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ON ORTHOSIMILAR SYSTEMS IN A SPACE OF ANALYTICAL FUNCTIONS AND THE PROBLEM OF DESCRIBING THE DUAL SPACE

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Abstract. We consider an orthosimilar system with the measure μ in the space of analytical functions H on the domain $G \subset \mathbb{C}$. Let $K_H(\xi, t)$, $\xi, t \in G$ be a reproduction kernel in the space H. We claim that a system $\{K_H(\xi, t)\}_{t \in G}$ is the orthosimilar system with the measure μ in the space H if and only if the space H coincides with the space $B_2(G, \mu)$. A problem of describing the dual space in terms of the Hilbert transform is considered. This problem is reduced to the problem of existence of a special orthosimilar system in $B_2(G, \mu)$. We prove that the space $\widetilde{B}_2(G, \mu)$ is the only space with a reproduction kernel and it consists of functions given on the domain $\mathbb{C} \backslash \overline{G}$ with an orthosimilar system $\{\frac{1}{(z-\xi)^2}\}_{\xi \in G}$ with the measure μ.

Keywords: Bergman space, Hilbert spaces, reproducing kernel, orthosimilar system, Hilbert transform.

1. Introduction

Let us assume that G is a simply connected domain in $\mathbb{C}$, and μ is a nonnegative Borel measure on G. Let us denote the space of functions holomorphic in G, for which

$$\|f\|^2_{B_2(G,\mu)} = \int_G |f(z)|^2 \, d\mu(z) < \infty,$$

by $B_2(G, \mu)$.

Let us impose the following condition on the measure μ. The space $B_2(G, \mu)$ should be a Hilbert space, i.e. the space $B_2(G, \mu)$ with the norm $\| \cdot \|_{B_2(G,\mu)}$ should be complete.

A scalar product in the space $B_2(G, \mu)$ has the form

$$(f, g)_{B_2(G,\mu)} = \int_G f(z) \cdot \overline{g(z)} \, d\mu(z).$$

In addition, we require that the system of functions $\{\frac{1}{(z-\xi)^2}, \xi \in \mathbb{C} \backslash \overline{G}\}$ is complete in the space $B_2(G, \mu)$.

Remark. *We do not require that the space $B_2(G, \mu)$ is separable. A detailed presentation of the theory of nonseparable Hilbert spaces can be found in* [1], [2].

Let us associate every linear continuous functional f^* on $B_2(G, \mu)$, generated by the function $f \in B_2(G, \mu)$, to the function

$$\widetilde{f}(\xi) \stackrel{def}{=} f^*\left(\tfrac{1}{(z-\xi)^2}\right) = (\tfrac{1}{(z-\xi)^2}, f(z))_{B_2(G,\mu)} = \int_G \overline{f(z)} \cdot \frac{1}{(z-\xi)^2} \, d\mu(z), \quad \xi \in \mathbb{C} \backslash \overline{G}.$$

Definition 1. *The function $\widetilde{f}$ is termed as the Hilbert transform of the functional generated by $f \in B_2(G, \mu)$.*

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Since the system of functions $\{\frac{1}{(z-\xi)^2}, \xi \in \mathbb{C}\backslash\overline{G}\}$ is complete in the space $B_2(G,\mu)$, the mapping $f^* \to \widetilde{f}$ is injective. The family of functions $\widetilde{f}$ constructs a space

$$\{\widetilde{f} : \widetilde{f}(\xi) = (\tfrac{1}{(z-\xi)^2}, f(z))_{B_2(G,\mu)}\} \overset{\text{об}}{=} \widetilde{B}_2(G,\mu),$$

where the induced structure of the Hilbert space is considered, i.e.

$$(\widetilde{f}, \widetilde{g})_{\widetilde{B}_2(G,\mu)} \overset{def}{=} (g, f)_{B_2(G,\mu)}$$

and

$$\|\widetilde{f}\|_{\widetilde{B}_2(G,\mu)} = \|f\|_{B_2(G,\mu)}.$$

The present paper considers the question when a norm of the form

$$\|\widetilde{f}\|_\nu = \sqrt{\int_{\mathbb{C}\backslash G} |\widetilde{f}(\xi)|^2 \, d\nu(\xi)},$$

where ν is a nonnegative measure on $\mathbb{C}\backslash\overline{G}$, equivalent to the induced norm $\|\widetilde{f}\|_{\widetilde{B}_2(G,\mu)}$, can be introduced in the space $\widetilde{B}_2(G,\mu)$. In more detail, are there a nonnegative Borel measure ν in $\mathbb{C}\backslash\overline{G}$ and constants $A_1, A_2 > 0$ such that the relations

$$A_1\|\widetilde{f}\|_{\widetilde{B}_2(G,\mu)} \le \|\widetilde{f}\| \le A_2\|\widetilde{f}\|_{\widetilde{B}_2(G,\mu)} \quad , \quad \forall \widetilde{f} \in \widetilde{B}_2(G,\mu)$$

hold. Thus, the problem describing a space dual to $B_2(G,\mu)$ in terms of the Hilbert transform is considered.

