# CAUCHY PROBLEM FOR NAVIER-STOKES EQUATIONS, FOURIER METHOD 

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#### Abstract

The Cauchy problem for the 3D Navier-Stokes equations with periodicity conditions on the spatial variables is investigated. The vector functions under consideration are decomposed in Fourier series with respect to eigenfunctions of the curl operator. The problem is reduced to the Cauchy problem for Galerkin systems of ordinary differential equations with a simple structure. The program of reconstruction for these systems and numerical solutions of the Cauchy problems are realized. Several model problems are solved. The results are represented in a graphic form which illustrates the flows of the liquid. The linear homogeneous Cauchy problem is investigated in Gilbert spaces. Operator of this problem realizes isomorphism of these spaces. For a general case, some families of exact global solutions of the nonlinear Cauchy problem are found. Moreover, two Gilbert spaces with limited sequences of Galerkin approximations are written out.


Keywords: Fourier series, eigenfunctions of the curl operator, Navier-Stokes equations, Cauchy problem, global solutions, Galerkin systems, Gilbert spaces.

## 1. Introduction

1.1. Problem statement. Let us consider $2 \pi$-periodic functions $f(x+2 \pi m)=f(x)$ for all $m \in \mathbb{Z}^{3}$ in the space $R^{3}$. There exists a natural realization of the factor space $R^{3} / 2 \pi \mathbb{Z}^{3}$ in the form of a 3-dimensional torus

$$
\mathbb{T}=\left\{\left(e^{i x_{1}}, e^{i x_{2}}, e^{i x_{3}}\right) \in C^{3} ;\left(x_{1}, x_{2}, x_{3}\right) \in R^{3}\right\}
$$

given by the mapping $\left(x_{1}, x_{2}, x_{3}\right) \mapsto\left(e^{i x_{1}}, e^{i x_{2}}, e^{i x_{3}}\right)$. Whence, the standard realization of functions periodic on $R^{3}$ follows in the form of functions on a 3-dimensional torus. Let the fundamental cube $Q^{3}$ be given by the inequality $0 \leqslant x_{j}<2 \pi$. Integration on $\mathbb{T}$ is determined by means of the Lebesgue integral on the cube $Q$. Namely $\left.\int_{\mathbb{T}} f\right|_{\mathbb{T}} d x=\int_{Q} f d x$, where $f$ is a restriction on $Q$ of the periodic function in $R^{3}$ generated by the function $f$ on the torus. $L_{p}$-spaces on $\mathbb{T}$ are identified with $L_{p}$-spaces on $Q$ and are denoted by $L_{p}(\mathbb{T})$. Note that the class of continuous functions $C(\mathbb{T})$ ) corresponds not to the class of all continuous functions on $Q$, but only to the functions that remain continuous when continued periodically on all $R^{3}$. The Banach space $C(\mathbb{T})$ is a subspace in $L_{\infty}(\mathbb{T})$ and is endowed with the $L_{\infty}$-norm ( see [1], Ch.10, [2], Ch.7).

In addition, consider a subspace of solenoidal vector functions in $\left[L_{2}(\mathbb{T})\right]^{3}$ and denote it by

$$
\widehat{V}^{0}=\left\{v(x) \in\left[L_{2}(\mathbb{T})\right]^{3}: \operatorname{div} v=0 ;\|v\|_{\widehat{V}^{0}}=(2 \pi)^{-3}\|v\|_{L_{2}(Q)}\right\}
$$

Let us assume that complex vector functions $g(x) \in \widehat{V}^{0}$ and $f(t, x) \in \widehat{V}^{0}$ are given for any $t \geq 0$.

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Problem 1. Find the velocity vector $v(t, x)=\left(v_{1}, v_{2}, v_{3}\right)$ and pressure $p(t, x)$ that are $2 \pi$ periodic with respect to the space variables $x_{j}$, continuous in $R_{+} \times R^{3}$, and that have the corresponding smoothness and satisfy the Navier-Stokes

$$
\begin{equation*}
\frac{\partial v}{\partial t}-\nu \Delta v+(v \cdot \nabla) v=-\nabla p+f, \quad \text { div } v=0 \quad \text { when }(t, x) \in R_{+} \times R^{3} \tag{1}
\end{equation*}
$$

and the initial condition:

$$
\begin{equation*}
v(0, x)=g(x) \tag{2}
\end{equation*}
$$

Here $\Delta, \nabla$, div are linear Laplace's, gradient and divergence operators, respectively and the nonlinear operator $(v \cdot \nabla) v=\sum_{j=1}^{3} v_{j} \partial_{j} v$.

In the classical statement, it is supposed that the functions $g$ and $f$ are smooth: $g \in\left[C^{\infty}(\mathbb{T})\right]^{3}$ and $f \in\left[C^{\infty}([0, \infty) \times \mathbb{T})\right]^{3}$. Physically meaningful solutions satisfy the conditions that functions $v$ and $p$ should be smooth and globally determined:

$$
\begin{equation*}
v(t, x) \in\left[C^{\infty}([0, \infty) \times \mathbb{T})\right]^{3}, \quad p(t, x) \in\left[C^{\infty}([0, \infty) \times \mathbb{T})\right]^{3} \tag{3}
\end{equation*}
$$

and that the kinetic energy of the solution should be globally bounded, i.e. there is a constant $E \in(0, \infty)$ such that

$$
\begin{equation*}
\int_{Q}|v(t, x)|^{2} d x<E \quad \text { for any } t \geq 0 \tag{4}
\end{equation*}
$$

Wikipedia, the free encyclopedia in the Internet, discusses a complicated problem in the article „Navier-Stokes existence and smoothness". Namely, prove either (A) or (B).
(A) Existence and smoothness of the Navier-Stokes solutions in $\mathbb{T}^{3}$. Let $f(t, x) \equiv 0$. For any smooth initial condition $g$ there exists a smooth and globally defined solution to the Navier-Stokes equations, i.e. there is a velocity vector $v(t, x)$ and a pressure $p(t, x)$, satisfying the conditions (3), (4);
(B) Breakdown of the Navier-Stokes solutions in $\mathbb{T}^{3}$. There exists an external force $f(t, x)$ and an initial condition $g(x)$ such that there exist no smooth and globally defined solutions to the Navier-Stokes equations, i.e. the velocity vector $v(t, x)$ and the pressure $p(t, x)$ do not satisfy the conditions (3), (4).

In §5, we generate families of classical solutions to the problem in the explicit form for any $\nu>0$ for particular cases of initial conditions $g$ and the right-hand sides $f$, corresponding to the eigenfunctions of the curl operator.

Let us turn to a generalized statement of the problem (see monographs [4, 5, 6, 7], we follow the notation used in $[8,9]$ ).
1.2. Function spaces of the problem. The basic space

$$
\begin{equation*}
V^{0}=\left\{v \in\left[L_{2}(\mathbb{T})\right]^{3}: \operatorname{div} v=0, \int_{Q} v d x=0 ;\|v\|_{V^{0}}=(2 \pi)^{-3}\|v\|_{L_{2}(Q)}\right\} \tag{5}
\end{equation*}
$$

where the relation $\operatorname{div} v=0$ is interpreted in the sense of the theory of distributions over the space $\Pi^{\infty}$ of infinitely differentiable $2 \pi$-periodic functions, where the convergence $\varphi_{n} \rightarrow 0$ indicates a uniform convergence of $\varphi_{n}$ to zero when $n \rightarrow \infty$ together with all derivatives (see [1], Ch. 10). That is

$$
\begin{equation*}
(\operatorname{div} v, \varphi) \equiv-(v, \nabla \varphi)=0 \text { for any } \varphi \in \Pi^{\infty} . \tag{6}
\end{equation*}
$$

Note the inclusion of spaces $V^{0} \subset \widehat{V}^{0}$.
The Fourier series

$$
\begin{equation*}
v(x)=v_{0}+\sum_{|k|^{2}=1}^{\infty} v_{k} e^{i k x}, \text { where } v_{k}=\frac{1}{(2 \pi)^{3}} \int_{Q} v(x) e^{-i k x} d x \tag{7}
\end{equation*}
$$

is used to represent the vector functions $v(x) \in\left[L_{2}(\mathbb{T})\right]^{3}$. Here $k=\left(k_{1}, k_{2}, k_{3}\right)$ are integer-valued vectors, $k^{2} \equiv|k|^{2}=k_{1}^{2}+k_{2}^{2}+k_{3}^{2}$, the sign $v(x)=\sum$ indicates that the series converges to $v(x)$ in a quadratic mean, i.e. in the norm $L_{2}(\mathbb{T})^{3}$.

The Parseval-Steklov equality

$$
\begin{equation*}
(2 \pi)^{-3}\|v\|^{2}=\left|v_{0}\right|^{2}+\sum_{k^{2}=1}^{\infty}\left|v_{k}\right|^{2} \tag{8}
\end{equation*}
$$

exists in the set $\Pi^{\infty}$, dense in $\left[L_{2}(\mathbb{T})\right]^{3}$. It allows one to determine the Fourier expansion (transform) for elements of $\left[L_{2}(\mathbb{T})\right]^{3}$ and for wider spaces of distributions [1].

The solenoidal condition $v(x)$ is reduced to equalities $\left(v_{k}, k\right)=0$, i.e. to orthogonality of vectors $v_{k}$ to wave vectors $k$ for any $k \neq 0$.

The integral condition $\int_{Q} v d x=0$ indicates that $v_{0}=0$.
Following the works $[1,8]$, let us introduce the Sobolev space of periodic vector functions $H_{s}$ with the norm, defined by the equality

$$
\begin{equation*}
\|v\|_{H_{s}}^{2}=\left|v_{0}\right|^{2}+\sum_{|k|^{2}=1}^{\infty}|k|^{2 s}\left|v_{k}\right|^{2}, \quad s \in R_{+} \tag{9}
\end{equation*}
$$

Further, the spaces $\widehat{V}^{s}$ and $V^{s}$ are intersections of $\widehat{V}^{0}$ and $V^{0}$ with $H_{s}$ :

$$
\begin{equation*}
V^{s}=\left\{v(x) \in H_{s} \cap V^{0} ;\|v\|_{V^{s}}=\|v\|_{H_{s}}\right\} . \tag{10}
\end{equation*}
$$

The squared norm in $\widehat{V}^{s}$ is defined by the previous formula, provided that $\left(v_{k}, k\right)=0$ for all $k \neq 0$. This causes certain difficulties in investigation.

This condition disappears with turning to the Fourier series of the curl operator. According to $\S 2$, the eigenfunctions of the curl operator have the form $c_{k}^{ \pm} e^{i k x}$ и $\widehat{k} e^{i k x}$, where $\widehat{k}=k /|k|$. Any vector function $v(x)$ from $L_{2}(\mathbb{T})^{3}$ can be expanded into series in terms of the eigenfunctions of the curl operator

$$
\begin{gather*}
v(x)=v_{0}+\sum_{|k|^{2}=1}^{\infty}\left(\gamma_{k} \widehat{k}+\gamma_{k}^{+} c_{k}^{+}+\gamma_{k}^{-} c_{k}^{-}\right) e^{i k x}  \tag{11}\\
\gamma_{k}=\frac{1}{(2 \pi)^{3}} \int_{Q}(v(x), \widehat{k}) e^{-i k x} d x  \tag{12}\\
\gamma_{k}^{ \pm}=\frac{1}{(2 \pi)^{3}} \int_{Q}\left(v(x), c_{k}^{ \pm}\right) e^{-i k x} d x \tag{13}
\end{gather*}
$$

Then,

$$
\begin{gather*}
\|v\|_{H_{s}}^{2}=\left|v_{0}\right|^{2}+\sum_{|k|^{2}=1}^{\infty}|k|^{2 s}\left(\left|\gamma_{k}\right|^{2}\left|+\gamma_{k}^{+}\right|^{2}+\left|\gamma_{k}^{-}\right|^{2}\right)  \tag{14}\\
\|v\|_{V^{s}}^{2}=\sum_{|k|^{2}=1}^{\infty}|k|^{2 s}\left(\left|\gamma_{k}^{+}\right|^{2}+\left|\gamma_{k}^{-}\right|^{2}\right) \tag{15}
\end{gather*}
$$

because according to Lemma 1 , the function $v(x) \in V^{s}$, if and only if all $\gamma_{k}=0$ and $v_{0}=0$.
The vector function $v(t, x)$, as a function of $t \in(0, T)$ with values in the space $V^{s}$, belongs to the space $L_{2}\left(0, T ; V^{s}\right)$ if it has a finite norm, the square of which equals to

$$
\begin{align*}
& \|v(t, x)\|_{L_{2}\left(0, T ; V^{s}\right)}^{2}=\int_{0}^{T}\|v(t, \cdot)\|_{V^{s}}^{2} d t=  \tag{16}\\
& \int_{0}^{T}\left(\sum_{|k|^{2}=1}^{\infty}|k|^{2 s}\left(\left|\gamma_{k}^{+}(t)\right|^{2}+\left|\gamma_{k}^{-}(t)\right|^{2}\right)\right) d t
\end{align*}
$$

The space $L_{2}\left(0, T ; V^{s}\right)$ is indicated as $L_{2}\left(R_{+} ; V^{s}\right)$ when $T=+\infty$.
The norm of $v(t, x)$ in the space $L_{\infty}\left(0, T ; V^{s}\right)$ is defined as follows

$$
\begin{equation*}
\|\varphi\|_{L_{\infty}\left(0, T ; V^{s}\right)}=\operatorname{ess}^{\sup _{t \in[0, T]}}\|v(t, \cdot)\|_{V^{s}} \tag{17}
\end{equation*}
$$

Finally, define the space $W^{1,2(s)}$, where $s \in[0, \infty)$, by the formula

$$
\begin{equation*}
W^{1,2(s)}=\left\{v(t, x) \in L_{2}\left(0, T ; V^{2+s}\right): \partial_{t} v(t, x) \in L_{2}\left(0, T ; V^{s}\right)\right\} . \tag{18}
\end{equation*}
$$

Note that usually, when physical fields $v$ are represented, the mean of the vector function $v$ with respect to a cube is considered to be equal to zero, i.e. the vector $v_{0}=0$. This condition is included in definition of the space $V^{0}$.
1.3. Generalized problem statement (see [7] §.3 Ch.3). Suppose that $(v, p)$ is a classical solution of the problem (1), (2) and

$$
v \in C^{2}([0, T] \times \mathbb{T}), p \in C^{1}([0, T] \times \mathbb{T})
$$

Evidently, $v \in L_{2}\left(0, T ; V^{2}\right), \partial_{t} v \in L_{2}\left(0, T ; V^{0}\right)$. Multiplication (scalar n $\left.L_{2}(Q)\right)$ of the first equation (1) by an arbitrary vector function $w$ of the class $V^{1}$ and integration by parts provides

$$
\begin{equation*}
\frac{d}{d t}(v, w)+\nu(\nabla v, \nabla w)+b(v, v, w)=(f, w), \quad w \in V^{1} \tag{19}
\end{equation*}
$$

where

$$
\begin{equation*}
b(u, v, w) \equiv \sum_{i, j=1}^{3} \int_{Q} u_{i}\left(\partial_{i} v_{j}\right) \overline{w_{j}} d x \tag{20}
\end{equation*}
$$

by definition. Note that $\nabla p$ falls out: $(\nabla p, w)=-(p$, divw $)=0$, since $w \in V^{1}$.
Poblem2. Given $f(t, x) \in L_{2}\left(0, T ; V^{0}\right), g \in V^{1}$. In the class

$$
\begin{equation*}
W^{1,2(0)}=\left\{v(t, x) \in L_{2}\left(0, T ; V^{2}\right): \partial_{t} v(t, x) \in L_{2}\left(0, T ; V^{0}\right)\right\} \tag{21}
\end{equation*}
$$

find a vector function $v(t, x)$, satisfying Equations (19) and the initial condition (2): $v(0, x)=g(x)$ for any $w \in V^{1}$.
1.4. Results. The present paper is devoted to investigation of the Cauchy problem for the Navier-Stokes system of equations in a three-dimensional space with periodicity conditions with respect to space variables on the basis of Fourier series of the curl operator.

