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CONSTRUCTION OF FUNCTIONS WITH DETERMINED BEHAVIOR $T_G(b)(z)$ AT A SINGULAR POINT

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Abstract. I.N. Vekua developed the theory of generalized analytic functions, i.e., solutions of the equation

$$\partial_{\bar{z}}w + A(z)w + B(z)\overline{w} = 0, \tag{0.1}$$

where $z \in G$ (G, for example, is the unit disk on a complex plane) and the coefficients $A(z)$, $B(z)$ belong to $L_p(G)$, $p > 2$. The Vekua theory for the solutions of (0.1) is closely related to the theory of holomorphic functions due to the so-called similarity principle. In this case, the T_G-operator plays an important role. The T_G-operator is right-inverse to $\frac{\partial}{\partial \bar{z}}$, where $\frac{\partial}{\partial \bar{z}}$ is understood in Sobolev's sense.

The author suggests a scheme for constructing the function $b(z)$ in the unit disk G with determined behavior $T_G(b)(z)$ at a singular point $z = 0$, where T_G is an integral Vekua operator. The paper states the conditions for $b(z)$ under which the function $T_G(b)(z)$ is continuous.

Keywords: T_G-operator, singular point, modulus of continuity.

1. Introduction

Methods based on expressing solutions in a complex form have been successfully used in the theory of partial differential equations of recent years. These methods are mainly developed in works of I.N. Vekua, L. Bers, S. Bergman and others. As an example let us consider the following expression for a system of equations with partial derivatives of the first order:

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta} + a_0 u + b_0 v, \frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi} + c_0 u + d_0 v, \tag{1.1}$$

where a_0, b_0, c_0, d_0 are continuous functions of variables ξ, η in a domain G.

The system (1.1) is a generalization of the Cauchy-Riemann conditions that are obtained when $a_0 = b_0 = c_0 = d_0 = 0$. The system was first considered by Carleman, who proved the uniqueness theorem for its solution. A detailed investigation of the system (1.1) and its applications was made by I.N. Vekua [1]. In what follows, the functions u, v are considered to have contunuous partial derivatives in the domain G for the sake of simplicity.

Let us introduce the notation

$$\frac{\partial}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta}\right), \frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right). \tag{1.2}$$

The formula for representing an arbitrary function $f(\zeta) = u + iv$, possessing continuous partial derivatives in a bounded domain G ([2, c. 317]) is known:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{\pi}\iint_G \frac{\partial f}{\partial \bar{\zeta}}\frac{d\xi d\eta}{\zeta - z}. \tag{1.3}$$

The double integral disappears for analytic functions and we turn to the integral Cauchy formula.

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Let us apply the formula to solving the system (1.1). By means of the symbol of differentiation (1.2), the system is written in the form of a complex equation

$$\frac{\partial f}{\partial \overline{\zeta}} = Af + B\overline{f}, \tag{1.4}$$

where $f = u + iv$, $A = \frac{1}{4}(a_0 + d_0 + ic_0 - ib_0)$, $B = \frac{1}{4}(a_0 - d_0 + ic_0 + ib_0)$. Therefore, the formula (1.3) provides the following complex representation of solutions to the system (1.1):

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{\pi}\iint_G \frac{A(\zeta)f(\zeta) + B(\zeta)\overline{f(\zeta)}}{\zeta - z}\,d\xi d\eta. \tag{1.5}$$

In the present work, behavior of a double integral in the formula (1.3) is considered in terms of a specific example. This integral is denoted in [1] by $T_G(f)(z)$:

$$T_G(f)(z) = -\frac{1}{\pi}\iint_G \frac{f(\zeta)d\xi d\eta}{\zeta - z}, \zeta = \xi + i\eta. \tag{1.6}$$

It is known (see e.g. [3]–[5]), that the Vekua theory for the system (1.4) breaks down if coefficients $A(\zeta)$, $B(\zeta)$ do not belong to the space $L_p(G)$ $(p > 2)$. Therefore, it is necessary to make a separate investigation for equations with such coefficients as $A(\zeta) = \frac{1}{\zeta}$, $B(\zeta) = \frac{1}{\zeta}$ and others. A large variety of results for such equations (1.4) with singular coefficients is obtained in the works [3]–[12]. It should be mentioned that a great part is played by the value of the operator $T_G(f)$ on this or that set of functions in all these works. It is known that T_G transforms the space $L_p(G)$ $(p > 2)$ into the Hölder space $C_\alpha(G)$ with the coefficients $\alpha = \frac{p-2}{p}$. The behavior of $T_G(f)$ on some other classes of functions is also known (see also the section 2).