Problems of describing analytical functions conjugate to various spaces in terms of the Cauchy, Hilbert, Fourier-Laplace transform were considered in earlier works by numerous authors. I mention here only the works that are most closely related to the topic of the present article [3, 4, 5, 6, 7, 8, 9] etc.

2. Auxiliary information

Definition 2. *(see* [10]*) Let us assume that H is a Hilbert space over a field $\mathbb{R}$ or $\mathbb{C}$, and Ω is a space with a countably additive measure μ (see* [15]*, p.109–116). The system of elements $\{e_\omega\}_{\omega\in\Omega}$ is called an orthosimilar system (similar to orthogonal) in H with the measure μ, if any element $y \in H$ can be represented in the form*

$$y = \int_\Omega (y, e_\omega)_H e_\omega \, d\mu(\omega).$$

Here the integral is interpreted as a proper or improper Lebesgue integral of a function with values in H. In the latter case there is such an exhaustion $\{\Omega_k\}_{k=1}^\infty$ of the space Ω possibly depending on y and called suitable for y, that the function $(y, e_\omega)_H \cdot e_\omega$ is Lebesgue integrable on Ω_k and

$$y = \int_\Omega (y, e_\omega)_H e_\omega \, d\mu(\omega) = \lim_{k\to\infty} (L) \int_{\Omega_k} (y, e_\omega)_H e_\omega \, d\mu(\omega).$$

Note that all Ω_k are measurable by μ, $\Omega_k \subset \Omega_{k+1}$ for $k \in \mathbb{N}$ and $\bigcup_{k=1}^\infty \Omega_K = \Omega$.

Examples:

1. Any orthogonal basis $\{e_k\}_{k=1}^{\infty} \subset H$ in an arbitrary Hilbert space H is an orthosimilar system; any element $y \in H$ can be represented in the form
$$y = \sum_{k=1}^{\infty} (y, e_k) e_k.$$
Here one can take a set $\mathbb{N}$ as Ω, and in the capacity of the measure μ, one can take a countable measure, i.e. a set measure from $\mathbb{N}$ is the amount of different natural numbers entering the set.
2. Let us assume that H is a Hilbert space, H_1 is a subspace of H, and P is the operator of orthogonal projection of elements from H onto H_1. Let $\{e_k\}_{k=1}^{\infty} \subset H$ be an orthogonal basis in H. Then, the system of elements $\{P(e_k)\}_{k=1}^{\infty} \subset H_1$ is an orthosimilar system in H_1. (see [10], Theorem 9). Note, that if $\{e_k\}_{k=1}^{\infty}$ is an orthogonal basis in H, then the system $\{P(e_k)\}_{k=1}^{\infty}$, generally speaking, is not an orthogonal basis in H_1.
3. Let $H = L_2(\mathbb{R})$. The function $\psi \in L_2(\mathbb{R})$, $\|\psi\|_{L_2(\mathbb{R})} = 1$. A system of Morlet wavelets $\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right)$, $a \in \mathbb{R} \backslash \{0\}$, $b \in \mathbb{R}$ is an orthosimilar system in the space $L_2(\mathbb{R})$; any function $f \in L_2(\mathbb{R})$ can be represented in the form
$$f(x) = \int_{\mathbb{R} \backslash 0} \int_{\mathbb{R}} (f(\tau), \psi_{a,b}(\tau))_{L_2(\mathbb{R})} \psi_{a,b}(x) \frac{dbda}{C_\psi |a|^2},$$
where $C_\psi > 0$ is a constant. The set $(\mathbb{R} \backslash \{0\}) \times \mathbb{R}$ with the measure $\frac{dbda}{C_\psi |a|^2}$ is taken as the space Ω here. (see [11],[10]).

Expansion of elements of the Hilbert space with respect to orthosimilar systems can be not the only one. At the same time, orthosimilar systems possess many properties of orthogonal ones, e.g., an analogue of the Parseval identity holds for them and they have the extremal property of coefficients for orthosimilar systems.

Definition 3. *([10]) An orthosimilar system is said to be nonnegative if the measure μ is nonnegative.*

We will need the following two theorems from the work [10], Theorems 1 and 3.