Some periodic eigenfunctions of the curl operator have long been known and used in works of V.I. Arnold [10] and his disciples, O. Bogoyavlenskii [11], physicists [12], [13],[14],[15] . See also V.V. Kozlov's monograph "General Theory of Vortices"[16] and surveys by V.V. Pukhnachev [17] and A.S. Makhalov and V.P. Nikolaenko [18].

In 2000, I managed to write out basis eigenfunctions of the curl operator in the space $\left[\mathrm{L}_{2}(\mathbb{T})\right]^{3}$ (see Theorem in §2) and informed O.A. Ladyzhenskaya about it during a seminar in Ufa (see [19],[20]). In 2003, Ladyzhenskaya tackled the problem "Construction of bases in spaces of solenoidal vector fields"[21]. On page 73 she writes about the Galerkin scheme: "It is good for proving existence theorems and further qualitative analysis of solutions. However, its numerical realization requires knowledge of a fundamental system $\left\{\varphi_{k}\right\}$ in $H(\Omega)$. In the present paper, we suggest a method for its construction". In particular, O.A. Ladyzhenskaya was interested in the possibility to calculate eigenfunctions of the Stokes operator in domains of simplest forms (cube, ball etc) and asked me about it.

It turns out that the periodic eigenfunctions $\left(v_{k}, p_{k}\right)$ of the Stokes operator are such that $p_{k}=$ const and the vector functions $v_{k}$ coincide with the solenoidal eigenfunctions of the curl operator $u_{k}^{ \pm}$for $k \neq 0$ and $u_{0}^{j}$ for $k=0$ [22].

Later on [23], I calculated eigenfunctions $\left(v_{n}, p_{n}\right)$ of the Stokes operator in a ball with the condition $v_{n}=0$ on the boundary. In this case, $p_{n}$ are also constants and every vector eigenfunction $v_{n}$ of the Stokes operator is the sum $v_{n}=u_{n}^{+}+u_{n}^{-}$of vector eigenfunctions of the
curl operator $u_{n}^{+}$and $u_{n}^{-}$with the eigenvalues having the same absolute values but opposite signs. These eigenfunctions at the ball boundary are located in the tangent plane and are oppositely directed. Thus, in [23] another approach to solving the problem on construction of bases in spaces of solenoid vector fields was obtained. I described the work [22] to O.A. Ladyzhenskaya in POMI in late 2003, and the basis of the paper [23] to researchers of her laboratory in early 2005.

In [24], the Cauchy problem for the Navier-Stokes system in the class of 2p-periodic functions was reduced to that for an infinite system of ordinary differential equations. Its explicit form suggests a method for constructing families of exact solutions.

In the present paper, we construct "approximations" $v_{l}$ of the velocity vector $v$, using the basis of periodic eigenfunctions of the curl operator in the space $V^{0}$. Coefficients $v_{l}$ satisfy the finite nonlinear Galerkin system $R S_{l}$. The system has a simple form in the given basis. Its linear part is diagonal, and the nonlinear part of every equation is a quadratic form (of unknown functions), with coefficients calculated explicitly via a scalar product of basis vectors of the curl operator (see $\S 3$ ). Programs for calculating coefficients of the systems $R S_{l}$, a numerical solution of the Cauchy problem and others are developed.

Some models are calculated. Figures in $\S 6$ give an idea of oscillation of the velocity vector in planes, orthogonal to wave vectors $k$. One can see that the motion of liquid is complicated noticeably with the decrease of the viscous parameter.

In $\S 4$, solvability of the Cauchy problem for a linear homogeneous Stokes system in the scale of spaces $W^{1,2(s)}(s \geq-1)$ is investigated. The operator of the problem $\left(\partial_{t}+A, \gamma_{0}\right) v \rightarrow\left(0, \gamma_{0} v\right)$ is proved to realize the isomorphism of spaces $W^{1,2(s)} \cap \operatorname{Ker}\left(\partial_{t}+A\right)$ and $V^{s+1}$ (Theorem 2).

In $\S 3$, solutions $v_{l}$ of the Cauchy problems for Galerkin equations $R S_{l}$ with given $S_{l} f, S_{l} g$ are considered in proposition that $f \in L_{2}\left(0, T ; V^{0}\right), g \in V^{1}$. The sequence $\left\{v_{l}\right\}_{l=1}^{\infty}$ is proved to be bounded both in the space $L_{2}\left(0, T ; V^{1}\right)$ and in the space $L_{\infty}\left(0, T ; V^{0}\right)$.

In $\S 5$, families of exact global solutions are singled out. For the sake of simplicity, we limit ourselves to four cases. Other families can be written out from [25], assuming that $\Omega=0$.
A.Babin, A.Makhalov and V. Nikolaenko published a number of works (see [8] and the survey [18]) devoted to investigation of the Cauchy problem for the Navier-Stokes system of equations in a space rotating uniformly (about a vertical vector with the angular velocity $\Omega$ ) with the initial data periodic with $2 \pi a_{j}$ periods along coordinate axes $e_{j}$. In this case, the Coriolis force equal to $\Omega\left[e_{3}, v\right]$ is introduced in the equations (1). Assuming that $g(\mathrm{x}) \in V^{\alpha}$ and $\|g(x)\|_{\alpha} \leq M_{\alpha}$, and the right-hand side of $f(t, x)$ belongs to the space $V^{\alpha-1}$ for $\alpha>1 / 2$ and

$$
\begin{equation*}
\sup _{T} \int_{T}^{T+1}\|f\|_{\alpha-1}^{2} d t \leq M_{\alpha f}^{2} \tag{22}
\end{equation*}
$$

they prove that there is a number $\Omega_{1}$, depending on $M_{\alpha}, M_{\alpha f}, \nu, a_{1}, a_{2}, a_{3}$ such that for $\Omega \geq \Omega_{1}$, the Navier-Stokes system has a global solution $U(t)$ with the value in $V^{\alpha}$, and $\|U(t)\|_{\alpha} \leq M_{\alpha}^{\prime}$ for all $t \geq 0$.

Studying the article [8], I restricted myself to the case when periodicity with respect to variables $x_{j}$ is the same, $a_{1}=a_{2}=a_{3}=1$, and expanded the given and sought vector functions in Fourier series in terms of eigenfunctions of the curl operator. This entailed significant simplification and gave a possibility to write out the explicit form of the Galerkin equations and various families of exact solutions of the Navier-Stokes equations (see [25, 26, 27]).
A.V.Fursikov [9] studied the initial boundary-value problem for the Navier-Stokes equations in a bounded domain with a smooth boundary and proved its local solvability in $V^{1,2(0)}$ with initial conditions from a unbounded ellipsoid $E l_{\rho}^{1 / 2}=\left\{g \in V^{1}:\|g\|_{V^{1 / 2}}<\rho\right\}$ with the small $\rho$.
S.S.Titov ( see, e.g., [28] Ch.4) studies the periodic Cauchy problem for the Navier-Stokes equations by the Cauchy-Kovalevskaya method in scales of the Banach spaces. Solution is
constructed in the form of a special power series. Existence of a solution is proved via L.V. Ovsyannikov's results with definite conditions of smallness $v(0, x)$ and time $t$.

## 2. Fourier series

2.1. Fourier series and eigenfunctions of the Laplace operator. The spectrum of the Laplace operator in the class of $2 \pi$-periodic functions consists of the numbers $|k|^{2}=k_{1}^{2}+k_{2}^{2}+k_{3}^{2}$, equal to squared lengths of integer valued vectors $k$. When $k \in \mathbb{Z}^{3}, j=1,2,3$, the eigenfunctions $(2 \pi)^{-3 / 2} e_{j} e^{i k x}$ generate an orthonormal basis in the space $\left[L_{2}(\mathbb{T})\right]^{3}$ of vector functions, integrable with a squared module in a cube Q .

Any vector function $f(x) \in\left[L_{2}(\mathbb{T})\right]^{3}$ can be expanded in a Fourier series

$$
\begin{equation*}
f(x)=f_{0}+\sum_{|k|^{2}=1}^{\infty} f_{k} e^{i k x}, \text { where } f_{k}=\frac{1}{(2 \pi)^{3}} \int_{Q} f(x) e^{-i k x} d x \tag{23}
\end{equation*}
$$

converging in the quadratic mean (see, e.g., $[1,2,3]$ ).
Note that trigonometric polynomials are dense in $\mathrm{C}(\mathbb{T})^{3}$ and $\mathrm{L}_{p}(\mathbb{T})^{3}, 1 \leqslant p<\infty$.
If all Fourier coefficients vanish for some $f \in \mathrm{~L}_{p}(\mathbb{T})^{3}$, then $f=0$.
If $f \in \mathrm{~L}_{2}(\mathbb{T})^{3}$ and $\sum_{k \in Z^{3}} f_{k} e^{i k x}$ is its Fourier series, then the Parseval-Steklov equality holds

$$
\begin{equation*}
(2 \pi)^{-3}\|f(x)\|^{2}=\sum_{k \in Z^{3}}\left|f_{k}\right|^{2} . \tag{24}
\end{equation*}
$$

The correspondence $f \leftrightarrow\left\{f_{k}\right\}$ is a unitary mapping of $\mathrm{L}_{2}\left(\mathbb{T}^{3}\right)^{3}$ on $\mathrm{l}_{2}\left(\mathbb{Z}^{3}\right)^{3}$.
The Fourier series (23) can be rewritten in the form of the Fourier integral

$$
\begin{equation*}
f(x)=\int_{Q}\left(\sum_{k} f_{k} \delta(y-k)\right) e^{i x y} d y \tag{25}
\end{equation*}
$$

The formula (25) indicates that a bornological function $\sum_{k} f_{k} \delta(y-k)$ serves as the Fourier transform of the periodic function $f(x)$ (see [1], ch. 10).
2.2. Fourier series on the basis of eigenfunctions of the curl operator. In [22], I proved that the spectrum of the curl operator consists of the number 0 of infinite multiplicity and of numbers $\pm|k|$ of finite multiplicity.

Let $k_{0} \in \mathbb{Z}^{3} \backslash\{0\}$. Denote by $u_{k}^{ \pm}(x)$ basis vector eigenfunctions of the curl operator, corresponding to their eigenvalues $\pm\left|k_{0}\right|$, respectively. They satisfy the equations

$$
\begin{equation*}
\operatorname{rot} u_{k}^{ \pm}(x)= \pm\left|k_{0}\right| u_{k}^{ \pm}(x) \tag{26}
\end{equation*}
$$

and have the form

$$
\begin{equation*}
u_{k}^{ \pm}(x)=(2 \pi)^{-3 / 2} c_{k}^{ \pm} e^{i k x} \tag{27}
\end{equation*}
$$

Here, points $k$ lie on the sphere of the radius $\left|k_{0}\right|$. Vectors $c_{k}^{ \pm}=a_{k}^{ \pm}+i b_{k}^{ \pm}$are chosen depending on whether the vector $k^{\prime}=\left(k_{1}, k_{2}\right)$ equals to zero:

$$
c_{k}^{ \pm}= \pm \frac{\sqrt{2}}{2\left|k^{\prime}\right|}\left(\begin{array}{c}
k_{2}  \tag{28}\\
-k_{1} \\
0
\end{array}\right)+i \frac{\sqrt{2}}{2|k|\left|k^{\prime}\right|}\left(\begin{array}{c}
k_{1} k_{3} \\
k_{2} k_{3} \\
-k_{1}^{2}-k_{2}^{2}
\end{array}\right) \text { when } k^{\prime} \neq 0
$$

and

$$
c_{k}^{ \pm}= \pm \frac{\sqrt{2}}{2|k|}\left(\begin{array}{c}
k_{3}  \tag{29}\\
0 \\
0
\end{array}\right)+i \frac{\sqrt{2}}{2}\left(\begin{array}{c}
0 \\
1 \\
0
\end{array}\right) \text { when } k^{\prime}=0 \text { and } k \neq 0 .
$$

One can readily verify that $b_{k}^{+}$is a vector product of $\widehat{k}=k /|k|$ and $a_{k}^{+}$and that for any $k \neq 0$ three vectors $c_{k}^{+}, c_{k}^{-}$and $\widehat{k}$ construct an orthonormal basis in a complex space $\mathbb{C}^{3}$, and three vectors $\sqrt{2} a_{k}^{+}, \sqrt{2} b_{k}^{+}, \widehat{k}$ construct it in a real space $R^{3}$, respectively.

A zero eigenvalue of the curl operator corresponds to the vector functions $u_{k}(x)=(2 \pi)^{-3 / 2} \widehat{k} e^{i k x}$ when $k \neq 0$ and the vectors $u_{0}^{j}=(2 \pi)^{-3 / 2} e_{j}$ when $k=0$.

Together with $u_{k}^{ \pm}(x)$ they construct an orthonormal basis in $\mathrm{L}_{2}(\mathbb{T})^{3}$ [22]. This result will be used in what follows and is represented in the form of the following theorem with the proof.

Theorem. Any vector function $f(x) \in\left[L_{2}(\mathbb{T})\right]^{3}$ can be expanded in a Fourier series

$$
\begin{equation*}
f(x)=f_{0}+\sum_{|k|^{2}=1}^{\infty}\left(\phi_{k} \widehat{k}+\phi_{k}^{+} c_{k}^{+}+\phi_{k}^{-} c_{k}^{-}\right) e^{i k x} \tag{30}
\end{equation*}
$$

with respect to eigenvalues of the curl operator. The vector $f_{0}$ is the integral

$$
\begin{equation*}
f_{0}=\frac{1}{(2 \pi)^{3}} \int_{Q} f(x) d x \tag{31}
\end{equation*}
$$

which is the mean $f$ with respect to a cube, the remaining coefficients $\phi_{k}, \phi_{k}^{+}, \phi_{k}^{-}$equal to

$$
\begin{align*}
& \phi_{k}=\frac{1}{(2 \pi)^{3}} \int_{Q}(f(x), \widehat{k}) e^{-i k x} d x  \tag{32}\\
& \phi_{k}^{ \pm}=\frac{1}{(2 \pi)^{3}} \int_{Q}\left(f(x), c_{k}^{ \pm}\right) e^{-i k x} d x \tag{33}
\end{align*}
$$

The brackets $(f, g)$ indicate scalar products in $\mathbb{C}^{3}$. The series converges in the quadratic mean, i.e. in the norm $\left[L_{2}(\mathbb{T})\right]^{3}$.