In this connection, the given paper is devoted to investigation of behavior of $T_G(f)$ for functions $f(z)$, having a singularity of this or that order at the point $z = 0$.

The function $b = b(\zeta)$, such that the behavior of $T_G(b)(z)$ at the point $z = 0$ is characterized at this point by the function μ, is constructed here for a unit circle $G = \{z : |z| < 1\}$ for a given module of continuity $\mu(x)$. Here T_G is the operator introduced by I.N. Vekua ((1.6)).

For this purpose, a scheme for calculating $T_G f$ by means of the theory of residues is developed. In section 3, auxiliary statements that allow one to calculate $T_G f$ for a sufficiently wide class of functions $f(z)$ with a singularity at the point $z = 0$ are proved.

Examples verify results published in L.G. Mikhailov's book without a proof. Moreover, they indicate that $T_G f$ can be a bounded and even (upon completing the definition at the point $z = 0$) a continuous function, although $f(z)$ has a singularity at zero.

2. Properties of the operator T_G

The given section presents basic properties of the function $T_G(f)(z)$ (see, e.g., [1, c. 39]).

Property 1. *Let G be a bounded domain. If $f \in L_p(\overline{G})$, $p > 2$, then the function $g = T_G f$ satisfies the conditions*

$$|g(z)| \leqslant M_1 L_p(f, \overline{G}), z \in E, \tag{2.1}$$

$$|g(z_1) - g(z_2)| \leqslant M_2 L_p(f, \overline{G})|z_1 - z_2|^\alpha, \alpha = \frac{p-2}{p}, \tag{2.2}$$

where z, z_1, z_2 are arbitrary points of the plane and M_1, M_2 are arbitrary constants, while M_1 depends on p and G, and M_2 depends only on p; $L_p(f, \overline{G})$ is the norm of f in the space $L_p(\overline{G})$.

The inequalities (2.1) and (2.2) demonstrate that T_G is a linear completely continuous operator in the space $L_p(\overline{G})$, mapping the space onto $C_\alpha(\overline{G})$, $\alpha = \frac{p-2}{p}$, $p > 2$ (such operators are sometimes termed as *strongly completely continuous* operators), and

$$C_\alpha(T_G f, \overline{G}) \leqslant M L_p(f, \overline{G}), \alpha = \frac{p-2}{p}, p > 2. \tag{2.3}$$

Property 2. *Let $f \in C(\overline{G})$. Then,*

$$g(z_1) - g(z_2) = \frac{z_1 - z_2}{\pi} = \iint_G \frac{f(\zeta)\, d\xi d\eta}{(\zeta - z_1)(\zeta - z_2)}, z_1 \neq z_2$$

entails

$$\begin{cases} |g(z)| \leqslant MC(f, \overline{G}), \\ |g(z_1) - g(z_2)| \leqslant MC(f, \overline{G}) |z_1 - z_2| \lg \frac{2d}{|z_1 - z_2|}, \end{cases}$$

where d is the diameter of the domain G, M is a constant.

If $f \in L_\infty(\overline{G})$, one has

$$\begin{cases} |g(z)| \leqslant ML_\infty(f, \overline{G}), \\ |g(z_1) - g(z_2)| \leqslant ML_\infty(f, \overline{G}) |z_1 - z_2| \lg \frac{2d}{|z_1 - z_2|}, \end{cases}$$

It follows from these inequalities that the operator T_G is continuous in the spaces $C(\overline{G})$ and $L_\infty(\overline{G})$, and it maps the spaces onto the class of functions satisfying the Dini condition.