Theorem A. (An analogue of the Parseval identity) *Let $\{e_\omega\}_{\omega \in \Omega} \subset H$ be a nonnegative orthosimilar system with the measure μ in H.*

Then, for any element $y \in H$ one has
$$\|y\|_H^2 = \int_\Omega |(y, e_\omega)|^2 \, d\mu(\omega),$$
and for any two elements $x, y \in H$ one has
$$(x, y)_H = \int_\Omega (x, e_\omega) \cdot \overline{(y, e_\omega)} \, d\mu(\omega).$$

Theorem B. (Extremal property of expansion coefficients) *Let $\{e_\omega\}_{\omega \in \Omega}$ be a nonnegative orthosimilar system in H, and $c(\omega)$ be a function on Ω with the value in $\mathbb{R}$ or $\mathbb{C}$ (depending on the field over which H is considered and*
$$y = \int_\Omega c(\omega) e_\omega \, d\mu(\omega),$$
where the integral is interpreted as a proper or improper Lebesgue integral of a function with the value in H. In the latter case there is an exhaustion $\{\Omega_k\}_{k=1}^{\infty}$ of the space Ω (all Ω_k are

measurable by μ, $\Omega_k \subset \Omega_{k+1}$ for $k \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} \Omega_K = \Omega$) such that the function $c(\omega) \cdot e_\omega$ is Lebesgue integrable on Ω_k and

$$y = \int_\Omega c(\omega) \cdot e_\omega \, d\mu(\omega) = \lim_{k\to\infty} (L) \int_{\Omega_k} c(\omega) \cdot e_\omega \, d\mu(\omega).$$

Then,

$$\|y\|_H^2 \leq \int_\Omega |c(\omega)|^2 \, d\mu(\omega),$$

and the equality holds only if $c(\omega) = (y, e_\omega)_H$ almost everywhere on Ω by the measure μ.

In the present paper, functional Hilbert spaces, consisting of functions in a domain $G \subset \mathbb{C}$ are considered.

Definition 4. *A Hilbert space H, consisting of functions $f(z) : E \to \mathbb{C}$, given on a set E, is said to be functional if the functional $\delta_{z_0} : f \to f(z_0)$ is linear and continuous over H for any $z_0 \in E$.*

According to the Riesz-Fisher theorem, any linear continuous functional over H is constructed by an element from H. Whence, there is a function $K_H(z, z_0) \in H$ such that the identity $f(z_0) = (f(z), K_H(z, z_0))_H$ holds.

Thus, the function $K_H(z, \xi)$, $z, \xi \in E$, which is called a reproducing kernel of the space H is defined (see, e.g., [12]). Basic properties of functional spaces and reproducing kernels are described in [12].

3. BASIC RESULTS

The following statement is proved in the present paper.

Theorem 1. *Let H be a functional Hilbert space of functions on the domain $G \subset \mathbb{C}$. A norm has an integral form*

$$\|f\|_H = \sqrt{\int_G |f(\xi)|^2 \, d\nu(\xi)} \tag{1}$$

in the space H if and only if the system of functions $\{K_H(\xi, t)\}_{t\in G}$ is a nonnegative orthosimilar system with the measure ν in the space H.

Remark. *Obviously, if a norm in the space H is determined as in (1), then*

$$(f, g)_H = \int_G f(\xi) \cdot \overline{g(\xi)} \, d\nu(\xi).$$

Proof. *Sufficiency.* Let us assume that a system of functions $\{K_H(\xi, t)\}_{t\in G}$ is a nonnegative orthosimilar system with the measure ν in the space H. It means that any element $f \in H$ can be represented in the form

$$f(\xi) = \int_G (f(\tau), K_H(\tau, t))_H K_H(\xi, t) \, d\nu(t), \quad \xi \in G.$$

By virtue of Theorem A,

$$\|f\|_H^2 = \int_G |(f(\tau), K_H(\tau, t))_H|^2 \, dv(t) = \int_G |f(t)|^2 \, d\nu(t).$$

Necessity. Let us assume that the equation

$$\|f\|_H^2 = \int_G |f(\xi)|^2 \, d\nu(\xi)$$

holds for any $f \in H$. Then,

$$f(\xi) = (f(t), K_H(t,\xi))_H = \int_G f(t) \cdot \overline{K_H(t,\xi)}\, d\nu(t).$$

By the property of reproducing kernels $\overline{K_H(t,\xi)} = K_H(\xi,t)$ (see [12]). Therefore,

$$f(\xi) = \int_G f(t) \cdot K_H(\xi,t)\, d\nu(t) =$$
$$= \int_G (f(\tau), K_H(\tau,t))_H \cdot K_H(\xi,t)\, d\nu(t),\ \xi \in G. \tag{2}$$

Thus, the system of functions $\{K_H(\xi,t)\}_{t\in G}$ is a nonnegative orthosimilar system in the space H with the measure ν.

