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NECESSARY CONDITIONS OF DARBOUX INTEGRABILITY FOR DIFFERENTIAL-DIFFERENCE EQUATIONS OF A SPECIAL KIND

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Abstract. This work dwells upon chains of differential equations of the form $\varphi(x, u_{i+1}, (u_{i+1})_x) = \psi(x, u_i, (u_i)_x)$, where u depends on the discrete variable i and the continuous variable x, and the functions $\varphi(x, y, z)$, $\psi(x, y, z)$ and x are functionally-independent. We demonstrate that necessary Darboux integrability conditions for chains of the above form can be easily derived from already known results. These conditions are not sufficient but may be useful for classification of Darboux-integrable differential-difference equations. As an auxiliary result, we also prove a proposition about structure of symmetries for differential-difference equations of a more general form.

Keywords: Darboux integrability, differential-difference equations

A class of integrable equations of the form

$$u_{xy} = F(x, y, u, u_x, u_y) \tag{1}$$

is generated by the following equations. For every such equation there is a differential substitution of the form $v = X(x, y, u, u_x, u_{xx}, \dots)$ as well as substitution of the form $w = Y(x, y, u, u_y, u_{yy}, \dots)$, that transform solutions of (1) to solutions of equations $v_y = 0$ and $w_x = 0$, respectively. Such equations are said to be Darboux integrable. They are also termed as equations of the Liouville type. A complete classification of Darboux integrable equations of the form (1) was made in [1].

A chain of differential equations of the form

$$(u_{i+1})_x = F(x, u_i, u_{i+1}, (u_i)_x),$$

where the unknown function u depends on an integer i and a real variable x, can be considered as a differential-difference analogue of Equation (1). Some of the chains are Darboux integrable as well. However, unlike equations of the form (1), the above mentioned “semidiscrete” Darboux integrable equations have not been classified completely, only separate examples (see, e.g., [2]) and results of classification for chains of a special form [3] are known.

In order to give a rigorous definition of Darboux integrability one has to introduce some notation. In what follows the index i is omitted in all formulae for the sake of brevity, in particular the above chain is written in the from

$$(u_1)_x = F(x, u, u_1, u_x). \tag{2}$$

We assume that $F_{u_x} \neq 0$ and therefore, Equation (2) can be written in the form

$$(u_{-1})_x = \tilde{F}(x, u, u_{-1}, u_x). \tag{3}$$

By virtue of Equations (2)–(3), derivatives $u_m^{(n)} := \partial^n u_{i+m}/\partial x^n$ of shifts of u for any nonzero $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ can be expressed via x and the so-called *dynamic variables* $u_l := u_{i+l}$,

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$u^{(k)} := \partial^k u_i/\partial x^k u_i$. The notation $g[u]$ indicates that the function g depends on x and a finite number of dynamic variables.

Lets us denote the shift operator with respect to i due to Equations (2) by T. The operator is given by the following rules: $T(f(a,b,\dots)) = f(T(a),T(b),\dots)$ for any function f; $T(u_m) = u_{m+1}$; $T(u^{(n)}) = D^{n-1}(F)$ (i.e. "mixed" variables $u_1^{(n)}$ are expressed via dynamic variables by virtue of Equation (2)). Here

$$D = \frac{\partial}{\partial x} + u^{(1)}\frac{\partial}{\partial u} + \sum_{k=1}^{\infty}\left(u^{(k+1)}\frac{\partial}{\partial u^{(k)}} + T^{(k-1)}(F)\frac{\partial}{\partial u_k} + T^{(1-k)}(\tilde F)\frac{\partial}{\partial u_{-k}}\right),$$

i.e. D designates an operator of a total derivative with respect to x by virtue of Equations (2)–(3). The backward shift operator T^{-1} is given likewise.