L.G. Mikhailov's book [3] contains the following table illustrating properties of the function $T_G(f)(z)$ by properties of the function $f(z)$:

Conditions on $f(\zeta)$	Properties of $T_G(f)(z)$
1) $L(G)$	$L_{2-\varepsilon}(G)$, $\varepsilon > 0$ is small
2) $L_p(G)$, $1 < p < 2$	$L_q(G)$, $q = \frac{2p}{p-2}$
3) $L_2(G)$	$L_s(G)$ for any $s \geq 1$
4) $L_p(G)$, $2 < p$	$C_\alpha(G)$, $\alpha = \frac{p-2}{p}$
5) $L_\infty(G)$, $C(G)$	$\Delta\omega = O(\lvert\Delta z\rvert \cdot \ln \lvert\Delta z\rvert)$, $\Delta\omega$ is the continuity module
6) Holomorphic in G	Holomorphic in G

3. Auxiliary statements

This section provides the proof for three lemmas that can be used in other investigations as well.

Lemma 1. *Consider the integral $J(z, r) = \displaystyle\int_0^{2\pi} \frac{d\phi}{re^{i\phi} - z}$, where $|z| < 1$ u $0 < r < 1$. If $0 < r < |z|$, then $J(z, r) = -\frac{2\pi}{z}$. If $|z| < r < 1$, then $J(z, r) = 0$.*

Proof. Let us change the variable $e^{i\phi} = t$ in the given integral. As a result, one obtains the following loop integral:

$$J(z, r) = \frac{1}{i} \int_{|t|=1} \frac{dt}{t(rt - z)}.$$

The integrand function has two singularities: $t_1 = 0$, $t_2 = \frac{z}{r}$, which are simple poles.

1. Let $t_2 < 1$, which is equivalent to the condition $|z| < r$, then

$$J(z, r) = 2\pi \left[\operatorname*{res}_{t_1=0} \frac{1}{t(rt - z)} + \operatorname*{res}_{t_2=\frac{z}{r}} \frac{1}{t(rt - z)} \right] = 0.$$

2. Let $|t_2| > 1$, this condition is equivalent to $|z| > r$, then

$$J(z, r) = 2\pi \cdot \operatorname*{res}_{t_1=0} \frac{1}{t(rt - z)} = -\frac{2\pi}{z}.$$

Thus,

$$J(z,r)=\begin{cases}-\frac{2\pi}{z}, \text{if } 0<r<|z|,\\ 0, \text{if } |z|<r<1.\end{cases}$$

Lemma 1 is proved.

Lemma 2. *Consider the integral*

$$J(z,r)=\int_0^{2\pi}\frac{d\phi}{e^{in\phi}(re^{i\phi}-z)},$$

where n is a natural number, $|z|<1$ and $0<r<1$. If $0<r<|z|$, then $J(z,r)=-\frac{2\pi r^n}{z^{n+1}}$. If $|z|<r<1$, then $J(z,r)=0$.

Proof. Repeating the scheme of the proof for Lemma 1, one obtains the following loop integral

$$J(z,r)=\frac{1}{i}\int_{|t|=1}\frac{dt}{t^{n+1}(rt-z)}.$$

The integrand function has two singularities: $t_1=0$ — $(n+1)$, which is a *multiple pole,* and $t_2=\frac{z}{r}$.

1. Let $t_2<1$, which is equivalent to the condition $|z|<r$, then

$$J(z,r)=2\pi\left[\operatorname*{res}_{t_1=0}\frac{1}{t^{n+1}(rt-z)}+\operatorname*{res}_{t_2=\frac{z}{r}}\frac{1}{t^{n+1}(rt-z)}\right]=$$
$$=2\pi\left[(-1)^n\cdot\frac{r^n}{(-z)^{n+1}}+\frac{r^n}{z^{n+1}}\right]=2\pi\left[-\frac{r^n}{z^{n+1}}+\frac{r^n}{z^{n+1}}\right]=0.$$

2. Let $|t_2|>1$, this condition is equivalent to $|z|>r$, then

$$J(z,r)=\frac{1}{i}\cdot 2\pi i\cdot\operatorname*{res}_{t_1=0}\frac{1}{t^{n+1}(rt-z)}=-\frac{2\pi r^n}{z^{n+1}}.$$

Thus,

$$J(z,r)=\begin{cases}-\frac{2\pi r^n}{z^{n+1}}, \text{if } 0<r<|z|,\\ 0, \text{if } |z|<r<1.\end{cases}$$

Lemma 2 is proved.