The theorem is proved.

Corollary. *A functional Hilbert space H, consisting of functions analytical on the domain G, coincides with the space $B_2(G,\mu)$ for a measure μ if and only if the family of reproducing kernels $\{K_H(\xi,t)\}_{t\in G}$ of the space H is a nonnegative orthosimilar system in H with the measure μ, i.e. any function $g \in H$ can be represented in the form*

$$g(\xi) = \int_G (g(\tau), K_H(\tau,t))_H K_H(\xi,t)\, d\mu(t), \quad \xi \in G.$$

Theorem 2. *A functional Hilbert space H, consisting of functions of a variable $\xi \in \mathbb{C}\backslash\overline{G}$, coincides with the space $\widetilde{B}_2(G,\mu)$ if and only if the family of functions $\{\frac{1}{(\xi-t)^2}\}_{t\in G}$ is an orthosimilar system in H with a measure μ, i.e. any function $g \in H$ can be represented in the form*

$$g(\xi) = \int_G (g(\tau), \tfrac{1}{(\tau-t)^2})_H \tfrac{1}{(\xi-t)^2}\, d\mu(t), \quad \xi \in \mathbb{C}\backslash\overline{G}. \tag{3}$$

Necessity. Let the space H coincide with $\widetilde{B}_2(G,\mu)$. The space $\widetilde{B}_2(G,\mu)$ consists of functions, that can be represented in the form

$$\widetilde{f}(\xi) = (\tfrac{1}{(\xi-t)^2}, f(t))_{B_2(G,\mu)} = \int_G \overline{f(t)} \tfrac{1}{(\xi-t)^2}\, d\mu(t), \quad f \in B_2(G,\mu). \tag{4}$$

Meanwhile, an induced structure of the Hilbert space

$$(\widetilde{f},\widetilde{g})_{\widetilde{B}_2(G,\mu)} = (g,f)_{B_2(G,\mu)}$$

is considered in $\widetilde{B}_2(G,\mu)$. Let us consider a function $K_{B_2(G,\mu)}(\xi,t)$ of ξ when t is fixed. In our notation

$$\widetilde{K}_{B_2(G,\mu)}(\xi,t) = (\tfrac{1}{(\tau-t)^2}, K_{B_2(G,\mu)}(\tau,\xi))_{B_2(G,\mu)} = \frac{1}{(\xi-t)^2}\,.$$

Therefore, for any $f \in B_2(G,\mu)$

$$\overline{f(t)} = \overline{(f(\tau), K_{B_2(G,\mu)}(\tau,t))}_{B_2(G,\mu)} =$$
$$= (K_{B_2(G,\mu)}(\tau,t), f(\tau))_{B_2(G,\mu)} = (\widetilde{f}(\tau), \tfrac{1}{(\tau-t)^2})_{\widetilde{B}_2(G,\mu)}.$$

This equation, together with (4), entails that

$$g(\xi) = \int_G (g(\tau), \tfrac{1}{(\tau-t)^2})_{\widetilde{B}_2(G,\mu)} \tfrac{1}{(\xi-t)^2}\, d\mu(t), \quad \xi \in \mathbb{C}\backslash G,\ g \in \widetilde{B}_2(G,\mu)$$

for any $g \in \widetilde{B}_2(G,\mu)$. Thus, the system of functions $\{\frac{1}{(\xi-t)^2}\}_{t\in G}$ is an orthosimilar system with the measure μ in the space $\widetilde{B}_2(G,\mu)$.

Sufficiency. Let the system of functions $\{\frac{1}{(\xi-t)^2}\}_{t\in G}$ be an orthosimilar system in the space H with the measure μ. It means that any element of the space H can be represented in the form

$$f(\xi)=\int_G (f(\tau),\tfrac{1}{(\tau-t)^2})_H \tfrac{1}{(\xi-t)^2}\,d\mu(t),\quad \xi\in\mathbb{C}\backslash\overline{G}.$$

Let us calculate the reproducing kernel of the space H:

$$K_H(\xi,\eta)=\int_G (K_H(\tau,\eta),\tfrac{1}{(\tau-t)^2})_H \tfrac{1}{(\xi-t)^2}\,d\mu(t)=$$

$$=\int_G \overline{\tfrac{1}{(\eta-t)^2}}\cdot\tfrac{1}{(\xi-t)^2}\,d\mu(t)=(\tfrac{1}{(\xi-t)^2},\tfrac{1}{(\eta-t)^2})_{B_2(G,\mu)},\quad \xi\in\mathbb{C}\backslash\overline{G}. \tag{5}$$

On the other hand, it follows from (3) that

$$K_{\widetilde{B}_2(G,\mu)}(\xi,\eta)=(\tfrac{1}{(\xi-t)^2},\tfrac{1}{(\eta-t)^2})_{B_2(G,\mu)}=K_H(\xi,\eta).$$

According to the Moore-Aronszajn theorem (see [13],[12]), the space H coincides with $\widetilde{B}_2(G,\mu)$.