Definition 1. *Equation* (2) *is said to be Darboux integrable if there are functions $I[u]$ and $X[u]$, everyone depending on at least one dynamic variable, such that the relations $D(I) = 0$ and $T(X) = X$ hold. In this case, the functions $I[u]$ and $X[u]$ are called the i-integral and the x-integral of Equation* (2)*, respectively.*

A special subclass of Equations (2) is considered in the present paper, namely equations of the form

$$\varphi(x,u_1,(u_1)_x) = \psi(x,u,u_x), \tag{4}$$

where functions $\varphi(x,y,z)$ and $\psi(x,y,z)$ satisfy the conditions $\varphi_y\psi_z - \varphi_z\psi_y \neq 0$ and $\varphi_z\psi_z \neq 0$. This subclass is of interest, e.g. since such equations admit an invertible transformation $v = \varphi(x,u,u_x)$ (for more details see [4]), translating (4) to an equation of the form

$$(v_1)_x = p(x,v,v_1)v_x + q(x,v,v_1). \tag{5}$$

Thus, studying a subclass of Equations (4), we also investigate equations of the form (5) at the same time. Moreover, some conditions of Darboux integrability for equations of the form (4) have already been obtained in earlier works. One can readily see it comparing results of the works [2], [4]–[6]. The present article is devoted to this observation.

To be more specific, its basic result is proof of the following statement.

Theorem 1. *If Equation* (4) *is Darboux integrable then, solving it with respect to $(u_1)_x$, one obtains equations of the form*

$$\xi_x(x,u_1) + \xi_{u_1}(x,u_1)(u_1)_x = \alpha(x,\psi)(\xi(x,u_1))^2 + \beta(x,\psi)\xi(x,u_1) + \gamma(x,\psi),$$

and solving (4) *with respect to u_x, one arrives to an equation of the form*

$$\eta_x(x,u) + \eta_u(x,u)u_x = \hat\alpha(x,\varphi)(\eta(x,u))^2 + \hat\beta(x,\varphi)\eta(x,u) + \hat\gamma(x,\varphi).$$

In other words, for any Darboux integrable equation (4) there exist pointwise substitutions of the variables $\tilde u = \xi(x,u)$ and $\bar u = \eta(x,u)$, reducing the equation to the form

$$(\tilde u_1)_x = \alpha(x,\tilde\psi)\tilde u_1^2 + \beta(x,\tilde\psi)\tilde u_1 + \gamma(x,\tilde\psi),$$

as well as to the form

$$\bar u_x = \hat\alpha(x,\bar\varphi)\bar u^2 + \hat\beta(x,\bar\varphi)\bar u + \hat\gamma(x,\bar\varphi).$$

It means that in notation of the form (2) and (3), the right-hand side of Equation (4) turns to be quadratic with respect to u_1 and u_{-1}, respectively upon an appropriate change of variables.

Before proceeding to the proof of Theorem 1, we have to give a definition and prove two auxiliary statements.

Definition 2. *An equation of the form $u_t = s[u]$ is called a symmetry of Equation* (2) *if the relation $L(s) = 0$, where*

$$L = TD - F_{u_x}D - F_{u_1}T - F_u,$$

holds.

Lemma 1. *Let us assume that $X[u] \in \ker(T-1)$ for an equation of the form* (2)*. Then, $X[u]$ does not depend on shifts of u (on dynamic variables of the form u_l, $l \neq 0$).*

Proof. Let us make a contrary proposition and denote by j and r respectively the largest positive and the smallest negative numbers for which $X[u]$ depends on u_j and u_r. Differentiation of the relation $T(X) = X$ with respect to u_{j+1} and u_r provides $T(X_{u_j}) = 0$ and $X_{u_r} = 0$, respectively. Thus, we arrive to a contradiction that proves the lemma.

Lemma 2. *Any symmetry $u_t = s[u]$ of Equation* (2) *has the form*

$$u_t = \hat{s}(x, u_l, u_{l+1}, \dots, u_{-1}, u, u_1, \dots, u_k) + \bar{s}(x, u, u^{(1)}, \dots, u^{(n)}), \tag{6}$$

i.e. $s[u]$ splits into a sum of two addends. One addend depends only on shifts, the other one depends only on derivatives of u.