Let us term the expansion (30) as a modified Fourier series. The Parseval-Steklov equality takes the form

$$
\begin{equation*}
(2 \pi)^{-3}\|f\|^{2}=\left|f_{0}\right|^{2}+\sum_{|k|^{2}=1}^{\infty}\left(\left|\phi_{k}\right|^{2}+\left|\phi_{k}^{+}\right|^{2}+\left|\phi_{k}^{-}\right|^{2}\right) \tag{34}
\end{equation*}
$$

Proof. The expansion

$$
\begin{equation*}
h=(h, \widehat{k}) \widehat{k}+\left(h, c_{k}^{+}\right) c_{k}^{+}+\left(h, c_{k}^{-}\right) c_{k}^{-} \tag{35}
\end{equation*}
$$

exists for any vector $h$ of $C^{3}$ when $k \neq 0$.
According to the notation (32) and (33), it has the following form for the vector $f_{k}$ of (23): $f_{k}=\phi_{k} \widehat{k}+\phi_{k}^{+} c_{k}^{+}+\phi_{k}^{-} c_{k}^{-}$. Substituting this expansion into the series (23), one obtains the series (30). Denote by $S_{l} f(x)$ a partial sum of the series (30), projection of the vector $f(x)$ onto a finite-dimensional space $G_{l}$, spanned on the basis vectors $u_{0}^{j}, u_{k}$ and $u_{k}^{ \pm}$for $|k|^{2} \leqslant l$. Then,

$$
\begin{equation*}
(2 \pi)^{-3}\left\|S_{l} f(x)\right\|^{2}=\left|f_{0}\right|^{2}+\sum_{|k|^{2}=1}^{l}\left(\left|\phi_{k}\right|^{2}+\left|\phi_{k}^{+}\right|^{2}+\left|\phi_{k}^{-}\right|^{2}\right)=\sum_{|k|^{2}=0}^{l}\left|f_{k}\right|^{2} \tag{36}
\end{equation*}
$$

The vector $f-S_{l} f(x)$ is orthogonal to $G_{l}$ and $\left\|f-S_{l} f(x)\right\|^{2}=\|f\|^{2}-\left\|S_{l} f(x)\right\|^{2}$ due to the Pythagoras theorem [4]. The Parseval equality entails that the difference norm $\left\|f-S_{l} f(x)\right\| \rightarrow 0$ when $l \rightarrow \infty$. Hence, the sequence $S_{l} f(x)$ converges to $f(x)$ in the norm of the space $L_{2}(\mathbb{T})^{3}$. Theorem is proved.
2.3. Summation of series. Multiplicity of eigenvalues $\pm\left|k_{0}\right|$ of the curl operator equals to the number $\kappa\left(\left|k_{0}\right|^{2}\right)$ of points of an integer lattice in $\mathbb{Z}^{3}$, located on a sphere of the radius $\left|k_{0}\right|$. The number $\left|k_{0}\right|^{2}$ is integer. Numbers $n=4^{m}(8 q+7)$, where $m, q \geq 0$ are integer, are not representable in the form of a sum of three squared integers, the number $n=7$ is the first one. For such $n$, we assume that $\kappa(n)=0$. Then, $N_{l}=\kappa(1)+\ldots+\kappa(l)$ is the number of nonzero points of an integer lattice located in a sphere of the radius $\sqrt{l}$. The sign $\sum_{|k|^{2}=1}^{\infty}$ indicates that the series (30) are summed over the levels $|k|^{2}=l$, where $l=1,2 \ldots$. Summation of lattice points is arbitrary on every such sphere, and convergence of the sphere is independent of the order of summation. The levels for which $\kappa(n)=0$ are omitted.

Calculating the Fourier coefficients (32), (33), one has to know the enumeration of the lattice points. A program for enumeration of nonzero points of the lattice

$$
k \rightarrow \vartheta(k): 1(-1,0,0), 2(0,-1,0), 3(0,0,-1), \ldots, 18(1,1,0), \ldots,
$$

where the point $(-1,0,0)$ is the first one, the point $(1,1,0)$ is the 18 th one etc is developed.
Calculating integrals over cubes, we used Sobolev's cubature formulae with a regular boundary layer ([1], ch.14).
2.4. Decomposition to orthogonal subspaces. The Fourier series (23) and (30) indicate that there are 2 ways to decompose the vector space $L_{2}(\mathbb{T})^{3}$ into orthogonal subspaces:

$$
\begin{equation*}
L_{2}(\mathbb{T})^{3}=\underset{j, k}{\oplus} F_{k}^{j} \quad \text { and } \quad L_{2}(\mathbb{T})^{3}=\underset{j}{\oplus} F_{0}^{j} \underset{k \neq 0}{\oplus}\left(R_{k} \oplus R_{k}^{+} \oplus R_{k}^{-}\right), \tag{37}
\end{equation*}
$$

where $F_{k}^{j}$ and $R_{k}, R_{k}^{ \pm}$are subspaces, generated by vector functions $e_{j} e^{i k x}(j=1,2,3)$, and $\widehat{k} \mathrm{e}^{\mathrm{i} k x}, c_{k}^{ \pm} \mathrm{e}^{i k x}$, respectively.

Indeed, choosing another basis in a complex space $F_{k}=F_{k}^{1} \oplus F_{k}^{2} \oplus F_{k}^{3}$, we obtain the decomposition $F_{k}=R_{k} \oplus R_{k}^{+} \oplus R_{k}^{-}$. These bases are equivalent, their choice is ambiguous (see [22]).

Lemma 1. Let $f(x)$ of $L_{2}(\mathbb{T})^{3}$ be represented by the series (30). It satisfies the equation divf $=0$ in terms of generalized functions if and only if its coefficients (32) vanish $\phi_{k}=0$.

Indeed, due to (6), the condition $\operatorname{div} f=0$ indicates that the vector $f$ is orthogonal to the gradient of any scalar periodic function $\psi(x)$ from $\Pi^{\infty}$.

An arbitrary function $\psi(x)$ of $\Pi^{\infty}$ is expanded into a Fourier series $\psi(x)=\psi_{0}+\sum_{k \neq 0} \psi_{k} e^{i k x}$, converging in $L_{2}(Q)$ together with derivatives of any order.

Its gradient equals to $\sum_{k \neq 0} i k \psi_{k} e^{i k x}$. The formula (32) indicates that $i(2 \pi)^{3}|k| \phi_{k}=$ $-\left(f, \nabla e^{i k x}\right)$ is a scalar product of $-f$ and $\nabla e^{i k x}$ in $L_{2}(Q)$. Therefore,

$$
\begin{equation*}
-(f(x), \nabla \psi(x))=i(2 \pi)^{3} \sum_{|k|^{2}=1}^{\infty}|k| \phi_{k} \bar{\psi}_{k} . \tag{38}
\end{equation*}
$$

Hence, $\operatorname{div} f=0$, if all $\phi_{k}=0$. The inverse statement follows from the arbitrary choice of $\psi_{k}$ and completeness of the exponential system $\left\{e^{i k x}\right\}$.

Lemma 1 implies that there is an expansion

$$
\begin{equation*}
f(x)=f_{0}+\sum_{|k|^{2}=1}^{\infty}\left(\phi_{k}^{+} c_{k}^{+}+\phi_{k}^{-} c_{k}^{-}\right) e^{i k x} \tag{39}
\end{equation*}
$$

for functions $f \in \widehat{V}^{0}$. The squared norm of the expansion equals to

$$
\begin{equation*}
\|f\|_{\hat{V}^{0}}^{2}=\left|f_{0}\right|^{2}+\sum_{|k|^{2}=1}^{\infty}\left(\left|\phi_{k}^{+}\right|^{2}+\left|\phi_{k}^{-}\right|^{2}\right) \tag{40}
\end{equation*}
$$

2.5. Gradient and solenoidal components of vector function. Let us expand the vector function $F(x) \in L_{2}(\mathbb{T})^{3}$ in the Fourier series (30) and represent it in the form of a sum $F(x)=$ $f(x)+\nabla q(x)$, where

$$
\begin{equation*}
f=f_{0}+\sum_{|k|^{2}=1}^{\infty}\left(\phi_{k}^{+}(t) c_{k}^{+}+\phi_{k}^{-}(t) c_{k}^{-}\right) e^{i k x}, \quad q=-i \sum_{|k|^{2}=1}^{\infty} \phi_{k}|k|^{-1} e^{i k x} \tag{41}
\end{equation*}
$$

Whence, one can observe that $f \in \widehat{V}^{0}$ and $q \in H_{1}(\mathbb{T})$. The vector functions $\nabla q(x)$ and $f(x)$ are mutually orthogonal. They are projections of $F$ on the spaces

$$
\begin{equation*}
G=\underset{k \neq 0}{\oplus} R_{k}, \text { and } \widehat{V}^{0}=F_{0} \underset{k \neq 0}{\oplus}\left(R_{k}^{+} \oplus R_{k}^{-}\right) ; \quad L_{2}(\mathbb{T})^{3}=G \oplus \widehat{V}^{0} \tag{42}
\end{equation*}
$$

Denote them by $\widehat{\Pi}_{0} F$ and $\Pi_{G} F$, i.e. $f=\widehat{\Pi}_{0} F, \nabla q=\Pi_{G} F$.
Another formulation of the Lemma 1 is as follows: $f \in \widehat{V}^{0} \Leftrightarrow \Pi_{G} f=0$.
If the function $F(t, x)$ depends on time $t$, then the functions $q$ and $f$ depend on $t$ as well. Substituting $\nabla q+f$ into the right-hand side of Equation (1) and assuming that $P=p-q$, one obtains equations of the same form, where $F=f$ and $f(t, x)$ is solenoidal for any $t \geq 0$.
2.6. Connection between eigenfunctions of the curl and Stokes operators. Periodic vector eigenfunctions ( $v_{n}, p_{n}$ ) of the Stokes operator satisfy the equations [5]:

$$
\begin{equation*}
-\nu \Delta v_{n}+\nabla p_{n}=\lambda v_{n}, \quad \text { div } v_{n}=0 \tag{43}
\end{equation*}
$$

Whence, one readily obtains that $p_{n}$ are harmonic functions, $\Delta p_{n}=0$. However, a harmonic function is periodic if and only if it is constant $p_{n}=$ const. Hence, $\nabla p_{n}=0$, and Equations (43) do not contain pressure. Solenoidal eigenfunctions of the curl operator $u_{0}^{j}(x)$ and $u_{k}^{ \pm}(x)$ satisfy these equations with $\lambda=0$ and $\lambda=\nu|k|^{2}$ when $k \neq 0$. According to Theorem 1, there are no other eigenfunctions. Thus, the series (39) is an expansion of the vector function $f(x) \in \widehat{V}^{0}$ over eigenfunctions of the curl operator as well as of the Stokes operator.
2.7. Hilbert space $H_{s}(\mathbb{T}), s \in R$. This is the notation for the Sobolev space of $2 \pi$-periodic vector functions with the norm

$$
\begin{equation*}
\|f\|_{H_{s}}^{2}=\left|f_{0}\right|^{2}+\sum_{|k|^{2}=1}^{\infty}|k|^{2 s}\left(\left|\phi_{k}\right|^{2}+\left|\phi_{k}^{+}\right|^{2}+\left|\phi_{k}^{-}\right|^{2}\right) \tag{44}
\end{equation*}
$$

when $s \geq 0$, and $H_{0}(\mathbb{T})$ is identified with $L_{2}(\mathbb{T})^{3}$. The space $H_{(-s)}$ for $s>0$ is defined as a space conjugate to $H_{s}$ with respect to a scalar product in $L_{2}(\mathbb{T})^{3}$. The norm in $H_{(-s)}$ is defined by the formula (44), where $s$ is negative. Thus, the spaces $H_{s}$ are defined for any $s$ of $R$ (see [1], ch. 12, [6], ch.1).
S.L. Sobolev defined and investigated these spaces when $s$ are integer.

In section 1.2, we determined the space $\widehat{V}^{s}=H_{s} \cap \widehat{V}^{0}$ and its subspace $V^{s}=H_{s} \cap V^{0}$, consisting of vector functions $f$ with the zero mean $f_{0}=S_{0} f=0$ when $s \geq 0$. Now they can be determined for all $s \in R$. Let us single out injections of the spaces. When $s>1$, one has

$$
V^{s} \subset V^{1} \subset V^{0} \subset V^{-1} \subset V^{-s}
$$

If $f \in V^{s}$ then, according to Lemma 1 ,

$$
\begin{equation*}
\|f\|_{V^{s}}^{2}=\sum_{|k|^{2}=1}^{\infty}|k|^{2 s}\left(\left|\phi_{k}^{+}\right|^{2}+\left|\phi_{k}^{-}\right|^{2}\right) . \tag{45}
\end{equation*}
$$

Note that the norm (45) of the function $f$ in the space $V^{s}$ for $s=1$ coincides with the norm $\|\nabla f\|_{V^{0}}$, and for $s=2$ it coincides with the norm $\|\triangle f\|_{V^{0}}$.

Furthermore, the norm of the function $f(t, x)=\widehat{\Pi}_{0} f(t, x)$ in the space $L_{2}\left(0, T ; \widehat{V}^{s}\right)$ is defined as follows:

$$
\begin{equation*}
\|f\|_{L_{2}\left(0, T ; \hat{V}^{s}\right)}^{2}=\int_{0}^{T}\left(\left|f_{0}(t)\right|^{2}+\sum_{|k|^{2}=1}^{\infty}|k|^{2 s}\left(\left|\phi_{k}^{+}(t)\right|^{2}+\left|\phi_{k}^{-}(t)\right|^{2}\right)\right) d t \tag{46}
\end{equation*}
$$

Substituting the series (45) into the formulae (17) and (18), we obtain the explicit expressions for the norms $f$ in spaces $L_{\infty}\left(0, T ; V^{s}\right)$ and $W^{1,2,(s)}$ via the modified Fourier coefficients and their derivatives.
2.8. Real eigenfunctions of the curl operator and vortex flows. Since a curl operator is a differential operator of the first order with real coefficients and its eigenvalues are real, the real and imaginary parts of its eigenfunctions are also eigenfunctions with the same eigenvalues. Let us write out the explicit form of their real parts.

Suppose that $\phi_{k}=\alpha_{k}+i \beta_{k}=\left|\phi_{k}\right| e^{i \theta_{k}}$,

$$
\phi_{k}^{ \pm}=\alpha_{k}^{ \pm}+i \beta_{k}^{ \pm}=\left|\phi_{k}^{ \pm}\right| e^{i \theta_{k}^{ \pm}}, \text {where }\left|\theta_{k}\right|, \quad\left|\theta_{k}^{ \pm}\right| \leq \pi
$$

then

$$
\begin{gather*}
\operatorname{Re}\left(\phi_{k} e^{i k x}\right) \overparen{k}=\left(\alpha_{k} \cos k x-\beta_{k} \sin k x\right) \overparen{k}=\left|\phi_{k}\right| \cos \left(k x+\theta_{k}\right) \overparen{k}, \\
\operatorname{Re}\left(\phi_{k}^{ \pm} c_{k}^{ \pm} e^{i k x}\right)=\left|\phi_{k}^{ \pm}\right|\left(\cos \left(k x+\theta_{k}^{ \pm}\right) a_{k}^{ \pm}-\sin \left(k x+\theta_{k}^{ \pm}\right) b_{k}^{ \pm}\right) \tag{47}
\end{gather*}
$$

The resulting expressions give a possibility to represent fluid dynamics in $\mathbb{R}^{3}$, definable by the stationary fields $d_{k}^{ \pm}(x)=2 \operatorname{Re}\left(\phi_{k}^{ \pm} c_{k}^{ \pm} e^{i k x}\right)$. Let the fluid velocity be $v(x)=d_{k}^{+}(x)$. Evidently, the vector $\mathrm{d}_{k}^{+}(x)$ belongs to the plane, generated by vectors $a_{k}^{+}, b_{k}^{+}$. Its length $\left|\mathrm{d}_{k}^{+}(x)\right|$ is independent of $x$ and equals $\sqrt{2}\left|\phi_{k}^{+}\right|$, and the direction is constant in every plane $P_{\delta+2 \pi n}$, where $k x=$ $\delta+2 \pi n, n \in Z$, because

$$
\begin{equation*}
\left.\mathrm{d}_{k}^{+}(x)\right|_{P_{\delta+2 \pi n}}=\left.\mathrm{d}_{k}^{+}(x)\right|_{P_{\delta}}=2\left|\phi_{k}^{+}\right|\left(\cos \left(\delta+\theta_{k}^{+}\right) a_{k}^{+}-\sin \left(\delta+\theta_{k}^{+}\right) b_{k}^{+}\right) \tag{48}
\end{equation*}
$$

The plane $P_{\delta}$ is orthogonal to the vector $k$ and the vector $\mathrm{d}_{k}^{+}(x)$, transferred to the point $x \in P_{\delta}$, is located inside the plane. Therefore, fluid flows uniformly in one and the same direction in every such plane. If the vectors $d_{k}^{+}(x)$ are laid off the points $x$, belonging to an axis of the vector $k$, then $k x=|k||x|$ and hence, they rotate when $x$ varies.