Theorem 2 is proved.

Definition 5. *([14], p. 280) A linearly continuous operator A, acting in the Hilbert space H, is said to be positive if the value $(x,Ax)_H$ is positive for any $x\in H$, $x\neq 0$.*

Definition 6. *([14], p. 281). The numbers*

$$C_1=\inf_{\substack{x\in H\\ x\neq 0}}\frac{(x,Ax)}{\|x\|^2},\quad C_2=\sup_{\substack{x\in H\\ x\neq 0}}\frac{(x,Ax)}{\|x\|^2}$$

are termed as the lower and the upper bound of a self-adjoint operator A.

Manifestly, the identities

$$C_1\|x\|^2\leq (x,Ax)\leq C_2\|x\|^2,\forall x\in H$$

hold.

Lemma 1. *Let us assume that H is a Hilbert space with a scalar product (x,y) and that another scalar product $(x,y)_1$ is defined in H.*

The following conditions are equivalent:

1. *The norms, determined by scalar products (x,y), $(x,y)_1$, are equivalent, i.e. there are such constants $C_1,C_2>0$ that for any element $x\in H$ the inequalities*

$$C_1\|x\|\leq\|x\|_1\leq C_2\|x\|$$

hold.

2. *There exists a linear continuous self-adjoint operator A, which is an automorphism of the Banach space H with a norm $\|\cdot\|$, such that*

$$\|x\|_1^2=(x,Ax),\quad \forall x\in H. \tag{6}$$

Proof. Let us prove that 2 follows from 1. Consider a linear functional

$$h\to(h,x)_1,\quad \forall h\in H$$

for an element $x\in H$ in the Hilbert space H.

Since

$$|(h,x)_1|\leq\|h\|_1\cdot\|x\|_1\leq C_2\cdot\|h\|_1\cdot\|x\|,$$

the functional $h\to(h,x)_1$ is a linear and continuous functional in the Hilbert space H. According to the Riesz-Fischer theorem, there exists only one element $y_x\in H$ such that the identity

$$(h,x)_1=(h,y_x),\quad \forall h\in H \tag{7}$$

holds. Let us determine the mapping $A : H \to H$ by the formula $A(x) = y_x$. Evidently, A is a linear operator. Moreover,

$$\|y_x\| = \sup_{\substack{h\in H\\ h\neq 0}} \frac{(h, y_x)}{\|h\|} \le C_2 \sup_{\substack{h\in H\\ h\neq 0}} \frac{(h, x)_1}{\|h\|_1} = C_2\|x\|_1 \le C_2^2\|x\|.$$

Likewise,

$$\|y_x\| = \sup_{\substack{h\in H\\ h\neq 0}} \frac{(h, y_x)}{\|h\|} \ge C_1 \sup_{\substack{h\in H\\ h\neq 0}} \frac{(h, x)_1}{\|h\|_1} = C_1\|x\|_1 \ge C_1^2\|x\|.$$

It follows from the latter two estimates that

$$C_1^2\|x\| \le \|Ax\| \le C_2^2\|x\|.$$

In particular, A is an injective linear bounded operator. The norms $\|\cdot\|$, $\|\cdot\|_1$ are equivalent in our considerations, therefore the operator A is surjective. Thus, A is an automorphism of the Banach space H (as well as of the space H with the norm $\|\cdot\|_1$).

Then, Definition (7) entails that

$$(h, x)_1 = (h, Ax)$$

and

$$C_1^2\|x\|^2 \le \|x\|_1^2 = (x, x)_1 = (x, Ax).$$

Hence, the operator A has a positive lower bound and thus, A is a positive self-adjoint operator (see [14], стр. 247).

Let us prove that the condition 1 follows from the condition 2. If A is a self-adjoint operator such that the equality (6) holds, then A is a positive operator and there exists a unique positive square root of the operator A, i.e. such an operator S, that $A = S \circ S$ (see, e.g., [14], p. 282). The operator S is one-to-one and conjugate as well (see [14], p. 247). Let us use the theorem on page 285 of [14].