Proof. Let the symmetry have the form

$$u_t = s(x, u_l, u_{l+1}, \dots, u_{-1}, u, u_1, \dots, u_k, u^{(1)}, \dots, u^{(n)}).$$

Differentiating the determining relation for the symmetry with respect to $u^{(n+1)}$, one obtains

$$(L(s))_{u^{(n+1)}} = F_{u_x}(T(s_{u^{(n)}}) - s_{u^{(n)}}) = 0.$$

Applying Lemma 1, one can see that $s_{u^{(n)}}$ is independent of shifts of u and hence, can depend only on variables x, u, $u^{(1)}$, $\dots$, $u^{(n)}$. Therefore, the right-hand side $s[u]$ of the symmetry is represented in the form

$$s[u] = g(x, u_l, u_{l+1}, \dots, u_k, u^{(1)}, \dots, u^{(n-1)}) + h(x, u, u^{(1)}, \dots, u^{(n)}).$$

Now, assume that

$$s[u] = g(x, u_l, u_{l+1}, \dots, u_k, u^{(1)}, \dots, u^{(m)}) + h(x, u, u^{(1)}, \dots, u^{(n)}), \tag{7}$$

where m is a natural number less than n. Then, differentiating the determining relation $L(s) = 0$ with respect to $u^{(m+1)}$, one obtains

$$(L(h))_{u^{(m+1)}} + F_{u_x}(T(g_{u^{(m)}}) - g_{u^{(m)}}) = 0. \tag{8}$$

Note that $L(h)$ can depend only on x, u, u_1 and derivatives of u. This allows us to apply the reasoning of the proof for Lemma 1 to (8) and demonstrate that $g_{u^{(m)}}$ is independent of shifts of u. Therefore, the right-hand side of the symmetry is written in the form

$$s[u] = \tilde{g}(x, u_l, u_{l+1}, \dots, u_k, u^{(1)}, \dots, u^{(m-1)}) + \tilde{h}(x, u, u^{(1)}, \dots, u^{(n)}).$$

It means that (7) provides validity of the relation with m one less, and by virtue of the principle of mathematical induction the expression (6) is true.

Proof of Theorem 1. It was proved in [2] that for any Darboux integrable equation of the form (2) there is a differential operator $R = \sum_{k=0}^{r} c_k[u]D^k$, such that $u_t = R(\omega)$ is a symmetry of this equation for any $\omega \in \ker(T-1)$. Let us demonstrate that coefficients c_k of the operator R can depend only on x, u and derivatives of u. Let us make a contrary assumption: suppose that coefficients of R depend on u_j for some $j \neq 0$. Denote by l the largest number for which $(c_l)_{u_j} \neq 0$. Let us consider the x-integral as ω, provided that the order m of the highest-order derivatives of u, on which the integral depends, is higher than orders of derivatives of u contained in coefficients of the operator R. (Since the operator D transforms x-integrals to x-integrals again, we can always construct the x-integral depending on derivatives of sufficiently high orders.) Then $(R(\omega))_{u_j u^{(m+l)}} = (c_l)_{u_j} \omega_{u^{(m)}} \neq 0$, which contradicts Lemma 2. Thus, the symmetry $u_t = R(\omega)$ for any $\omega \in \ker(T-1)$ has the form

$$u_t = s(x, u, u^{(1)}, \dots, u^{(n)}). \tag{9}$$

It was proved [1] in the work [4] that any symmetry of the form (9) for Equation (4) is transformed into an equation of the same form (9) by means of a differential substitution $v = \varphi(x, u, u_x)$, as well as the substitution $\tilde{v} = \psi(x, u, u_x)$. On the other hand, according to [5], Equation (9) is transformed by a differential substitution $v = f(x, u, u_x)$ to an equation of the same form again if and only if (9) is a symmetry of equation $u_{xy} = -f_u u_y / f_{u_x}$. Recall that (9) is said to be a symmetry of Equation (1), if s belongs to the kernel of the operator

$$M = D_x D_y - F_{u_x} D_x - F_{u_y} D_y - F_u,$$

where D_x and D_y denote total derivatives by virtue of Equation (1) with respect to x and y, respectively. Note that operators D_x and D coincide with each other on a set of functions depending only on x, u and derivatives of u with respect to x.

