transformations for symmetries in direction n, which also turn out to be rational. We write most of these symmetries, but we give the corresponding discrete equation in only one important case.

In contrast to transformations (65), we here have not linearizable transformations but transformations of the Miura type. Indeed, in terms of $\hat{v}_{n,m}$, from relations (68) and

$$\check{v}_{n,m} = \frac{\beta_N - v_{n,m}}{\beta_N + v_{n,m}}, \tag{76}$$

we obtain

$$w_{n,m} = -\frac{3}{4}(\hat{v}_{n+1,m}-1)(\hat{v}_{n,m}+1), \qquad w_{n,m+1} = -\frac{3}{4}(\check{v}_{n+1,m}+1)(\check{v}_{n,m}-1). \tag{77}$$

These are known discrete Miura transformations relating the Volterra equation and its known modification. They are obtained here by point transformations of the function $v_{n,m}$.

In the case $N=1$, $\beta_N=1$, using the first transformation in (77), we obtain the higher symmetry in the direction n

$$-3\partial_{\theta_1} w_{n,m} = w_{n,m}(a_{n+3}w_{n+1,m} - a_n w_{n-1,m} + c w_{n,m} - 3c), \qquad a_n = b + cn, \tag{78}$$

from symmetry (71) written in terms of $\hat{v}_{n,m}$. The strategy for rewriting differential–difference equations using such transformations is explained in Appendix A.2 in [18]. For any fixed m, symmetry (78) is the Volterra equation with its known master symmetry [19].

If $N=2$, i.e., $\beta_N=-1$, from symmetry (72) using the first transformation in (75), we obtain the higher symmetry

$$\partial_{\theta_2} w_{n,m} = w_{n,m}(T_n+1)\left[\frac{a_{n+3}w_{n,m}}{2w_{n+1,m}-3} - \frac{a_n w_{n-1,m}}{2w_{n-2,m}-3} + \frac{1}{2}(T_n-1)\left(a_n - \frac{3a_{n+1}}{2w_{n-1,m}-3}\right)\right], \tag{79}$$

where the coefficient a_n is defined as in (7).

We can obtain a known equation in terms of

$$\widetilde{w}_{n,m} = -\frac{3}{2w_{n,m}-3}. \tag{80}$$

We write symmetry (79) as

$$\begin{aligned} -2\partial_{\theta_2}\widetilde{w}_{n,m} = (\widetilde{w}_{n,m}-1)\Bigg[& a_{n+4}\frac{\widetilde{w}_{n+2,m}(\widetilde{w}_{n+1,m}-1)\widetilde{w}_{n,m}}{\widetilde{w}_{n+1,m}} - a_n\frac{\widetilde{w}_{n,m}(\widetilde{w}_{n-1,m}-1)\widetilde{w}_{n-2,m}}{\widetilde{w}_{n-1,m}} - \\ & - a_{n+3}\widetilde{w}_{n+1,m} + a_{n+1}\widetilde{w}_{n-1,m} + (a_n - a_{n+2})\widetilde{w}_{n,m}\Bigg]. \end{aligned} \tag{81}$$

In the particular case $a_n \equiv 1$ for any fixed m, this is a five-point analogue of the Volterra equation. This equation was previously found in [14] (Eq. (39) there); it coincides with our equation up to a dilation θ_2 and $\widetilde{w}_{n,m}$. It was noted in [20] that this equation plays a crucial role for a special class of differential–difference equations.

Formula (81) with the same fixed m for this equation gives a nonautonomous symmetry for $a_n = (-1)^n$ and the master symmetry for $a_n = n$. Moreover, for this known equation, we can write a corresponding discrete equation determining the Bäcklund autotransformation:

$$\begin{aligned} &\frac{\widetilde{w}_{n+1,m+1} - 1}{\widetilde{w}_{n+1,m+1}(\widetilde{w}_{n+1,m} + 1)}(\Theta_{n+1,m} - \widetilde{w}_{n+1,m+1} + \widetilde{w}_{n+1,m} + 2) + \\ &\qquad + \frac{\widetilde{w}_{n+1,m} - 1}{\widetilde{w}_{n+1,m}(\widetilde{w}_{n,m+1} - 1)}(\Theta_{n,m} + \widetilde{w}_{n,m+1} - \widetilde{w}_{n,m} - 2) = 4, \\ &\Theta_{n,m} = \sqrt{4\widetilde{w}_{n,m+1}\widetilde{w}_{n,m} + (\widetilde{w}_{n,m+1} + \widetilde{w}_{n,m} + 2)^2}. \end{aligned} \tag{82}$$

For $N = 3$, we can construct a seven-point analogue of the Volterra equation. Using the first transformation in (75), from (74), we obtain the higher symmetry

$$\begin{aligned} &3\partial_{\theta_3} w_{n,m} = w_{n,m}(1 + T_n)[A_{n,m} + (1 - T_n)(w_{n-1,m}B_{n,m})], \\ &A_{n,m} = a_{n+4}\frac{\beta_3 w_{n+1,m}w_{n,m}}{W_{n+2,m}} - a_{n+2}\frac{(\beta_3 + 1)w_{n,m}w_{n-1,m}}{W_{n,m}} + \\ &\qquad + a_n\frac{w_{n-1,m}w_{n-2,m}}{W_{n-2,m}} - 3c\left(w_{n,m} - \frac{3}{2}\right), \\ &B_{n,m} = a_{n+2}\frac{2\beta_3 + 1}{W_{n,m}} - a_{n+1}\frac{\beta_3 + 2}{W_{n-1,m}} + a_n, \\ &W_{n,m} = \beta_3 w_{n-1,m} - w_{n,m} + 1 - \beta_3, \end{aligned} \tag{83}$$

where the coefficients a_n, c, and β_3 are defined as in (8). If we fix m, then from symmetry (83), we obtain the abovementioned analogue with $a_n \equiv 1$, its nonautonomous commuting symmetries with $a_n = \beta_3^n$ and $a_n = \beta_3^{2n}$, and also its master symmetry with $a_n = n$.

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