Theorem C. *The necessary and sufficient condition for a linear operator T in a Hilbert space to have an inverse operator is that there be such a constant $C_1 > 0$ that the following inequalities hold:*

$$(T^* \circ Tx, x) \ge C_1\|x\|^2, \quad (T \circ T^*x, x) \ge C_1\|x\|^2,$$

where T^* is an operator conjugate to T.

Let us apply Theorem C to the operator S. Let us take the self-adjoint linear continuous one-to-one operator S as operator T. Applying Theorem C and invoking that the operator A is bounded, one concludes that the operator $S \circ S^* = S \circ S = A$ has positive lower and upper bounds, i.e. there are constants $C_1, C_2 > 0$ such that the inequalities

$$C_1\|x\|^2 \le (x, x)_1 = (x, Ax) \le \|A\|\|x\|^2 = C_2\|x\|^2, \ \forall x \in H$$

hold. The latter indicates that the condition 1 is met. This proves the Lemma.

The following theorem specifies Theorem 1 in the case when the Hilbert space H is a space $\widetilde{B}_2(G, \mu)$.

Theorem 3. *In order to introduce into the space $\widetilde{B}_2(G, \mu)$ a norm equivalent to the original one*

$$\|\widetilde{f}\|_\nu = \sqrt{\int_{\mathbb{C}\setminus G} |\widetilde{f}(\xi)|^2\, d\nu(\xi)},$$

where ν is a nonnegative Borel measure on $\mathbb{C}\setminus G$, it is necessary and sufficient that there exist a linear continuous operator S, governing an automorphism of the Banach space $B_2(G, \mu)$, such *that the system* $\{S\left(\frac{1}{(z-\xi)^2}\right)\}_{\xi\in\mathbb{C}\setminus\overline{G}}$ *is an orthosimilar system with the measure* ν *in the space* $B_2(G,\mu)$*, i.e. any element* $f\in B_2(G,\mu)$ *can be represented in the form*

$$f(z)=\int_{\mathbb{C}\setminus\overline{G}}(f(\tau),S_\tau\tfrac{1}{(\tau-\xi)^2})_{B_2(G,\mu)}S_z\tfrac{1}{(z-\xi)^2}\,d\nu(\xi),\quad z\in\mathbb{C}\setminus G.$$

Proof. *Necessity.* Suppose that one can introduce an equivalent integral norm of the form

$$\|\widetilde{f}\|_\nu=\sqrt{\int_{\mathbb{C}\setminus G}|\widetilde{f}(\xi)|^2\,d\nu(\xi)},$$

in the space $\widetilde{B}_2(G,\mu)$, i.e. the Banach spaces $\widetilde{B}_2(G,\mu)$ and $B_2(\mathbb{C}\setminus G,\nu)$ are isomorphic. Consider the following operator on functions $f\in B_2(G,\mu)$:

$$Tf(\xi)\overset{def}{=}(f(z),\tfrac{1}{(z-\xi)^2})_{B_2(G,\mu)}.$$

Let us introduce the notation

$$J_2(\mathbb{C}\setminus\overline{G},\nu)=\{f,\quad\overline{f}\in B_2(\mathbb{C}\setminus\overline{G},\nu)\},$$

where the bar over f indicates complex conjugation.

A Hilbert space $J_2(\mathbb{C}\setminus\overline{G},\nu)$ can be considered as a Banach space with the norm $\|\cdot\|_\nu$.

The function $\widetilde{f}(\xi)=(\frac{1}{(\xi-z)^2},f(z))_{B_2(G,\mu)}$ belongs to the space $\widetilde{B}_2(G,\mu)$. By condition of the norm, $\|\cdot\|_{B_2(G,\mu)}$ and $\|\cdot\|_\nu$ are equivalent, therefore the spaces $\widetilde{B}_2(G,\mu)$ and $B_2(\mathbb{C}\setminus\overline{G},\nu)$ are isomorphic. It means that $\widetilde{f}(\xi)$ belongs to the space $B_2(\mathbb{C}\setminus\overline{G},\nu)$ and hence, $\overline{\widetilde{f}(\xi)}$ belongs to the space $J_2(\mathbb{C}\setminus\overline{G},\nu)$.

The equality

$$Tf(\xi)=(f(z),\tfrac{1}{(z-\xi)^2})_{B_2(G,\mu)}=\overline{(\tfrac{1}{(z-\xi)^2},f(z))}_{B_2(G,\mu)}=\overline{\widetilde{f}(\xi)},$$

implies that the operator T acts from the space $B_2(G,\mu)$ into the space $J_2(\mathbb{C}\setminus G,\nu)$ and that it is a linear continuous one-to-one operator.