The vector rot $d_{k}^{ \pm}(x)$ at the point $x$ is called the vorticity of the flow given by the field $\mathrm{d}_{\mathrm{k}}^{ \pm}(x)$. Since

$$
\begin{equation*}
\text { rot } d_{k}^{ \pm}(x)= \pm|k| d_{k}^{ \pm}(x) \tag{49}
\end{equation*}
$$

for any $k \in Z^{3} \backslash\{0\}$, and the length of the vectors $\mathrm{d}_{k}^{ \pm}(x)$ is constant in $x$, then the vorticity of such fluid flows is not vanishing at every point $x \in \mathbb{R}^{3}$. Let us call them vortices. Note that vector functions $\widehat{d}_{k}^{ \pm}(x)$ of a unit length satisfy Equation (49) as well. Vorticity of the flows increases with $|k|$.
2.9. Fourier series of a real function. Consider an integer lattice $\mathbb{Z}^{3}$ and its subsets $M_{1}=\left\{k: \mathrm{k}_{1} \in N, \mathrm{k}_{2}=\mathrm{k}_{3}=0\right\}, M_{2}=\left\{k: \mathrm{k}_{1} \in \mathbb{Z}, \mathrm{k}_{2} \in N, \mathrm{k}_{3}=0\right\}, M_{3}=$ $\left\{k:\left(\mathrm{k}_{1}, \mathrm{k}_{2}\right) \in \mathbb{Z}^{2}, \mathrm{k}_{3} \in N\right\}$. Denote the union of the subsets, and a set centrally symmetric with $M$ by $M=M_{1} \cup M_{2} \cup M_{3}$, and by $M^{*}$, respectively. The vector $-k \in M^{*}$, if $k \in M$ and vice versa. $M \cup M^{*}=\mathbb{Z}^{3} /\{0\}$.

Let $k \in M$. The formulae (28), (29) manifest that $\bar{c}_{k}^{+}=-c_{-k}^{+}=-c_{k}^{-}$, where the bar indicates a complex conjugation. Hence, $\overline{\phi_{k}^{+} c_{k}^{+} e^{i k x}}=\phi_{-k}^{+} c_{-k}^{+} e^{-i k x}$ if and only if $\overline{\phi_{k}^{+}}=-\phi_{-k}^{+}$.

The difference $f(x)-\overline{f(x)}=0$ for a real function $f(x)$. Taking into account that representation of $f(x)$ in the form of a Fourier series is unique, one arrives at the the following statement.

Lemma 2. Let us assume that a vector function $f(x) \in L_{2}(\mathbb{T})$ is represented by the series (30). It is real, $f(x)=\overline{f(x)}$ if and only if its Fourier coefficients satisfy the relations

$$
\bar{\phi}_{k}=-\phi_{-k}, \quad \bar{\phi}_{k}^{+}=-\phi_{-k}^{+}, \quad \bar{\phi}_{k}^{-}=-\phi_{-k}^{-}, k \in M, \quad \text { and } f_{0}^{j}=\bar{f}_{0}^{j}, j=1,2,3 .
$$

For a real function, the series (30) takes the form:

$$
\begin{equation*}
f(x)=f_{0}+2 R e \sum_{k \in M}\left(\phi_{k} \widehat{k}+\phi_{k}^{+} c_{k}^{+}+\phi_{k}^{-} c_{k}^{-}\right) e^{i k \cdot x} . \tag{50}
\end{equation*}
$$

The norm of the function $\widehat{\Pi}_{0} f$, projection of $f$ on $\widehat{V}^{0}$, is given by the formula

$$
\begin{equation*}
\left\|\widehat{\Pi}_{0} f\right\|_{\widehat{V}^{0}}^{2}=\left|f_{0}\right|^{2}+2 \sum_{k \in M}\left(\left|\phi_{k}^{+}\right|^{2}+\left|\phi_{k}^{-}\right|^{2}\right) . \tag{51}
\end{equation*}
$$

## 3. The Faedo-Galerkin method

3.1. The generalized Cauchy problem. Let us use the Faedo-Galerkin method [5, 6, 7]. Consider eigenfunctions of the Stokes operator

$$
u_{0}^{j}=\varpi^{-1} e_{j} \text { and } u_{k}^{ \pm}(x)=\varpi^{-1} c_{k}^{ \pm} e^{i k x} \text {, where } j=1,2,3, k \neq 0 \text {, }
$$

$\varpi=(2 \pi)^{3 / 2}$, being also eigenfunctions of the curl operator, as a fundamental orthonormal system in $\widehat{V}^{0}$

The condition $f(t, x) \in L_{2}\left(0, T ; \widehat{V}^{0}\right)$ means that the vector function $f=\widehat{\Pi}_{0} f$, represented by the series (39), has a finite norm

$$
\begin{equation*}
\|f(t, x)\|_{L_{2}\left(0, T ; \hat{V}^{0}\right)}^{2}=\int_{0}^{T}\left(\left|f_{0}(t)\right|^{2}+\sum_{|k|^{2}=1}^{\infty}\left(\left|\phi_{k}^{+}(t)\right|^{2}+\left|\phi_{k}^{-}(t)\right|^{2}\right)\right) d t \tag{52}
\end{equation*}
$$

Since $V^{0} \subset V^{-1}$, the norm $f$ in $L_{2}\left(0, T ; \widehat{V}^{-1}\right)$ is aso finite.
The condition $g \in \widehat{V}^{1}$ indicates that $g=\widehat{\Pi}_{0} g$ and is expanded into the series

$$
\begin{equation*}
g(x)=g_{0}+\sum_{|k|^{2}=1}^{\infty}\left(\psi_{k}^{+} c_{k}^{+}+\psi_{k}^{-} c_{k}^{-}\right) e^{i k \cdot x} \tag{53}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{0}=\frac{1}{(2 \pi)^{3}} \int_{Q} g(x) d x, \quad \psi_{k}^{ \pm}=\frac{1}{(2 \pi)^{3}} \int_{Q}\left(g(x), c_{k}^{ \pm}\right) e^{-i k \cdot x} d x . \tag{54}
\end{equation*}
$$

Meanwhile,

$$
\begin{equation*}
\|g(x)\|_{\hat{V}^{1}}^{2}=\left|g_{0}\right|^{2}+\sum_{|k|^{2}=1}^{\infty}|k|^{2}\left(\left|\psi_{k}^{+}\right|^{2}+\left|\psi_{k}^{-}\right|^{2}\right)<\infty \tag{55}
\end{equation*}
$$

Hence, the sequence of partial sums of the series (53)

$$
\begin{equation*}
S_{l} g(x)=g_{0}+\varpi \sum_{|k|^{2}=1}^{l}\left(\psi_{k}^{+} u_{k}^{+}(x)+\psi_{k}^{-} u_{k}^{-}(x)\right) \tag{56}
\end{equation*}
$$

converges to $g$ in the norm $\widehat{V}^{1} \subset \widehat{V}^{0}$. At the same time, the quantities $\left\|g-S_{l} g\right\|_{\widehat{V}^{0}}^{2}=$

$$
\begin{equation*}
\|g\|_{\hat{V}^{0}}^{2}-\left\|S_{l} g\right\|_{\widehat{V}^{0}}^{2}=\varpi^{-2} \int_{Q}|g|^{2} d x-\left|g_{0}\right|^{2}-\sum_{|k|^{2}=1}^{l}\left(\left|\psi_{k}^{+}\right|^{2}+\left|\psi_{k}^{-}\right|^{2}\right), \tag{57}
\end{equation*}
$$

also tend to zero when $l \rightarrow \infty$.

Let $f(t, x) \in L_{2}\left(0, T ; \widehat{V}^{0}\right), g \in \widehat{V}^{1}$. Let us find the "approximate" solution to the problem 2 in the form

$$
\begin{equation*}
v_{l}(x, t)=\varpi \sum_{j=1}^{3} v_{0}^{j}(t) e_{0}^{j}+\varpi \sum_{|k|^{2}=1}^{l}\left(\gamma_{k, l}^{+}(t) u_{k}^{+}(x)+\gamma_{k, l}^{-}(t) u_{k}^{-}(x)\right) \tag{58}
\end{equation*}
$$

where the functions $v_{0}^{j}(t)$ and $\gamma_{k, l}^{ \pm}(t)$ are obtained from conditions

$$
\begin{equation*}
v_{0}^{j}(0)=g_{0}^{j}, \quad \gamma_{k, l}^{ \pm}(0)=\psi_{k}^{ \pm}, \quad j=1,2,3, \quad 0<|k|^{2} \leqslant l, \tag{59}
\end{equation*}
$$

and equations

$$
\begin{align*}
\frac{d}{d t}\left(v_{l}, u_{0}^{j}\right)+\nu\left(\nabla v_{l}, \nabla u_{0}^{j}\right)+b\left(v_{l}, v_{l}, u_{0}^{j}\right)=\left(f, u_{0}^{j}\right), & j=1,2,3  \tag{60}\\
\frac{d}{d t}\left(v_{l}, u_{k}^{ \pm}\right)+\nu\left(\nabla v_{l}, \nabla u_{k}^{ \pm}\right)+b\left(v_{l}, v_{l}, u_{k}^{ \pm}\right)=\left(f, u_{k}^{ \pm}\right), & 0<|k|^{2} \leqslant l \tag{61}
\end{align*}
$$

Here, the brackets $(\cdot, \cdot)$ indicate scalar products in $L_{2}(Q)$. The functions

$$
\begin{equation*}
p_{l}(x, t)=p_{0}(t)+\sum_{|k|^{2}=1}^{l} p_{k, l}(t) e^{i k x} \tag{62}
\end{equation*}
$$

are derived from equations

$$
\begin{equation*}
\left(L\left(v_{l}\right)+\nabla p_{l}-f, \widehat{u}_{k}\right)=0,0<|k|^{2} \leqslant l . \tag{63}
\end{equation*}
$$

Manifestly, Equations (60) coincide with equations

$$
\begin{equation*}
\frac{\partial v_{0}}{\partial t}=f_{0}(t) \text { и } v_{0}(0)=g_{0} \tag{64}
\end{equation*}
$$

for the vector function $v_{0}(t)=\left(\mathrm{v}_{0}^{1}, \mathrm{v}_{0}^{2}, \mathrm{v}_{0}^{3}\right)$.
Taking into account that $v_{l}(t, x)$ are smooth in $x$, one concludes that Equations (61) coincide with equations

$$
\begin{equation*}
\left(L\left(v_{l}\right)+\nabla p_{l}-f, u_{k}^{ \pm}\right)=0,0<|k|^{2} \leqslant l . \tag{65}
\end{equation*}
$$

Let us write them out in more detail. Note that the vector function

$$
\begin{equation*}
\nabla p_{l}(x, t)=i \sum_{|k|^{2}=1}^{l} p_{k, l}(t) k e^{i k \cdot x}=i \sum_{|k|^{2}=1}^{l}|k| p_{k, l}(t) \overparen{k} e^{i k \cdot x} \tag{66}
\end{equation*}
$$

is orthogonal to basis vector functions $u_{k}^{ \pm}$of $V^{0}$, i.e. $\left(\nabla p_{l}, u_{k}^{ \pm}\right)=0$.
The operator $L(v)$ in Equations (1) is a sum of the linear and nonlinear operators: $S v=\partial_{t} v-\nu \Delta v$ and $N(v, v)=(v \cdot \nabla) v$.

Calculate the values of the operators in the sum (58).

$$
\begin{equation*}
S v_{l}=\partial_{t} v_{0}+\sum_{|k|^{2}=1}^{l}\left(\left(\partial_{t}+\nu|k|^{2}\right) \gamma_{k, l}^{+}(t) c_{k}^{+}+\left(\partial_{t}+\nu|k|^{2}\right) \gamma_{k, l}^{-}(t) c_{k}^{-}\right) e^{i k x} \tag{67}
\end{equation*}
$$

Denoting the vector $\gamma_{k, l}^{+}(t) c_{k}^{+}+\gamma_{k, l}^{-}(t) c_{k}^{-}$by $w_{k}$, one has $N\left(w_{n} e^{i n x}, w_{m} e^{i m x}\right)=i e^{i k x}\left(w_{n}, m\right) w_{m}$ when $n+m=k$. Hence,

$$
\begin{equation*}
N\left(v_{l}, v_{l}\right)=\left(v_{0} \cdot \nabla\right) v_{l}+i \sum_{|k|^{2}=1}^{\rho_{l}} e^{i k x} \sum_{n+m=k}\left(w_{n}, m\right) w_{m}, \tag{68}
\end{equation*}
$$

where $\rho_{l}=\max |n+m|^{2}$ if $|n|^{2} \leqslant l$ and $|m|^{2} \leqslant l$,

$$
\begin{equation*}
\left(v_{0} \cdot \nabla\right) v_{l}=i \sum_{|k|^{2}=1}^{l}\left(\left(v_{0}, k\right) \gamma_{k, l}^{+}(t) c_{k}^{+}+\left(v_{0}, k\right) \gamma_{k, l}^{-}(t) c_{k}^{-}\right) e^{i k x} \tag{69}
\end{equation*}
$$

The vector $w_{m}$ for $k \neq m$ is decomposed in basis vectors $\overparen{k}, c_{k}^{+}, c_{k}^{-}$:

$$
\begin{equation*}
w_{m}=\left(w_{m}, \widehat{k}\right) \widehat{k}+\left(w_{m}, c_{k}^{+}\right) c_{k}^{+}+\left(w_{m}, c_{k}^{-}\right) c_{k}^{-} . \tag{70}
\end{equation*}
$$

Substituting the expressions (69), (70) into (68), and using the formula (67), one readily obtains the explicit form of Equations (65):

$$
\begin{gather*}
\frac{\partial \gamma_{k, l}^{+}}{\partial t}+\left(\nu|k|^{2}+i\left(v_{0}, k\right)\right) \gamma_{k, l}^{+}+  \tag{71}\\
+i \sum_{m^{2}=1}^{l}\left[\gamma_{k-m, l}^{+}\left(c_{k-m}^{+}, m\right)+\gamma_{k-m, l}^{-}\left(c_{k-m}^{-}, m\right)\right]\left[\gamma_{m, l}^{+}\left(c_{m}^{+}, c_{k}^{+}\right)+\gamma_{m, l}^{-}\left(c_{m}^{-}, c_{k}^{+}\right)\right]=\phi_{k}^{+}(t) \\
\frac{\partial \gamma_{k, l}^{-}}{\partial t}+\left(\nu|k|^{2}+i\left(v_{0}, k\right)\right) \gamma_{k, l}^{-}+  \tag{72}\\
+i \sum_{m^{2}=1}^{l}\left[\gamma_{k-m, l}^{+}\left(c_{k-m}^{+}, m\right)+\gamma_{k-m, l}^{-}\left(c_{k-m}^{-}, m\right)\right]\left[\gamma_{m, l}^{+}\left(c_{m}^{+}, c_{k}^{-}\right)+\gamma_{m, l}^{-}\left(c_{m}^{-}, c_{k}^{-}\right)\right]=\phi_{k}^{-}(t)
\end{gather*}
$$

$0<|k-m|^{2} \leqslant l$, with respect to unknown functions $\gamma_{k, l}^{+}$and $\gamma_{k, l}^{-}$, satisfying the initial conditions

$$
\begin{equation*}
\gamma_{k, l}^{ \pm}(0)=\psi_{k}^{ \pm}, 0<|k|^{2} \leqslant l \tag{73}
\end{equation*}
$$

Equations (63) are reduced to algebraic equations and according to (66), functions $p_{k, l}(t)$ are determined via $\gamma_{k, l}^{ \pm}$:

$$
\begin{gather*}
p_{k, l}(t)=-i|k|^{-1}\left(\phi_{k}(t)-\right.  \tag{74}\\
\left.\sum_{m^{2}=1}^{l}\left[\gamma_{k-m, l}^{+}\left(c_{k-m}^{+}, m\right)+\gamma_{k-m, l}^{-}\left(c_{k-m}^{-}, m\right)\right]\left[\gamma_{m, l}^{+}\left(c_{m}^{+}, \widehat{k}\right)+\gamma_{m, l}^{-}\left(c_{m}^{-}, \widehat{k}\right)\right]\right)
\end{gather*}
$$

The function $p_{0}(t)$ is not defined and not taken into account in Equations (1), because $\nabla p_{0}(t)=0$. In order to determine the pressure $p(x, t)$ uniquely, assume that $p_{0}(t)=$ $\varpi^{-2} \int_{Q} p(x, t) d x=0$, as usually.