Lemma 3. *Consider the integral*

$$J(z,r)=\int_0^{2\pi}\frac{e^{in\phi}\,d\phi}{re^{i\phi}-z},$$

where n is a natural number, $|z|<1$ and $0<r<1$. If $0<r<|z|$, then $J(z,r)=0$. If $|z|<r<1$, then $J(z,r)=\frac{2\pi z^{n-1}}{r^n}$.

Proof. Let us substitute the variable $e^{i\phi}=t$ in the given integral and obtain the following loop integral:

$$J(z,r)=\frac{1}{i}\int_{|t|=1}\frac{t^{n-1}\,dt}{rt-z}.$$

The integrand function has one singularity $t=\frac{z}{r}$. Applying the basic theorem of the theory of residues, one obtains

$$J(z,r)=\begin{cases}0, \text{if } 0<r<|z|,\\ \frac{2\pi z^{n-1}}{r^n}, \text{if } |z|<r<1.\end{cases}$$

Lemma 3 is proved.

4. $T_G(f)(z)$ for radially dependent functions

4.1. $T_G(b)(z)$ for radially dependent functions $b = b(|\zeta|)$. Let us find $T_G(b)(z)$ in an explicit form if $b = b(|\zeta|) = b(\rho)$, $0 < \rho < 1$.

Suppose that the fixed number $z \neq 0$, $|z| < 1$. Then, if $\zeta = \xi + i\eta$, one has

$$T_G(b)(z) = -\frac{1}{\pi}\iint_{|\zeta|<1} \frac{b(\rho)}{\zeta - z}\, d\xi\, d\eta =$$
$$= -\frac{1}{\pi}\iint_{|\zeta|<|z|} \frac{b(\rho)}{\zeta - z}\, d\xi\, d\eta - \frac{1}{\pi}\iint_{|\zeta|>|z|} \frac{b(\rho)}{\zeta - z}\, d\xi\, d\eta = J_1(z) + J_2(z).$$

First, let us calculate $J_1(z)$:

$$J_1(z) = -\frac{1}{\pi}\int_0^{|z|} b(\rho)\cdot\rho\cdot\left(\int_0^{2\pi} \frac{d\varphi}{\rho e^{i\varphi} - z}\right) d\rho = -\frac{1}{\pi}\int_0^{|z|} b(\rho)\cdot\rho\cdot I(\rho, z)\, d\rho.$$

Since $I(\rho, z) = -\frac{2\pi}{z}$ for $|z| > \rho$ according to Lemma 1, then

$$-\frac{1}{\pi}\int_0^{|z|} b(\rho)\cdot\rho\cdot\left(-\frac{2\pi}{z}\right) d\rho = \frac{2}{z}\cdot\int_0^{|z|} b(\rho)\cdot\rho\, d\rho.$$

Likewise,

$$J_2(z) = -\frac{1}{\pi}\int_{|z|}^{1} b(\rho)\cdot\rho\cdot I(\rho, z)\, d\rho.$$

By Lemma 1, $I(\rho, z) = 0$ for $|z| < \rho$, then $J_2(z) = 0$.

Thus,

$$T_G(b)(z) = \frac{2}{z}\int_0^{|z|} b(\rho)\cdot\rho\, d\rho. \tag{4.1}$$

Examples.