The operator T^*, conjugate to the operator T, is determined by the equality

$$(Tf(\xi),h(\xi))_\nu=(f(z),T^*h(z))_{B_2(G,\mu)},\quad f\in B_2(G,\mu),\ h\in J_2(\mathbb{C}\setminus\overline{G},\nu).$$

Let us find the explicit form of the operator T^*

$$\begin{aligned}(Tf(\xi),h(\xi))_\nu&=\int_{\mathbb{C}\setminus\overline{G}}Tf(\xi)\cdot\overline{h(\xi)}\,d\nu(\xi)=\\
&=\int_{\mathbb{C}\setminus\overline{G}}\int_G f(z)\overline{\frac{1}{(z-\xi)^2}}\,d\mu(z)\cdot\overline{h(\xi)}\,d\nu(\xi)=\\
&=\int_G f(z)\int_{\mathbb{C}\setminus\overline{G}}\overline{\frac{1}{(z-\xi)^2}}\cdot\overline{h(\xi)}\,d\nu(\xi)\,d\mu(z)=\\
&=\int_G f(z)\overline{\int_{\mathbb{C}\setminus\overline{G}}\frac{1}{(z-\xi)^2}\cdot h(\xi)\,d\nu(\xi)}\,d\mu(z)=\end{aligned}$$

$$=\int_G f(z)\cdot\overline{T^*h(z)},\,d\mu(z)=(f(z),T^*h(z))_{B_2(G,\mu)}.\tag{8}$$

Thus, the operator T^* conjugate to T acts from the space $J_2(\mathbb{C}\setminus\overline{G},\nu)$ into the space $B_2(G,\mu)$ and has the form

$$T^*h(z)=\int_{\mathbb{C}\setminus\overline{G}}h(\xi)\frac{1}{(z-\xi)^2}\,d\nu(\xi),\quad h\in J_2(\mathbb{C}\setminus\overline{G},\nu).$$

In particular, it means that the operator $T^*\circ T \stackrel{def}{=} \mathcal{E}$ is a self-adjoint operator (see, e.g., [2], p. 222), acting in the space $B_2(G)$:

$$\mathcal{E}f(z)=\int_{\mathbb{C}\setminus\overline{G}}(f(\tau),\tfrac{1}{(\tau-\xi)^2})_{B_2(G,\mu)}\tfrac{1}{(z-\xi)^2}\,d\nu(\xi).$$

Moreover, the operator $\mathcal{E}$ is an automorphism of the space $B_2(G,\mu)$. Operator $\mathcal{E}$ being a self-adjoint operator, has a unique positive square root $R: B_2(G,\mu)\to B_2(G,\mu)$ (see, e.g., [14], pp.281, 282) such that $\mathcal{E}=R\circ R$. The operator R is an automorphism of the space $B_2(G,\mu)$ as well.

Then,

$$R\circ Rf(z)=\int_{\mathbb{C}\setminus\overline{G}}(f(\tau),\tfrac{1}{(\tau-\xi)^2})_{B_2(G)}\tfrac{1}{(z-\xi)^2}\,d\nu(\xi).$$

Invoking that the operator $\mathcal{E}$ is one-to-one and using the same argumentation as in ([15], p. 128), one can demonstrate that

$$f(z)=\int_{\mathbb{C}\setminus\overline{G}}(f(\tau),\tfrac{1}{(\tau-\xi)^2})_{B_2(G,\mu)}S\circ S\tfrac{1}{(z-\xi)^2}\,d\nu(\xi)=$$
$$=\int_{\mathbb{C}\setminus\overline{G}}(f(\tau),S\tfrac{1}{(\tau-\xi)^2})_{B_2(G,\mu)}S\tfrac{1}{(z-\xi)^2}\,d\nu(\xi), \tag{9}$$

where operator S is an inverse operator to the operator R, i.e. $R^{-1}\stackrel{\text{об}}{=}S$. The necessity is proved.

Sufficiency. Let the system $\{S\frac{1}{(z-\xi)^2}\}_{\xi\in\mathbb{C}\setminus\overline{G}}$ be an orthosimilar system in the space $B_2(G,\mu)$. It means that any element $f\in B_2(G,\mu)$ can be written in the form

$$f(z)=\int_{\mathbb{C}\setminus\overline{G}}(f(\tau),S_\tau\tfrac{1}{(\tau-\xi)^2})_{B_2(G,\mu)}S_z\tfrac{1}{(z-\xi)^2}\,d\nu(\xi),\quad z\in G.$$