In statement of Problem 2 in section 1.3, we made an assumption that $f(t, x) \in L_{2}\left(0, T ; V^{0}\right)$, $g \in V^{1}$. In this case $S_{0} f=f_{0}(t)=0$ and $S_{0} g=g_{0}=0$. Problem (64) has only a trivial solution $v_{0}(t) \equiv 0$, and Equations (71), (72) are reduced.

The system of equations (71), (72) with resect to unknown $\gamma_{k, l}^{+}(t)$ and $\gamma_{k, l}^{-}(t)$ is denoted by $R S_{l}$ (from "reduced system"). Note, that the system $R S_{1}$ is linear and is integrated elementarily, and the systems $R S_{l}$ are nonlinear when $l \geq 2$. They compose a complex Galerkin system of equations on the basis of eigenfunctions of the Stokes operator.
3.2. Expansion of the curl operator in terms of real eigenfunctions. Let us assume that the vector functions $f \in L_{2}\left(0, T ; V^{0}\right)$ and $g \in V^{1}$ are real and

$$
\begin{align*}
S_{l} f(t, x) & =2 \varpi R e \sum_{k \in M_{l}}\left(\phi_{k}^{+}(t) u_{k}^{+}+\phi_{k}^{-}(t) u_{k}^{-}\right),  \tag{75}\\
S_{l} g(x) & =2 \varpi R e \sum_{k \in M_{l}}\left(\psi_{k}^{+} u_{k}^{+}+\psi_{k}^{-} u_{k}^{-}\right) . \tag{76}
\end{align*}
$$

Then, the Galerkin approximations are obtained in the form

$$
\begin{equation*}
v_{l}=\operatorname{Re} v_{l}(x, t)=2 \varpi \operatorname{Re} \sum_{k \in M_{l}}\left(\gamma_{k, l}^{+}(t) u_{k}^{+}+\gamma_{k, l}^{-}(t) u_{k}^{-}\right) \tag{77}
\end{equation*}
$$

where the complex functions $\gamma_{k, l}^{ \pm}$and their conjugates satisfy the equations (71), (72), where $k \in M_{l}$ and $v_{0}(t)=0$. The set $M_{l}$ is an intersection of the set $M$ with a sphere of the radius
$\sqrt{l}$, it contains $N_{l} / 2$ points. The remaining $N_{l}$ equations are complex conjugate to the previous ones. This can be readily verified via the explicit form of Equations (71), (72), since due to section 3.10

$$
\begin{equation*}
c_{-k}^{ \pm}=-\overline{c_{k}^{ \pm}}, \quad \phi_{-k}^{ \pm}=-\overline{\phi_{k}^{ \pm}}, \quad k \in M \tag{78}
\end{equation*}
$$

Suppose that $\gamma_{k, l}^{ \pm}=\alpha_{k, l}^{ \pm}+i \beta_{k, l}^{ \pm}$and change to the real variables $\alpha, \beta$. Then, upon calculating the real and the imaginary part of complex equations, one obtains the system $2 N_{l}$ of real equations with $2 N_{l}$ real unknown variables. Denote it by $G S_{l}$. It is a real form of the Galerkin equations.

One the other hand, it can be easily verified that using an orthonormal basis of real eigenfunctions of the curl operator in the space $V^{0}$, one arrives at the same equations $G S_{l}$ :

$$
\begin{equation*}
\sqrt{2} \operatorname{Re} u_{k}^{ \pm}(x) \text { and } \sqrt{2} \operatorname{Im} u_{k}^{ \pm}(x), k \in M \tag{79}
\end{equation*}
$$

Let us write out analogues of Equations (61).

$$
\begin{align*}
& \frac{d}{d t}\left(v_{l}, \operatorname{Re} u_{k}^{ \pm}\right)+\nu\left(\nabla v_{l}, \nabla \operatorname{Re} u_{k}^{ \pm}\right)+b\left(v_{l}, v_{l}, \operatorname{Re} u_{k}^{ \pm}\right)=\left(f, \operatorname{Re} u_{k}^{ \pm}\right)  \tag{80}\\
& \frac{d}{d t}\left(v_{l}, \operatorname{Im} u_{k}^{ \pm}\right)+\nu\left(\nabla v_{l}, \nabla \operatorname{Im} u_{k}^{ \pm}\right)+b\left(v_{l}, v_{l}, \operatorname{Im} u_{k}^{ \pm}\right)=\left(f, \operatorname{Im} u_{k}^{ \pm}\right) \tag{81}
\end{align*}
$$

where $k \in M_{l}$. Note, that in this basis

$$
\begin{gather*}
v_{l}=2 \varpi \sum_{k \in M_{l}}\left(\alpha_{k, l}^{+} \operatorname{Re} u_{k}^{+}-\beta_{k, l}^{+} \operatorname{Im} u_{k}^{+}+\alpha_{k, l}^{-} \operatorname{Re} u_{k}^{-}-\beta_{k, l}^{-} \operatorname{Im} u_{k}^{-}\right),  \tag{82}\\
\left\|v_{l}(t, \cdot)\right\|_{V^{0}}^{2}=2 \sum_{k \in M_{l}}\left(\left(\alpha_{k, l}^{+}\right)^{2}+\left(\beta_{k, l}^{+}\right)^{2}+\left(\alpha_{k, l}^{-}\right)^{2}+\left(\beta_{k, l}^{-}\right)^{2}\right) . \tag{83}
\end{gather*}
$$

In practice, it is more convenient to work with the complex equations $R S_{l}$. If the functions $f$ and $g$ are real, one automatically obtains a real solution $v_{l}$ and according to Lemma 2 , it has the form (77).
3.3. Basic relations between $v_{l}(t, x), v_{l}(0, x)$ and $f(t, x)$ [5, 6, 7]. Multiplying Equations (80) by $2 \varpi \alpha_{k, l}^{ \pm}$, and Equations (81) by $-2 \varpi \beta_{k, l}^{ \pm}$, and adding them together, one obtains the basic relation

$$
\begin{equation*}
\left(\frac{d v_{l}}{d t}, v_{l}\right)+\nu\left\|\nabla v_{l}\right\|_{V^{0}}^{2}+b\left(v_{l}, v_{l}, v_{l}\right)=\left(f, v_{l}\right) \tag{84}
\end{equation*}
$$

where $b\left(v_{l}, v_{l}, v_{l}\right)=0$ because the vector $v_{l}$ is periodic and solenoidal. Indeed, omitting the index $l$ temporally, one obtains $b(v, v, v) \equiv$

$$
\sum_{i, j=1}^{3} \int_{Q} v_{i}\left(\partial_{i} v_{j}\right) v_{j} d x=\frac{1}{2} \sum_{i, j=1}^{3} \int_{Q} \partial_{i}\left(v_{i} v_{j}^{2}\right) d x-\frac{1}{2} \sum_{j=1}^{3} \int_{Q} v_{j}^{2}\left(\sum_{i=1}^{3} \partial_{i} v_{i}\right) d x=0
$$

Furthermore, invoking the formlae (82), (83), one has

$$
\left(\frac{d v_{l}}{d t}, v_{l}\right)=\frac{1}{2} \frac{d}{d t}\left\|v_{l}\right\|_{V^{0}}^{2} \quad \text { and } \quad\left\|\nabla v_{l}\right\|_{V^{0}}^{2}=\left\|v_{l}\right\|_{V^{1}}^{2}
$$

Therefore, the formula (84) takes the form

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t}\left\|v_{l}\right\|_{V^{0}}^{2}+\nu\left\|v_{l}\right\|_{V^{1}}^{2}=\left(f, v_{l}\right) \tag{85}
\end{equation*}
$$

3.4. Two a priori estimates. Multiply the equality (85) by 2 and note that the right-hand side of the resulting equations is bounded by

$$
\begin{equation*}
2\left|\left(f, v_{l}\right)\right| \leqslant 2\left\|v_{l}\right\|_{V^{1}}\|f\|_{V^{-1}} \leqslant \nu\left\|v_{l}\right\|_{V^{1}}^{2}+\nu^{-1}\|f\|_{V^{-1}}^{2} . \tag{86}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\frac{d}{d t}\left\|v_{l}\right\|_{V^{0}}^{2}+\nu\left\|v_{l}\right\|_{V^{1}}^{2} \leqslant \nu^{-1}\|f\|_{V^{-1}} \tag{87}
\end{equation*}
$$

Integrating (87) from 0 to $s, 0<s<T$, and invoking that $\left\|v_{l}(0, \cdot)\right\|_{V^{0}}^{2} \leqslant\|g\|_{V^{0}}^{2}$, one obtains

$$
\left\|v_{l}(s, \cdot)\right\|_{V^{0}}^{2} \leqslant\left\|v_{l}(0, \cdot)\right\|_{V^{0}}^{2}+\frac{1}{\nu} \int_{0}^{s}\|f(t, \cdot)\|_{V^{-1}}^{2} d t \leqslant\|g\|_{V^{0}}^{2}+\frac{1}{\nu} \int_{0}^{T}\|f(t, \cdot)\|_{V^{-1}}^{2} d t .
$$

Hence,

$$
\begin{equation*}
\sup _{s \in[0, T]}\left\|v_{l}(s, \cdot)\right\|_{V^{0}}^{2} \leqslant\|g\|_{V^{0}}^{2}+\nu^{-1}\|f\|_{L_{2}\left(0, T ; V^{-1}\right)}^{2} . \tag{88}
\end{equation*}
$$

The right-hand part is finite and independent of $l$. Therefore, the sequence of vector functions $v_{l}(t, x)$ is bounded in the space $L_{\infty}\left(0, T ; V^{0}\right)$.

Upon integration of (87) from 0 to $T$, one has

$$
\left\|v_{l}(T, \cdot)\right\|_{V^{0}}^{2}+\nu \int_{0}^{T}\left\|v_{l}(t, \cdot)\right\|_{V^{1}}^{2} d t \leqslant\left\|v_{l}(0, \cdot)\right\|_{V^{0}}^{2}+\frac{1}{\nu} \int_{0}^{T}\|f(t, \cdot)\|_{V^{-1}}^{2} d t
$$

Then,

$$
\begin{equation*}
\nu\left\|v_{l}\right\|_{L_{2}\left(0, T ; V^{1}\right)}^{2} \leqslant\|g\|_{V^{0}}^{2}+\nu^{-1}\|f\|_{L_{2}\left(0, T ; V^{-1}\right)}^{2} . \tag{89}
\end{equation*}
$$

Hence, the sequence of vector functions $v_{l}(t, x)$ is bounded in the space $L_{2}\left(0, T ; V^{1}\right)$ as well.
The nonlinear system $G S_{l}$ with boundary value conditions has a solution, defined in some maximal interval $\left[0, t_{l}\right]$.

If $t_{l}<T$, then the norm $\left\|v_{l}(t, \cdot)\right\|_{V^{0}}$ (see (83)) should tend to $+\infty$ when $t \rightarrow t_{l}$. However, the first a priori estimate (88) demonstrates that this is impossible and therefore, $t_{l}=T$.

An assumption was made that $f(t, x) \in L_{2}\left(0, T ; V^{0}\right), g(x) \in V^{1}$, and solution to Problem 2 is sought in the space

$$
\begin{equation*}
W^{1,2(0)}=\left\{v(t, x) \in L_{2}\left(0, T ; V^{2}\right): \partial_{t} v(t, x) \in L_{2}\left(0, T ; V^{0}\right)\right\} . \tag{90}
\end{equation*}
$$

Therefore, the right-hand parts in the system $G S_{l}$, belong, generally speaking, only to the space $L_{2}(0, T)$, and the derivatives are interpreted as generalized ones.

An important part of justification of the Faedo-Galerkin method is the existence proof of a converging sequence of the sequence $v_{l}$ in some space. In the monographs [5, 6, 7], devoted to boundary value initial problems for the Navier-Stokes equations in domains of various dimensions, this fact follows compact injection of definite Hilbert spaces. Some statements proved there are extended the Cauchy problem (1), (2) with periodic boundary value conditions. This matter is to be studied in a separate work.
3.5. Orthogonal projection method. Let $\Pi_{0}$ be an orthoprojector of the space $L_{2}(\mathbb{T})^{3}$ on $V^{0}$ (see section 3.7). Applying it to both parts of the first equation (1), we dispense with the vector $\nabla p$. This leads to an operator equation in vector functions of $t$ with values in the space $V^{0}$ :

$$
\begin{equation*}
\partial_{t} v(t, \cdot)+\nu A v+B(v, v)=\Pi_{0} f \quad \text { with the condition }\left.\quad \gamma_{0} v \equiv v\right|_{t=0}=g, \tag{91}
\end{equation*}
$$

where

$$
\begin{equation*}
A=-\Pi_{0} \Delta, \quad B(v, w)=\Pi_{0}\left(\sum_{j=1}^{3} v_{j} \partial_{j} w\right) \tag{92}
\end{equation*}
$$

Projections of the equation to orthogonal subspaces $R_{k}^{+}, R_{k}^{-}$coincide with Equations (71), (72), where $l=\infty$ and $v_{0}(t)=0$.

Problem (91) corresponds to the operator

$$
\begin{equation*}
\left(\partial_{t}+\nu A+B, \gamma_{0}\right): W^{1,2(s)} \rightarrow L_{2}\left(0, T ; V^{s}\right) \times V^{1+s} \tag{93}
\end{equation*}
$$

whose invertibility indicates that the problem is solvable.

## 4. Solution of the Cauchy problem for the Stokes system

4.1. The problem is as follows. Given $f$ and $g$. Find a vector function $(v, p), 2 \pi$-periodic in $x_{j}$, and satisfying the conditions

$$
\begin{equation*}
\frac{\partial v}{\partial t}-\nu \Delta v=-\nabla p+f, \quad \text { div } v=0, v(0, x)=g(x) \tag{94}
\end{equation*}
$$

The classical and generalized statement of the problem are the same as for the nonlinear system in §1. Operator (93), with $B=0$, corresponds to the problem.