Thus, if equation (4) is Darboux integrable, then there is an operator $R = \sum_{k=0}^{r} c_k[u] D_x^k$ such that $u_t = R(\omega)$ is a symmetry for two equations simultaneously

$$u_{xy} = -\frac{\varphi_u(x, u, u_x)}{\varphi_{u_x}(x, u, u_x)} u_y \qquad \text{and} \qquad u_{xy} = -\frac{\psi_u(x, u, u_x)}{\psi_{u_x}(x, u, u_x)} u_y \tag{10}$$

for any $\omega \in \ker(T-1)$. In particular, any function $g(x)$ can be taken as ω. Setting the coefficients of derivatives of g of the same order in the relation $M(R(g)) = 0$ equal to zero, one obtains that $u_t = R(g(x))$ is a symmetry of Equation (1) for any function g if and only if the following chain holds

$$(D_y - F_{u_x})(c_r) = 0,$$
$$M(c_k) + (D_y - F_{u_x})(c_{k-1}) = 0, \quad k = \overline{1, r},$$
$$M(c_0) = 0.$$

One can easily verify that $u_t = R(\omega)$ is a symmetry of Equation (1) for any ω not only from $\ker(T-1)$, but from $\ker D_y$ as well, provided that this chain holds.

Equations of the form

$$u_{xy} = -\frac{f_u(x, u, u_x)}{f_{u_x}(x, u, u_x)} u_y,$$

for which there is a differential operator R, such that $u_t = R(\omega)$ is a symmetry of the equation for any $\omega \in \ker D_y$ are described in the work [6]. It was proved in it that any such equation can be reduced to an equation of the same form $\tilde{u}_{xy} = -\tilde{f}_{\tilde{u}}/\tilde{f}_{\tilde{u}_x} \tilde{u}_y$, where $\tilde{f} = f(x, \lambda, D_x(\lambda))$ satisfies the relation $\tilde{u}_x = \alpha(x, \tilde{f})\tilde{u}^2 + \beta(x, \tilde{f})\tilde{u} + \gamma(x, \tilde{f})$ by means of a pointwise substitution of variables $u = \lambda(x, \tilde{u})$. Application of the result to Equations (10) proves the theorem.

In conclusion note that conditions obtained in the theorem are necessary but not sufficient for Darboux integrability of Equation (4). In order to illustrate this we assume that φ equals to $(u_1)_x$, and ψ equals to $u_x + c_2 u^2 + c_1 u + c_0$, where c_j are some constants and consider the equation

$$(u_1)_x = u_x + c_2 u^2 + c_1 u + c_0. \tag{11}$$

One can easily see that conditions of Darboux integrability hold for any values of the constants c_j (in notation of Theorem 1, $\xi = u_1$, $\alpha = \beta = 0$, $\gamma = \psi$ and $\eta = u$, $\hat{\alpha} = -c_2$, $\hat{\beta} = -c_1$, $\hat{\gamma} = \varphi - c_0$). However, (11) is Darboux integrable only if $c_2 = c_1 = 0$, because there is no a single equation with d, independent of u_1, and different from the constant in the list of Darboux integrable equations of the form $(u_1)_x = u_x + d(u, u_1)$ composed in [3].

[1]In the strict sense, the work [4] dwelled upon Equations (4) independent of x explicitly. However, one can readily verify that reasoning behind this work can be carried over to the case of explicit dependence of Equation (4) on x without serious changes.

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