1. $b(\zeta) = \frac{1}{|\zeta|^\alpha}$, $\alpha < 2$. The formula (4.1) provides

$$T_G(b)(z) = \frac{2}{2-\alpha}\cdot\frac{|z|^{2-\alpha}}{z}.$$

   In particular, if $\alpha = 1$,

$$T_G(b)(z) = 2\cdot\frac{|z|}{z}$$

   is a bounded function.
2. $b(\zeta) = \frac{1}{\zeta} = \frac{1}{\rho e^{i\varphi}}$. Repeating the argumentation of 4.1 with application of Lemma 2 for $n = 1$, one obtains that

$$T_G(b)(z) = \frac{\overline{z}}{z}$$

   is a bounded function.
3. $b(\zeta) = \frac{1}{\overline{\zeta}} = \frac{e^{i\varphi}}{\rho}$. Repeating the argumentation of 4.1 with the use of Lemma 3 for $n = 1$, one obtains

$$T_G(b)(z) = \ln|z|^2.$$

Examples 1–3 have been presented earlier in [3, c. 123–124] without explanation.

4.2. Behavior of $T_G(b)(z)$ at the point $z = 0$ for radially dependent $b(\zeta)$. Consider the difference

$$T_G(b)(z) - T_G(b)(0) = -\frac{z}{\pi} \int_{|\zeta|<1} \frac{b(\zeta)}{\zeta(\zeta - z)}\, d\xi\, d\eta \tag{4.2}$$

for $z \neq 0$. Let us introduce the notation

$$I(\rho, z) = \frac{1}{i\rho} \int_{|t|=1} \frac{dt}{t^2(\rho t - z)},$$

then, by Lemma 2,

$$I(z, r) = \begin{cases} 0, |z| < \rho, \\ -\frac{2\pi}{z^2}, |z| > \rho. \end{cases}$$

Applying this in (4.2), one obtains

$$T_G(b)(z) - T_G(b)(0) = \frac{2}{z} \int_0^{|z|} b(\rho) \cdot \rho\, d\rho. \tag{4.3}$$

Making an auxiliary assumption that $b(\rho) > 0$, one obtains from (4.3) that

$$|T_G(b)(z) - T_G(b)(0)| = \frac{2}{|z|} \int_0^{|z|} b(\rho) \cdot \rho\, d\rho. \tag{4.4}$$

4.3. Constructing functions with the given behavior $T_G(b)(z)$ at the point $z = 0$. The formula (4.4) allows one to construct functions $b = b(\rho)$, such that $T_G(b)(z)$ has a given behavior at the point $z = 0$ in the sense of the continuity module.

Recall that a continuity module is a function $\mu(t)$, given in the interval $(0, \delta)$, and satisfying the conditions

1. $\mu(t) \geq 0$, $t \geq 0$;
2. $\lim\limits_{t\to+0} \mu(t) = 0$;
3. $\mu(t)$ does not decrease for $t > 0$;
4. for any $t_1, t_2 \in (0, \delta)$ $\mu(t_1 + t_2) \leqslant \mu(t_1) + \mu(t_2)$.

The condition 4 is known to hold if we assume that $\frac{\mu(t)}{t}$ does not increase for $t > 0$.

Suppose that

$$\int_0^1 b(\rho) \cdot \rho\, d\rho < +\infty.$$

Let us introduce the notation $x = |z|$,

$$\mu(x) = \frac{2}{x} \int_0^x b(\rho) \cdot \rho\, d\rho \tag{4.5}$$

Then (4.4) can be written as

$$|T_G(b)(z) - T_G(b)(0)| = \mu(|z|). \tag{4.6}$$

Assuming that $\mu(x)$ is differentiable, one obtains

$$b(x) = \frac{1}{x} \cdot \left(\frac{x \cdot \mu(x)}{2} \right)' \tag{4.7}$$

The formula (4.7) allows one to construct $b = b(|\zeta|)$ for a given function μ so that (4.6) holds.

4.4. Examples.

4.4.1. $\mu(x) = x^\lambda$, $0 < \lambda < 1$. Then, according to the formula (4.7), one obtains that

$$b(\zeta) = \frac{\lambda+1}{2} \cdot \frac{1}{|\zeta|^{1-\lambda}}.$$

Here $b = b(\zeta)$ has a weak singularity at the point $\zeta = 0$, $T_G(b)(z)$, and satisfies the Hölder condition at the origin of coordinates.

4.4.2. $\mu(x) = x$. Then $b(\rho) = 1$. There are no nonconstant functions among radially dependent functions $b = b(\rho)$, such that $T_G(b)(z)$ satisfies the Lipschitz condition at the point $z = 0$.