The Cauchy problem for the Galerkin system

$$
\begin{equation*}
\partial_{t} \gamma_{k}^{ \pm}+\nu|k|^{2} \gamma_{k}^{ \pm}=\phi_{k}^{ \pm}(t), \quad \gamma_{k}^{ \pm}(0)=\psi_{k}^{ \pm} \tag{95}
\end{equation*}
$$

splits into separate problems, that can be solved trivially.
When $k \neq 0$ is fixed,

$$
\begin{equation*}
\gamma_{k}^{ \pm}(t)=\psi_{k}^{ \pm} e^{-\nu|k|^{2} t}+\rho_{k}^{ \pm}(t), \quad \text { where } \quad \rho_{k}^{ \pm}(t)=\int_{0}^{t} e^{\nu|k|^{2}(\tau-t)} \phi_{k}^{ \pm}(\tau) d \tau \tag{96}
\end{equation*}
$$

The function $\gamma_{k}^{ \pm}(t) \in C^{1}[0, T]$, if $\phi_{k}^{ \pm}(t) \in C[0, T]$.
If $\phi_{k}^{ \pm}(t) \in L_{2}[0, T]$, then $\gamma_{k}^{ \pm}(t) \in C[0, T]$ and has generalized derivatives from $L_{2}[0, T]$.
First, assume that $f=f_{k}, g=g_{k}$, where

$$
\begin{equation*}
f_{k}=\phi_{k}^{+}(t) u_{k}^{+}(x)+\phi_{k}^{-}(t) u_{k}^{-}(x), \quad g_{k}=\psi_{k}^{+} u_{k}^{+}+\psi_{k}^{-} u_{k}^{-} \tag{97}
\end{equation*}
$$

$u_{k}^{ \pm}(x)=\varpi^{-1} c_{k}^{ \pm} e^{i k x}$ are eigenfunctions of the curl operator and $\phi_{k}^{ \pm}(t) \in C[0, T]$. Then, the vector function ( $v_{k}, p_{k}$ ), where

$$
\begin{equation*}
v_{k}=\gamma_{k}^{+}(t) u_{k}^{+}(x)+\gamma_{k}^{-}(t) u_{k}^{-}(x), \quad \nabla p_{k}=0 \tag{98}
\end{equation*}
$$

is a classical solution of Problem (94).
If $f$ and $g$ are real and

$$
\begin{equation*}
f=\operatorname{Re}\left(\phi_{k}^{+}(t) u_{k}^{+}(x)+\phi_{k}^{-}(t) u_{k}^{-}(x)\right), g=\operatorname{Re}\left(\psi_{k}^{+} u_{k}^{+}+\psi_{k}^{-} u_{k}^{-}\right) \tag{99}
\end{equation*}
$$

then $\left(R e v_{k}, R e p_{k}\right)$ is the real solutions of the Problem (94).
Definition. Vector functions $\left(v_{k}, p_{k}\right)$, $\left(\operatorname{Re} v_{k}, \operatorname{Re} p_{k}\right)$, and $\left(\operatorname{Im} v_{k}, \operatorname{Im} p_{k}\right)$ are called basis solutions of the linear problem (94).

Finite sums of basis solutions are also solutions to the problem.
4.2. General case. Let $f \in L_{2}\left(0, T ; V^{s}\right), g \in V^{1+s}, s \geq-1$. Then $f$ and $g$ are represented by the series

$$
\begin{gather*}
f(x, t)=\sum_{|k|^{2}=1}^{\infty}\left(\phi_{k}^{+}(t) u_{k}^{+}(x)+\phi_{k}^{-}(t) u_{k}^{-}(x)\right),  \tag{100}\\
g(x)=\sum_{|k|^{2}=1}^{\infty}\left(\psi_{k}^{+} u_{k}^{+}(x)+\psi_{k}^{-} u_{k}^{-}(x)\right) . \tag{101}
\end{gather*}
$$

Let

$$
\begin{equation*}
v=\sum_{|k|^{2}=1}^{\infty}\left(\psi_{k}^{+} e^{-\nu|k|^{2} t}+\rho_{k}^{+}(t)\right) u_{k}^{+}(x)+\left(\psi_{k}^{-} e^{-\nu|k|^{2} t}+\rho_{k}^{-}(t)\right) u_{k}^{-}(x), \tag{102}
\end{equation*}
$$

be a formal series, $\nabla p=0$. If partial sums $S_{l} v$ of the series (102) converge in the space $W^{1,2(s)}$, then the vector function $(v, p)$ is a solution to Problem (94). The nonhomogeneous problem will be studied in a separate work.

Let us dwell upon a homogeneous problem.
Theorem 1. Let $f=0, g \in V^{1+s} \subset \widehat{V}^{1+s}, s \geq-1$,

$$
\begin{equation*}
v_{g}(t, x)=\varpi^{-1} \sum_{n=1}^{\infty} e^{-\nu n t} \sum_{|k|^{2}=n}\left(\psi_{k}^{+} c_{k}^{+}+\psi_{k}^{-} c_{k}^{-}\right) e^{i k x}, \quad p_{g}(t, x)=0 \tag{103}
\end{equation*}
$$

Then the vector function $\left(v_{g}, p_{g}\right)$ is a unique solution to the homogeneous problem (94). Meanwhile, if $t>0$, then $\left\|\partial_{t}^{m} v_{g}\right\|_{V^{q}} \leqslant M\|g\|_{V^{0}}$ for any $q$ and $m \geq 0$. Partial sums $S_{l} v_{g}$ and $S_{l} \partial_{t}^{m} v_{g}$ of the series $v_{g}$ and its derivative with respect to $t$ of the order $m$ converge when $l \rightarrow \infty$ in a norm of the Sobolev space $H_{q}$. If $t \geq 0$, then $S_{l} v_{g}$ converge when $l \rightarrow \infty$ in spaces $W^{1,2(s)}$ and $L_{\infty}\left(0, T ; V^{s+1}\right)$ for any $T>0$. Moreover, if $t \rightarrow 0$, the norm of the difference $\left\|v_{g}(t, \cdot)-g\right\|_{V^{s+1}} \rightarrow 0$, and if $t \rightarrow+\infty$, the norm $\left\|v_{g}(t, \cdot)\right\|_{V^{s+1}} \rightarrow 0$.

The proof of the theorem is based on following estimates of the row (103) and its formal derivatives. Let $t>0$. For any $q \geq 0$ and an integer $m \geq 0$, one has

$$
\begin{equation*}
\left\|\partial_{t}^{m} v_{g}\right\|_{V^{q}}^{2}=\nu^{2 m} \sum_{n=1}^{\infty} n^{2 m+q} e^{-2 \nu n t} \sum_{|k|^{2}=n}\left(\left|\psi_{k}^{+}\right|^{2}+\left|\psi_{k}^{-}\right|^{2}\right) . \tag{104}
\end{equation*}
$$

Introduce the notation

$$
\begin{equation*}
M^{2}(\nu, m, q, t)=\nu^{2 m} \max _{n \in N}\left(n^{2 m+q} e^{-2 \nu n t}\right) . \tag{105}
\end{equation*}
$$

If $t>0$, the constant $M<\infty$ for any $\nu>0, m \geq 0, q \geq 0$. Therefore,

$$
\begin{equation*}
\left\|\partial_{t}^{m} v_{g}\right\|_{V^{q}} \leqslant M\|g\|_{V^{0}} \tag{106}
\end{equation*}
$$

Let $m=1, q=s$ in (104). Integrating this series term by term and invoking that

$$
\int_{0}^{T} e^{-2 \nu n t} d t<\int_{0}^{\infty} e^{-2 \nu n t} d t=(2 \nu n)^{-1}
$$

we obtain

$$
\begin{equation*}
\left\|\partial_{t} v_{g}\right\|_{L_{2}\left(0, T ; V^{s}\right)}^{2}=\int_{0}^{T}\left\|\partial_{t} v_{g}\right\|_{V^{s}}^{2} d t<\frac{1}{2} \nu\|g\|_{V^{s+1}}^{2} \tag{107}
\end{equation*}
$$

Since $-\Delta u_{k}^{ \pm}(x)=|k|^{2} u_{k}^{ \pm}(x)$, then

$$
\begin{equation*}
\left\|\Delta v_{g}\right\|_{L_{2}\left(0, T ; V^{s}\right)}^{2}=\int_{0}^{T}\left\|\Delta v_{g}\right\|_{V^{s}}^{2} d t<\frac{1}{2 \nu}\|g\|_{V^{s+1}}^{2} \tag{108}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\left\|v_{g}\right\|_{W^{1,2(s)}}^{2}=\left\|v_{g}\right\|_{L_{2}\left(0, T ; V^{s+2}\right)}^{2}+\left\|\partial_{t} v_{g}\right\|_{L_{2}\left(0, T ; V^{s}\right)}^{2}<\frac{1}{2}\left(\nu+\nu^{-1}\right)\|g\|_{V^{s+1}}^{2} . \tag{109}
\end{equation*}
$$

The exponent $e^{-2 \nu n t} \leqslant 1$ when $t \geq 0, \nu>0, n \geq 1$ in (104). Therefore,

$$
\begin{equation*}
\left\|v_{g}\right\|_{L_{\infty}\left(0, T ; V^{s+1}\right)}=\text { ess }_{\sup }^{t \in[0, T]},\left\|v_{g}(t, \cdot)\right\|_{V^{s+1}} \leqslant\|g\|_{V^{s+1}} . \tag{110}
\end{equation*}
$$

Finally, the inequality

$$
\begin{equation*}
\left\|v_{g}\right\|_{V^{s+1}} \leqslant e^{-\mu t}\|g\|_{V^{s+1}}, \quad 0<\mu<\nu \tag{111}
\end{equation*}
$$

is provided by (104), since $0<e^{-2(\nu n-\mu) t}<1$ when $\nu>\mu>0, n \geq 1, t>0$.
The resulting inequalities yield estimated deviations $v_{g}-S_{l} v_{g}$ and their derivatives via deviations $g-S_{l} g$. Thus, when $t>0$, one has

$$
\begin{equation*}
\left\|\partial_{t}^{m} v_{g}-S_{l} \partial_{t}^{m} v_{g}\right\|_{V^{q}} \leqslant M\left\|g-S_{l} g\right\|_{V^{0}} \text { for any } q, m \geq 0 \text { и } l \geq 1 \tag{112}
\end{equation*}
$$

according to (106). By virtue of (109),(110), if $t \geq 0$,

$$
\begin{gather*}
\left\|v_{g}-S_{l} v_{g}\right\|_{W^{1,2(s)}}^{2}<\frac{1}{2}\left(\nu+\nu^{-1}\right)\left\|g-S_{l} g\right\|_{V^{s+1}}^{2},  \tag{113}\\
\left\|v_{g}-S_{l} v_{g}\right\|_{L_{\infty}\left(0, T ; V^{s+1}\right)}<\left\|g-S_{l} g\right\|_{V^{s+1}} . \tag{114}
\end{gather*}
$$

The space $V^{q}=H_{q} \cap V^{0}$ by definition, $q>0$. By condition of the theorem $g \in V^{1+s}, 1+s \geq 0$ and $V^{1+s} \subset V^{0}$. Hence, the sequence $S_{l} v_{g}$ converges to $v_{g}$ when $l \rightarrow \infty$ in spaces $W^{1,2(s)}$ and $L_{\infty}\left(0, T ; V^{s+1}\right)$ for any $T>0$. According to (112), if $t>0$, the sequence $S_{l} v_{g}$ of partial sums of the series $v_{g}$ (as well as the sequences $S_{l} \partial_{t}^{m} v_{g}$ of its derivatives with respect to $t$ ) converge when $l \rightarrow \infty$ to $v_{g}$ (and to $\partial_{t}^{m} v_{g}$ ) in the norm of the Sobolev space $H_{q}$. If $q \geq 2$, the Sobolev spaces $H_{q}(Q)$ are embedded in the Hilbert spaces $C^{q-1,5}(\bar{Q})$.

Hence, the series (103) has continuous derivatives with respect to $x_{j}$ and $t$ of any order and the superposition principles holds when $t>0$. According to Section 5.1, the series $v_{g}$ satisfies the equations (94), where $p=0$ and $f=0$. Further,

$$
\begin{equation*}
\left\|g-v_{g}(t, \cdot)\right\|_{V^{s+1}}^{2}=\sum_{n=1}^{\infty} n^{s+1}\left(1-e^{-\nu n t}\right)^{2} \sum_{|k|^{2}=n}\left(\left|\psi_{k}^{+}\right|^{2}+\left|\psi_{k}^{-}\right|^{2}\right), \tag{115}
\end{equation*}
$$

therefore $\left\|g-v_{g}(t, \cdot)\right\|_{V^{s+1}}^{2} \rightarrow 0$ when $t \rightarrow 0$. Finally, the estimate (111) entails that $\left\|v_{g}(t, \cdot)\right\|_{V^{s+1}} \rightarrow 0$ when $t \rightarrow+\infty$.

Single valued solvability of the problem follows from the uniqueness of expanding $g$ into a Fourier series. If $g=0$, then $g_{0}=0$ and $\psi_{k}^{ \pm}=0$ for all $k \neq 0$ and hence, $v_{g}=0$. The theorem is proved.
4.3. Theorem 1 holds if $t \in R_{+}=(0,+\infty)$. The formulae (104)-(109) reveal interesting inequalities:

$$
\begin{gather*}
\left\|v_{g}\right\|_{L_{2}\left(R_{+} ; V^{s+2}\right)}^{2}=(2 \nu)^{-1}\|g\|_{V^{s+1}}^{2}  \tag{116}\\
\left\|\partial_{t} v_{g}\right\|_{L_{2}\left(R_{+} ; V^{s}\right)}^{2}=2^{-1} \nu\|g\|_{V^{s+1}}^{2} \tag{117}
\end{gather*}
$$

Hence,

$$
\begin{equation*}
\left\|v_{g}\right\|_{W^{1,2(s)}}^{2}=2^{-1}\left(\nu+\nu^{-1}\right)\|g\|_{V^{s+1}}^{2} \tag{118}
\end{equation*}
$$

If $\nu=1$

$$
\begin{equation*}
\left\|v_{g}\right\|_{W^{1,2(s)}}=\|g\|_{V^{s+1}} . \tag{119}
\end{equation*}
$$

Recall that isomorphism of Euclidian spaces is a one-to one correspondence that preserves linear operations defined in the spaces as well as a scalar product.

If $s \geq-1, \nu=1$, the following theorem is true.
Theorem 2 a. A linear operator $g \mapsto v_{g}$, defined by the series (103), realizes its isomorphism of the spaces $V^{s+1}$ and $W^{1,2(s)} \cap \operatorname{Ker}\left(\partial_{t}+A\right)$.

Indeed, every element $g$ of $V^{s+1}$ with $g_{0}=0$ corresponds to a unique element $v_{g}$ of $W^{1,2(s)}$ such that $\left(\partial_{t}+A\right) v_{g}=0, \gamma_{0} v_{g}=g$. Conversely, the series $v$ determines the series $g$, equal to $\left.v\right|_{t=0}$, and according to (119) the lengths of these vector functions coincide. Thus, $g \leftrightarrow v$. Let $h, w$ be another pair such that $h \leftrightarrow w, h_{0}=0$ and $\phi_{k}^{ \pm}$are Fourier coefficients $h$. The relations $\alpha g \leftrightarrow \alpha v$ and $g+h \leftrightarrow v+w$ follow from linearity of the operator. The scalar product of vector functions $g$ and $h$ in $V^{s+1}$ has the form

$$
\begin{equation*}
(g, h)_{V^{s+1}}=\sum_{n=1}^{\infty} n^{s+1} \sum_{|k|^{2}=n}\left(\left(\psi_{k}^{+}, \phi_{k}^{+}\right)+\left(\psi_{k}^{-}, \phi_{k}^{-}\right)\right) \tag{120}
\end{equation*}
$$

Whence, one can readily see that if $\nu=1$

$$
\begin{equation*}
(v, w)_{W^{1,2(s)}}=(g, h)_{V^{s+1}} . \tag{121}
\end{equation*}
$$

The theorem is proved. It can be formulated differently.
Theorem 2. A linear operator $\left(\partial_{t}+A, \gamma_{0}\right) v \rightarrow\left(0, \gamma_{0} v\right)$ realizes the isomorphism of spaces $W^{1,2(s)} \cap \operatorname{Ker}\left(\partial_{t}+A\right)$ and $V^{s+1}$.