4.4.3. Conversely, let

$$b(|\zeta|) = \frac{1}{|\zeta|^2 \ln^2 \frac{1}{|\zeta|}},$$

then $\mu(|z|) = \frac{2}{|z| \ln \frac{1}{|z|}} \to +\infty$ when $z \to 0$.

4.4.4. One has

$$\mu(x) = \frac{2}{\ln^{1+\varepsilon} \frac{1}{x^*}}, 0 \leqslant x^* \leqslant x$$

for the function $b(|\zeta|) = \frac{1}{|\zeta| \ln^{1+\varepsilon} \frac{1}{|\zeta|}}$, belonging to $S_p(G)$ (see [12]). Therefore, invoking the monotonous character of the function, one obtains

$$\mu(x) \leqslant \frac{2}{\ln^{1+\varepsilon} \frac{1}{x}}.$$

Hence, $\mu(|z|) \to 0$, $z \to 0$.

4.5. Continuity of $T_G(b)(z)$ at the point $z = 0$. Let us find out when $\mu(|z|)$, involved in (4.6), tends to 0 if $z \to 0$. This is equivalent to the condition

$$\frac{\int_0^x b(\rho) \cdot \rho \, d\rho}{x} \to 0, x \to 0. \tag{4.8}$$

According to the L'Hospital rule, (4.8) is equivalent to

$$b(x) \cdot x \to 0, x \to 0. \tag{4.9}$$

Whence,

$$\int_0^1 b(\rho) \cdot \rho \, d\rho < +\infty,$$

of course.

Thus, the function $b = b(|\zeta|)$ has the function $T_G(b)(z)$ continuous at the point $z = 0$ if and only if (4.9) holds.

Example. The function

$$b(\zeta) = \frac{1}{|\zeta| \cdot \underbrace{\ln \ln \ln \ldots \frac{1}{|\zeta|}}_{\text{k times}}} \quad (|\zeta| \leqslant d < 1)$$

does not belong even to $L_2(U_d)$. Nevertheless, the condition (4.9) holds and $T_G(b)(z)$ is continuous at the origin of coordinates.

4.6. Connection with the space $S_p(G)$. It is known that $T_G(b)(z)$ is a continuous function at the point $z = 0$ for functions $b(\zeta) \in S_p(G)$ (see [12]). Denote by $S_1(G)$ a set of radial functions $b(\zeta)$ with the property (4.9).

Let us compare these spaces considering radially dependent functions $b(\zeta)$.

Results of the work [13] provide that there exists $\delta_0(\varepsilon)$ for any $\varepsilon > 0$ with the property

$$\frac{1}{p\left(\frac{1}{t}\right)} < \varepsilon \cdot t, t > \delta_0(\varepsilon),$$

i.e.

$$\frac{1}{p(x)} < \varepsilon \cdot \frac{1}{x}, x < \frac{1}{\delta_0} = \delta_1(\varepsilon).$$

Finally,

$$\frac{x}{p(x)} < \varepsilon, \text{for all } 0 < x < \delta_1(\varepsilon). \tag{4.10}$$

Now, let us take an arbitrary function $b = b(|\zeta|) \in S_p(G)$. Hence, there is $p = p(r)$:

$$\sup_{|\zeta|<1} |b(|\zeta|)| \cdot p(r) = C_0 < +\infty.$$

Then, by virtue of (4.10),

$$|b(r)| \cdot r = |b(r)| \cdot p(r) \cdot \frac{r}{p(r)} \leqslant C_0\varepsilon, r < \delta_1(\varepsilon).$$

Thus, $b(|\zeta|) \in S_1(G)$.

Moreover, $b(|\zeta|) = \frac{1}{|\zeta| \ln \frac{1}{|\zeta|}} \in S_1(0 < |\zeta| < t_0)$, but $b(|\zeta|) \notin S_p(G)$.

Finally, $b = b(|\zeta|)$ $S_p(G) \subset S_1(G)$ for radially dependent functions and the inclusion is strict.

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