Using the theorem and following the work [9], one can prove local solvability of a nonlinear problem in the class $W^{1,2(0)}$ with the initial conditions from an unbounded ellipsoid $E l_{\rho}^{1 / 2}=$ $\left\{g \in V^{1}:\|g\|_{V^{1 / 2}}<\rho\right\}$ when $\rho$ is sufficiently small.

Boundary value problems for equations rot $u+\lambda u=h$ with $\lambda \neq 0$, Stokes and Sobolev equations (in a stationary case), their Fredholm solvability in a domain with a smooth boundary have been studied by me earlier in [29, 30, 31].

In what follows, we provide families of explicit solutions to the nonlinear problem that are used in testing a program of numerical solution.

## 5. Explicit global solutions to the nonlinear problem

5.1. Basis solutions. Basis solutions are solutions $\left(v_{k}, p_{k}\right)$ to the nonlinear problem with the data $f_{k}, g_{k}$, that correspond to eigenfunctions of the curls operator $u_{k}^{ \pm}(x)=\varpi^{-1} c_{k}^{ \pm} e^{i k x}$ with eigenfunctions $\pm|k|$ for any $k \neq 0$ :

$$
\begin{gather*}
f_{k}=\phi_{k}^{+}(t) u_{k}^{+}(x)+\phi_{k}^{-}(t) u_{k}^{-}(x), \quad g_{k}=\psi_{k}^{+} u_{k}^{+}+\psi_{k}^{-} u_{k}^{-},  \tag{122}\\
v_{k}=\gamma_{k}^{+}(t) u_{k}^{+}(x)+\gamma_{k}^{-}(t) u_{k}^{-}(x), \quad \nabla p_{k}=0, \tag{123}
\end{gather*}
$$

where

$$
\begin{equation*}
\gamma_{k}^{ \pm}(t)=\psi_{k}^{ \pm} e^{-\nu|k|^{2} t}+\int_{0}^{t} e^{\nu|k|^{2}(\tau-t)} \phi_{k}^{ \pm}(\tau) d \tau, \quad \text { and } \phi_{k}^{ \pm}(t) \in C[0, T] \tag{124}
\end{equation*}
$$

for example. Let us fix the vector $k \neq 0$.
Theorem 3. Any basis solution ( $v_{k}, p_{k}$ ) (corr., (Re $v_{k}$, Re $p_{k}$ )) to the linear problem (94) with the data (122) (corr., (99)) is a classical solution to the nonlinear problem (1), (2) with the same data.

Proof. The pair $\left(v_{k}, p_{k}\right)$ is a solution to the linear problem $\nabla p_{k}=0$. It remains to demonstrate that $N\left(v_{k}, v_{k}\right)=0$. Obviously, $\partial_{j} v_{k}=i k_{j} v_{k}$ and $\left(v_{k}, k\right)=0$, since $\left(c_{k}^{ \pm}, k\right)=0$. Therefore,

$$
\begin{equation*}
N\left(v_{k}, v_{k}\right) \equiv \sum_{j=1}^{3} v_{k, j} \partial_{j} v_{k}=i\left(v_{k}, k\right) v_{k}=0 \tag{125}
\end{equation*}
$$

Let $\left(w_{k}, q_{k}\right)=\left(\operatorname{Re} v_{k}, \operatorname{Re} p_{k}\right)$. Since $\partial_{j} w_{k}=k_{j} \operatorname{Re}\left(i v_{k}\right)$ and $\left(w_{k}, k\right)=0$, then $N\left(w_{k}, w_{k}\right)=$ $\left(w_{k}, k\right) R e\left(i v_{k}\right)=0$.

Then, the pair $\left(\operatorname{Re} v_{k}, \operatorname{Re} p_{k}\right)$ is also a solution to the nonlinear problem.
Let us single out an important particular case. Let
$\phi_{k}^{ \pm}(t)=\beta_{k}^{ \pm} e^{-\sigma_{k}^{ \pm} t}, \psi_{k}^{ \pm}=0$, then

$$
\gamma_{k}^{ \pm}(t)=\left\{\begin{array}{c}
\beta_{k}^{ \pm} t e^{-\nu|k|^{2} t} \quad \text { when } \sigma_{k}^{ \pm}=\nu|k|^{2},  \tag{126}\\
\frac{\beta_{k}^{ \pm}}{\nu|k|^{2}-\sigma_{k}^{ \pm}}\left(e^{-\sigma_{k}^{ \pm} t}-e^{-\nu|k|^{2} t}\right) \quad \text { when } \sigma_{k}^{ \pm} \neq \nu|k|^{2} .
\end{array}\right.
$$

This formula shows that if $t \rightarrow+\infty$, the velocity modulus $\left|v_{k}^{ \pm}(x, t)\right|$ tends to zero, if $R e \sigma_{k}^{ \pm}>$ 0 ,

$$
\left|v_{k}^{ \pm}(x, t)\right| \rightarrow \frac{\left|\beta_{k}^{ \pm}\right|}{\left|\nu k^{2}-\sigma_{k}^{ \pm}\right|}, \text {if } \quad \operatorname{Re} \sigma_{k}^{ \pm}=0
$$

and $\left|v_{k}^{ \pm}(x, t)\right| \rightarrow+\infty$ exponentially, if $R e \sigma_{k}^{ \pm}<0$.
The resonance occurs when $\sigma_{k}^{ \pm}=\nu|k|^{2}$.
5.2. Let $\Lambda_{k}$ be a ray, given by the vector $k$. The points $k$ and $\lambda k \in \Lambda_{k}$, if $\lambda$ is a natural number. In this case, basis solutions can be summed up. For example,

Theorem 4. Let $f=0, g=g_{(k)} \in V^{1+s}, s \geq-1$, where

$$
\begin{equation*}
g_{(k)}=\sum_{\lambda=1}^{\infty}\left(\psi_{\lambda k}^{+} u_{\lambda k}^{+}(x)+\psi_{\lambda k}^{-} u_{\lambda k}^{-}(x)\right) . \tag{127}
\end{equation*}
$$

Suppose that

$$
\begin{equation*}
v_{(k)}=\varpi^{-1} \sum_{\lambda=1}^{\infty}\left(\psi_{\lambda k}^{+} c_{\lambda k}^{+}+\psi_{\lambda k}^{-} c_{\lambda k}^{-}\right) e^{i \lambda k x-\nu \lambda^{2}|k|^{2} t}, \quad p=0 \tag{128}
\end{equation*}
$$

Then the pair $\left(v_{(k)}, 0\right)$, is a solution to the problem (1), (2) with $f=0, g=g_{(k)}$.
Proof. The data of the problem satisfy the conditions of Theorem 1. Therefore, the pair $\left(v_{(k)}, 0\right)$ is a solution to the linear problem. It remans to demonstrate that $N\left(v_{(k)}, v_{(k)}\right)=0$. Since $c_{\lambda k}^{ \pm}=c_{k}^{ \pm}$for $\lambda \in \mathbb{N}$, then

$$
\begin{equation*}
v_{(k)}=\alpha_{k}^{+} c_{k}^{+}+\alpha_{k}^{-} c_{k}^{-}, \quad \text { where } \quad \alpha_{k}^{ \pm}(t, x)=\sum_{\lambda=1}^{\infty} \psi_{\lambda k}^{ \pm} e^{i \lambda k x-\nu \lambda^{2}|k|^{2} t} \tag{129}
\end{equation*}
$$

and the series $\alpha_{k}^{ \pm}(t, x)$ and their derivatives with respect to $x_{j}$ converge when $t>0, g \in V^{0}$.
One has

$$
\begin{align*}
& \nabla \alpha_{k}^{ \pm}=i k \alpha_{k, 1}^{ \pm}, \quad \text { where } \quad \alpha_{k, 1}^{ \pm}(t, x)=\sum_{\lambda=1}^{\infty} \lambda \psi_{\lambda k}^{ \pm} e^{i \lambda k x-\left.\nu \lambda^{2}|k|\right|^{2} t}  \tag{130}\\
& N\left(v_{(k)}, v_{(k)}\right)=\alpha_{k}^{+} N\left(c_{k}^{+}, \alpha_{k}^{+}\right) c_{k}^{+}+\ldots+\alpha_{k}^{-} N\left(c_{k}^{-}, \alpha_{k}^{-}\right) c_{k}^{-}=0 \tag{131}
\end{align*}
$$

since every addend of the sum equals to zero. Indeed, invoking that $\left(c_{k}^{ \pm}, k\right)=0$, one obtains for the first one

$$
\begin{equation*}
N\left(\alpha_{k}^{+} c_{k}^{+}, \alpha_{k}^{+} c_{k}^{+}\right)=\alpha_{k}^{+}\left(c_{k}^{+} \cdot \nabla\right) \alpha_{k}^{+} c_{k}^{+}=i \alpha_{k}^{+}\left(c_{k}^{+}, k\right) \alpha_{k, 1}^{+} c_{k}^{+}=0, \tag{132}
\end{equation*}
$$

which was to be proved.
5.3. Let $q$ be a plane given by vectors $m$ and $n$. Some solutions to the linear problem ( $v_{k}, p_{k}$ ) when $k \in q$, can be summed up. For example,

Theorem 5. Let $f=0, g=g_{q} \in V^{1+s}, s \geq-1$, where

$$
\begin{equation*}
g_{q}=\sum_{k \in q} \psi_{k} e^{i k x} m \times n . \tag{133}
\end{equation*}
$$

Assume that

$$
\begin{equation*}
v_{q}=\sum_{k \in q} \psi_{k} e^{i k x-\nu|k|^{2} t} m \times n, p=0 . \tag{134}
\end{equation*}
$$

Then, the pair $\left(v_{q}, 0\right)$ is a solution to the problem (1), (2) with $f=0, g=g_{q}$.
Proof. Let us expand the vector $m \times n$ in terms of the basis $\widehat{k}, c_{k}^{+}, c_{k}^{-}$and take into account that any vector $k \in q$ is orthogonal to the vector product $m \times n$. Then,

$$
\begin{equation*}
m \times n=\left(c_{k}^{+}, m \times n\right) c_{k}^{+}+\left(c_{k}^{-}, m \times n\right) c_{k}^{-} . \tag{135}
\end{equation*}
$$

Substituting this expression to the formulae (133), (134), one obtains the expansion $g_{q}$ and $v_{q}$ in terms of eigenfunctions of the curls operator.

The series (133) satisfies conditions of Theorem 1. It remans to demonstrate that $N\left(v_{q}, v_{q}\right)=$ 0 . This can be achieved similarly to the previous theorem:

$$
\begin{gather*}
v_{q}=\alpha m \times n, \text { where } \alpha(t, x)=\sum_{k \in q} \psi_{k} e^{i k x-\nu|k|^{2} t} .  \tag{136}\\
N\left(v_{q}, v_{q}\right)=i \alpha(t, x) \sum_{k \in q}(m \times n, k) \psi_{k} e^{i k x-\nu|k|^{2} t} m \times n=0 . \tag{137}
\end{gather*}
$$

5.4. Let $\sigma$ be a sphere of the radius $\sqrt{n}$ and $n=\left|k_{0}\right|^{2}$. The sum of solutions of the linear problem $\left(v_{k}, p_{k}\right)$ for $k \in \sigma$ is not a solution to the nonlinear problem. However, the following theorem holds for eigenfunctions with the same eigenvalues.

Theorem 6. Let $\phi_{k}^{+}(t) \in C[0, T]$,

$$
\begin{equation*}
f_{\sigma}^{+}=\sum_{|k|^{2}=n} \phi_{k}^{+}(t) u_{k}^{+}(x), \quad g_{\sigma}^{+}=\sum_{|k|^{2}=n} \psi_{k}^{+} u_{k}^{+} . \tag{138}
\end{equation*}
$$

Assume that

$$
\begin{gather*}
\gamma_{k}^{+}(t)=\psi_{k}^{+} e^{-\nu|k|^{2} t}+\int_{0}^{t} e^{\nu|k|^{2}(\tau-t)} \phi_{k}^{+}(\tau) d \tau,  \tag{139}\\
v_{\sigma}^{+}=\sum_{|k|^{2}=n} \gamma_{k}^{+}(t) u_{k}^{+}(x), \quad p_{\sigma}^{+}=\sum_{|k|^{2}=n} p_{k}^{+}-\frac{1}{2}\left(v_{\sigma}^{+}\right)^{2}, \quad \nabla p_{k}^{+}=0 . \tag{140}
\end{gather*}
$$

Then, the pair $\left(v_{\sigma}^{+}, p_{\sigma}^{+}\right)$is a classical solution to the nonlinear problem (1), (2) with $f=f_{\sigma}^{+}, g=g_{\sigma}^{+}$.

Proof. By construction, one has

$$
\begin{equation*}
S v_{\sigma}^{+}=f_{\sigma}^{+}-\nabla \sum_{|k|^{2}=n} p_{k}^{+} \tag{141}
\end{equation*}
$$

for the linear operator $S$. While calculating the nonlinear operator $N$ from $v_{\sigma}^{+}$, let us use the correlation $N(v, v)=($ rot $v) \times v+\nabla \frac{1}{2} v^{2}$ and take into account that rot $v_{\sigma}^{+}=\sqrt{n} v_{\sigma}^{+}$. Then,

$$
\begin{equation*}
N\left(v_{\sigma}^{+}, v_{\sigma}^{+}\right)=\left(\operatorname{rot} v_{\sigma}^{+}\right) \times v_{\sigma}^{+}+\nabla \frac{1}{2}\left(v_{\sigma}^{+}\right)=\nabla \frac{1}{2}\left(v_{\sigma}^{+}\right) . \tag{142}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
S v_{\sigma}^{+}+N\left(v_{\sigma}^{+}, v_{\sigma}^{+}\right)=f_{\sigma}^{+}-\nabla \sum_{|k|^{2}=n} p_{k}^{+}+\nabla \frac{1}{2}\left(v_{\sigma}^{+}\right)=f_{\sigma}^{+}-\nabla p_{\sigma}^{+} \tag{143}
\end{equation*}
$$

which was to be proved.
Note that a similar theorem holds for eigenvalues with negative eigenvalues $-\sqrt{n}$. Generally speaking, it is not true for eigenfunctions with various eigenvalues $\pm \sqrt{n}$.

Families of global solutions to the Cauchy problem for the nonlinear Navier-Stokes system in a uniformly rotating space with the angular velocity $\Omega$ are published in my work [25]. Assuming that $\Omega=0$, one can easily write out other families of exact global solutions to the nonlinear problem.

## 6. Numerical solution of model problems

A model problem is the Cauchy problem for the Galerkin system $R S_{l}$.
6.1. Standard form of the Galerkin equations. A program for enumerating nonzero points of the lattice is developed

$$
k \rightarrow \vartheta(k): 1(-1,0,0), 2(0,-1,0), 3(0,0,-1), \ldots, 18(1,1,0), \ldots,
$$

where the point $(-1,0,0)$ is the first one, the point $(1,1,0)$ is the eighteenth one and so on. Using this numbering, let us introduce the numbering $\vartheta(k)$ for the known and unknown functions $\gamma_{k}^{ \pm}(t)$. Thus, e.g., $\gamma_{1(-1,0,0)}^{+}$indicates that the element $\gamma_{(-1,0,0)}^{+}$is the first one in the numbering system and so on. If $|k|^{2} \leqslant l$, the last element has the number $N_{l}$, equal to the number of nonzero points of the lattice in the sphere with the radius $\sqrt{l}$.

Let us introduce a vector-row into our consideration. Let

$$
\begin{equation*}
\gamma=\left(\gamma^{+}, \gamma^{-}\right)=\left(\gamma_{1}^{+}, \ldots, \gamma_{N_{l}}^{+}, \gamma_{1}^{-}, \ldots, \gamma_{N_{l}}^{-}\right) \tag{144}
\end{equation*}
$$

and $B_{\vartheta(k)}^{+}, B_{\vartheta(k)}^{-}$be matrices of quadratic forms. In this notation $2 N_{l}$ of complex equations $R S_{l}$ and the initial conditions have the form

$$
\begin{equation*}
\frac{d}{d t} \gamma_{\vartheta(k)}^{ \pm}=-\nu|k|^{2} \gamma_{\vartheta(k)}^{ \pm}-\gamma B_{\vartheta(k)}^{ \pm} \gamma^{T}+\phi_{\vartheta(k)}^{ \pm}, \gamma_{\vartheta(k)}^{ \pm}(0)=\psi_{\vartheta(k)}^{ \pm} \tag{145}
\end{equation*}
$$

where $\vartheta(k)=1, \ldots, N_{l}$. Quadratic forms

$$
\begin{array}{r}
\gamma B_{\vartheta(k)}^{ \pm} \gamma^{T}=i \sum_{m^{2}=1}^{l}\left(\gamma_{k-m}^{+}\left(c_{k-m}^{+}, m\right)\left(c_{m}^{+}, c_{k}^{ \pm}\right) \gamma_{m}^{+}+\gamma_{k-m}^{+}\left(c_{k-m}^{+}, m\right)\left(c_{m}^{-}, c_{k}^{ \pm}\right) \gamma_{m}^{-}+\right. \\
\left.\gamma_{k-m}^{-}\left(c_{k-m}^{-}, m\right)\left(c_{m}^{+}, c_{k}^{ \pm}\right) \gamma_{m}^{+}+\gamma_{k-m}^{-}\left(c_{k-m}^{-}, m\right)\left(c_{m}^{-}, c_{k}^{ \pm}\right) \gamma_{m}^{-}\right)
\end{array}
$$

split into four similar quadratic forms $\gamma^{+} C^{++}\left(\gamma^{+}\right)^{\mathrm{T}}, \ldots, \gamma^{-} C^{--}\left(\gamma^{-}\right)^{\mathrm{T}}$. Nonzero elements in matrices of these forms hold the same positions $(k-m, m)$ and are situated "perpendicularly" to the main diagonal. These matrices and the matrices $B_{k}^{ \pm}$are sparse.

A program for calculating the modified Fourier coefficients (32), (33) of vector functions $f(t, x)$ and $g(x)$, contained in the equations and the initial data of the problem (145) is developed.

The Sobolev cubature formulae with a regular boundary layer were used in calculating the integrals with respect to a cube.

Programs for calculating coefficients of the system $R S_{l}$ and numeric solution of the Cauchy problem by the Runge-Kutta method were developed.

The printout of the system $R S_{2}$ consists of approximately 20 A 4 pages. Therefore, only the first plus equation from $R S_{2}$ is represented:

$$
\begin{aligned}
& \frac{d}{d t} \gamma_{1(-1,0,0)}^{+}=-\nu \gamma_{1(-1,0,0)}^{+}- \\
& \left\{\gamma_{2(0,-1,0)}^{+}(0.603553 i) \gamma_{10(-1,1,0)}^{+}+\gamma_{2(0,-1,0)}^{+}(0.103553 i) \gamma_{10(-1,1,0)}^{-}\right. \\
& +\gamma_{3(0,0,-1)}^{+}(0.603553 i) \gamma_{9(-1,0,1)}^{+}+\gamma_{3(0,0,-1)}^{+}(-0.103553 i) \gamma_{9(-1,0,1)}^{-} \\
& +\gamma_{4(0,0,1)}^{+}(-0.603553 i) \gamma_{8(-1,0,-1)}^{+}+\gamma_{4(0,0,1)}^{+}(0.103553 i) \gamma_{8(-1,0,-1)}^{-} \\
& +\gamma_{5(0,1,0)}^{+}(-0.603553 i) \gamma_{7(-1,-1,0)}^{+}+\gamma_{5(0,1,0)}^{+}(-0.103553 i) \gamma_{7(-1,-1,0)}^{-} \\
& +\gamma_{7(-1,-1,0)}^{+}(0.25 i) \gamma_{5(0,1,0)}^{+}+\gamma_{7(-1,-1,0)}^{+}(0.25 i) \gamma_{5(0,1,0)}^{-} \\
& +\gamma_{8(-1,0,-1)}^{+}(0.25 i) \gamma_{4(0,0,1)}^{+}+\gamma_{8(-1,0,-1)}^{+}(0.25 i) \gamma_{4(0,0,1)}^{-} \\
& +\gamma_{9(-1,0,1)}^{+}(-0.25 i) \gamma_{3(0,0,-1)}^{+}+\gamma_{9(-1,0,1)}^{+}(-0.25 i) \gamma_{3(0,0,-1)}^{+} \\
& +\gamma_{10,(-1,1,0)}^{+}(-0.25 i) \gamma_{2(0,-1,0)}^{+}+\gamma_{10(-1,1,0)}^{+}(-0.25 i) \gamma_{2(0,-1,0)}^{-} \\
& +\gamma_{2(0,-1,0)}^{-}(-0.603553 i) \gamma_{10(-1,1,0)}^{+}+\gamma_{2(0,-1,0)}^{+}(-0.103553 i) \gamma_{10(-1,1,0)}^{-} \\
& +\gamma_{3(0,0,-1)}^{-}(-0.603553 i) \gamma_{9(-1,0,1)}^{+}+\gamma_{3(0,0,-1)}^{-}(0.103553 i) \gamma_{9(-1,0,1)}^{-} \\
& +\gamma_{4(0,0,1)}^{-}(0.603553 i) \gamma_{8(-1,0,-1)}^{+}+\gamma_{4(0,0,1)}^{-}(-0.103553 i) \gamma_{8(-1,0,-1)}^{-} \\
& +\gamma_{5(0,1,0)}^{-}(0.603553 i) \gamma_{7(-1,-1,0)}^{+}+\gamma_{5(0,1,0)}^{-}(0.103553 i) \gamma_{7(-1,-1,0)}^{-} \\
& +\gamma_{7(-1,-1,0)}^{-}(-0.25 i) \gamma_{5(0,1,0)}^{+}+\gamma_{7(-1,-1,0)}^{-}(-0.25 i) \gamma_{5(0,1,0)}^{-} \\
& +\gamma_{8(-1,0,-1)}^{-}(0.25 i) \gamma_{4(0,0,1)}^{+}+\gamma_{8(-1,0,-1)}^{-}(0.25 i) \gamma_{4(0,0,1)}^{-} \\
& +\gamma_{9(0,0,1)}^{+}(-0.25 i) \gamma_{3(0,0,-1)}^{+}+\gamma_{9(-1,0,1)}^{+}(-0.25 i) \gamma_{3(0,0,-1)}^{-} \\
& \left.+\gamma_{10(-1,1,0)}^{-}(0.25 i) \gamma_{2(0,-1,0)}^{+}+\gamma_{10(-1,1,0)}^{-}(0.25 i) \gamma_{2(0,-1,0)}^{-}\right\}+\phi_{1(-1,0,0)}^{+}(t) .
\end{aligned}
$$

Its coefficients are calculated with the sixth order of accuracy.
6.2. Solvability of Problem (145). In $\S 4$, two a priori estimates

$$
\begin{equation*}
\sup _{s \in[0, T]}\left\|v_{l}(s, \cdot)\right\|_{V^{0}}^{2} \leqslant\|g\|_{V^{0}}^{2}+\nu^{-1}\|f\|_{L_{2}\left(0, T ; V^{-1}\right)}^{2} \tag{146}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|v_{l}\right\|_{L_{2}\left(0, T ; V^{1}\right)}^{2} \leqslant \nu^{-1}\|g\|_{V^{0}}^{2}+\nu^{-2}\|f\|_{L_{2}\left(0, T ; V^{-1}\right)}^{2} \tag{147}
\end{equation*}
$$

have been proved assuming that $f(t, x) \in L_{2}\left(0, T ; V^{0}\right), g(x) \in V^{1}$.
Whence, one concludes that Problem (145) is solvable on the whole interval ( $0, T$ ), and its solution $v_{l}(t, x)$ is bounded in spaces $L_{\infty}\left(0, T ; V^{0}\right)$ and $L_{2}\left(0, T ; V^{1}\right)$ for any $l=1,2, \ldots$. Note that vector functions $f$ and $g$ are real, $f_{0}(t)=0, g_{0}=0$ and $S_{l} f(t, x), S_{l} g(x)$ are their partial sums:

$$
\begin{align*}
S_{l} f(t, x) & =2 \varpi R e \sum_{k \in M_{l}}\left(\phi_{k}^{+}(t) u_{k}^{+}(x)+\phi_{k}^{-}(t) u_{k}^{-}(x)\right),  \tag{148}\\
S_{l} g(x) & =2 \varpi R e \sum_{k \in M_{l}}\left(\psi_{k}^{+} u_{k}^{+}(x)+\psi_{k}^{-} u_{k}^{-}(x)\right), \tag{149}
\end{align*}
$$

and the Galerkin approximations $v_{l}$, solutions to equations $G S_{l}$, have the form

$$
\begin{equation*}
v_{l}=\operatorname{Re} v_{l}(x, t)=2 \varpi \operatorname{Re} \sum_{k \in M_{l}}\left(\gamma_{k, l}^{+}(t) u_{k}^{+}(x)+\gamma_{k, l}^{-}(t) u_{k}^{-}(x)\right) . \tag{150}
\end{equation*}
$$

6.3. Visualization. Consider a flow whose velocity is given by a real eigenfunction of the curl operator in the form

$$
\mathrm{d}_{k}^{+}(t, x)=2 \operatorname{Re}\left(\phi_{k}^{+}(t) c_{k}^{+} e^{i k x}\right)
$$

when $k \neq 0$. It is defined by the complex function $\phi_{k}^{+}(t)=\left|\phi_{k}^{+}(t)\right| e^{i \theta_{k}^{+}(t)}$.
Let us assume that $P_{k, \delta}$ is a plane given by the equation $k x=\delta$. By construction, the vectors $\widehat{k}, \sqrt{2} a_{k}^{+}, \sqrt{2} b_{k}^{+}$generate an orthonormal basis in the space $R^{3}$, which induces a basis in the planes $P_{k, \delta}$. The velocity of the flow on every plane is

$$
\begin{equation*}
v_{k, \delta}^{+}(t)=\left.\mathrm{d}_{k}^{+}\right|_{P_{k, \delta}}=2\left|\phi_{k}^{+}(t)\right|\left(\cos \left(\delta+\theta_{k}^{+}(t)\right) a_{k}^{+}-\sin \left(\delta+\theta_{k}^{+}(t)\right) b_{k}^{+}\right), \tag{151}
\end{equation*}
$$

and it is independent of $x \in P_{k, \delta}$.
Its basis coordinates $\left(\sqrt{2} a_{k}^{+}, \sqrt{2} b_{k}^{+}\right)$are the same as that of the complex function $\phi_{k, \delta}^{+}(t)=\sqrt{2} \overline{\phi_{k}^{+}(t) e^{i \delta}}$ in the basis $(1, i)$. In other words, the curve $\phi_{k, \delta}^{+}(t)$ can be obtained from the curve $\phi_{k}^{+}(t)$ by means of dilation on $\sqrt{2}$, rotation at the angle $\alpha$ and reflection from a real axis.

Therefore, the drawing of the curve $\phi_{k}^{ \pm}(t)$ on a complex plane gives one a possibility to imagine the behavior of the velocity vector $v_{k, \delta}^{+}(t)$ in the plane $P_{k, \delta}$, orthogonal to the vector $k$. Let us term it as the ch ar t of the flow $\mathrm{d}_{k}^{+}(t, x)$.

Let us chart every addend of the solution (150). A set of such charts gives an idea of the flow as a whole.
6.4. Solution to the Cauchy problem for the system $R S_{2}$. Let us assume that $\nu=0.1$, $f=0$ and $\psi_{k}^{+}$are given when $k=(0,0,1), k=(0,1,1)$, and $\psi_{k}^{-}$is given when $k=(0,1,0)$, i.e. the initial flow $g$ consists of three stationary vortex flows:

$$
g=\left.2 \operatorname{Re}\left(\psi_{k}^{+} c_{k}^{+} e^{i k x}\right)\right|_{k=(0,0,1)}+\left.2 \operatorname{Re}\left(\psi_{k}^{+} c_{k}^{+} e^{i k x}\right)\right|_{k=(0,1,1)}+\left.2 \operatorname{Re}\left(\psi_{k}^{-} c_{k}^{-} e^{i k x}\right)\right|_{k=(0,1,0)}
$$

As a result of calculating the problem, one obtains 8 nonzero functions $\gamma_{k}^{ \pm}(t)$ for $k \in M^{\prime}=\{(0,1,0),(0,0,1),(0,-1,1),(0,1,1)\}$, and the resulting solution $v$ has the form

$$
\begin{equation*}
v(t, x)=2 \operatorname{Re} \sum_{k \in M^{\prime}}\left(\gamma_{k}^{+}(t) c_{k}^{+}+\gamma_{k}^{-}(t) c_{k}^{-}\right) e^{i k x} . \tag{152}
\end{equation*}
$$

Further, the calculations are repeated for $\nu=0.01$ and one can see the difference of the curves.

Example. The variable $t$ varies from 0 to 10 . The initial data are

$$
\psi_{4(0,0,1)}^{+}=-3, \psi_{14(0,1,1)}^{+}=14 i, \psi_{5(0,1,0)}^{-}=2,
$$

and by symmetry

$$
\psi_{3(0,0,-1)}^{+}=3, \psi_{11(0,-1,-1)}^{+}=14 i, \psi_{2(0,-1,0)}^{-}=-2 .
$$

The figures represent charts of vortex flows, i.e. the curves $\gamma_{k}^{ \pm}(t)$ in a complex plane that give an idea of behavior of the velocity vector in the plane orthogonal to the vector $k$.

$$
\nu=0,01, \gamma_{4(0,0,1)}^{+}(0)=-3
$$



Fig. 1.

$$
\nu=0,01, \quad \gamma_{5(0,1,0)}^{+}(0)=0
$$



Fig. 3.


Fig. 5.
$\nu=0,1, \gamma_{4(0,0,1)}^{+}(0)=-3$.


Fig.2.


Fig. 4.
$\nu=0,1, \quad \gamma_{12(0,-1,1)}^{+}(0)=0$


Fig.6.
$\nu=0,01, \quad \gamma_{14(0,1,1)}^{+}(0)=14 i$


Fig. 7.


Fig.9.
$\nu=0,01, \quad \gamma_{5(0,1,0)}^{-}(0)=2$


Fig. 11.
$\nu=0,01, \quad \gamma_{12(0,-1,1)}^{-}(0)=0$


Fig. 13.
$\nu=0,01, \quad \gamma_{14(0,1,1)}^{-}(0)=0$


Fig. 15.


Fig. 8.


Fig. 10.


Fig. 12.

$$
\nu=0,1, \quad \gamma_{12(0,-1,1)}^{-}(0)=0
$$



Fig. 14.

$$
\nu=0,1, \quad \gamma_{14(0,1,1)}^{-}(0)=0
$$



Fig. 16.

In conclusion, note that the idea of the given method for numerical solution of the problem belongs to O.A. Ladyzhenskaya, who also suggested calculating model problems and interpreting the character of solutions breakdown with the decrease of the viscosity coefficient of the system and with other conditions in a numerical experiment. The issue remains open.
A.G. Khaibullin did a great job in programming mentioned in the present article.

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