Using ([15], p 128), one can demonstrate that

$$f(z)=\int_{\mathbb{C}\setminus\overline{G}}(f(\tau),S_\tau\tfrac{1}{(\tau-\xi)^2})_{B_2(G,\mu)}S_z\tfrac{1}{(z-\xi)^2}\,d\nu(\xi)=$$
$$=\int_{\mathbb{C}\setminus\overline{G}}(S\circ Sf(\tau),\tfrac{1}{(\tau-\xi)^2})_{B_2(G,\mu)}\tfrac{1}{(z-\xi)^2}\,d\nu(\xi),\quad z\in G. \tag{10}$$

Let us introduce the notation $S\circ S\stackrel{\text{об}}{=}A$. Since the operator S has a continuous inverse operator then, due to Theorem C, the operator A has a positive lower bound and hence, one can introduce in the space $B_2(G,\mu)$ an equivalent integral norm

$$\|f\|_1=\sqrt{(Af,f)_{B_2(G,\mu)}}, \tag{11}$$

which constructs a scalar product

$$(f,g)_1=(Af,g)_{B_2(G,\mu)},\quad f,g\in B_2(G,\mu).$$

Note that

$$(A^{-1}f,g)_1=(f,g)_{B_2(G,\mu)}.$$

One has

$$f(z)=\int_{\mathbb{C}\setminus\overline{G}}(f(\tau),\tfrac{1}{(\tau-\xi)^2})_1\tfrac{1}{(z-\xi)^2}\,d\nu(\xi),\quad z\in G$$

for any $f \in B_2(G,\mu)$. That is the system of functions $\{\frac{1}{(\tau-\xi)^2}\}_{\xi\in\mathbb{C}\backslash\overline{G}}$ is an orthosimilar system with respect the measure ν in the space $B_2(G,\mu)$ with the norm $\|\cdot\|_1$. According to Theorem A

$$\|A^{-1}f\|_1^2 = \int_{\mathbb{C}\backslash\overline{G}} |(A^{-1}f(\tau), \tfrac{1}{(\tau-\xi)^2})_1|^2\, d\nu(\xi) =$$

$$= \int_{\mathbb{C}\backslash\overline{G}} |(f(\tau), \tfrac{1}{(\tau-\xi)^2})_{B_2(G,\mu)}|^2\, d\nu(\xi) = \int_{\mathbb{C}\backslash\overline{G}} |\widetilde{f}(\xi)|^2\, d\nu(\xi) = \|\widetilde{f}\|_\nu^2. \tag{12}$$

Furthermore, $\|\widetilde{f}\|_{\widetilde{B}_2(G,\mu)} = \|f\|_{B_2(G,\mu)}$. Due to Lemma 1 (see the equality 11), the norms $\|\cdot\|_{B_2(G,\mu)}$ and $\|\cdot\|_1$ are equivalent. Manifestly, there are such constants $C_3, C_4 > 0$ that

$$C_3\|f\|_1 \le \|A^{-1}f\|_1 \le C_4\|f\|_1, \quad f \in B_2(G,\mu).$$

Equation (12) entails that the norms $\|\cdot\|_{\widetilde{B}_2(G,\mu)}$ and $\|\cdot\|_\nu$ are equivalent. The theorem is proved.

Theorem 4. *Let us assume that there is an operator S, realizing the automorphism of the space $B_2(G,\mu)$, which transforms the family of reproducing kernels $\{K_H(z,t)\}_{t\in G}$ into the family of Hilbert kernels $\{\frac{1}{(z-\tau)^2}\}_{\tau\in\mathbb{C}\backslash\overline{G}}$. Then, in the space $\widetilde{B}_2(G,\mu)$, one can introduce an equivalent integral norm of the form*

$$\|\widetilde{f}\|_\nu = \sqrt{\int_{\widetilde{B}_2(G,\mu)} |\widetilde{f}(\xi)|^2\, d\nu(\xi)},$$

where the measure ν is determined as follows. The operator S defines the mapping

$$\tau = \rho(t);\ \rho : G \to \mathbb{C}\backslash\overline{G}$$

from the equality

$$SK_H(z,t) = \tfrac{1}{(z-\rho(t))^2}, \quad t \in G.$$

Let P be a manifold in G. Then $Q \overset{def}{=} \rho(P)$ is a manifold in $\mathbb{C}\backslash\overline{G}$, and the measure $\nu(Q) \overset{def}{=} \mu(P)$.

Proof. The system of elements $\{K_H(z,t)\}_{t\in G}$ is an orthosimilar system in the space $B_2(G,\mu)$ with the measure μ (see corollary to Theorem 1). It means that any function $f \in B_2(G,\mu)$ can be represented in the form

$$f(z) = \int_G (f(\tau), K_H(\tau,t))_H K_H(z,t)\, d\mu(t), \quad z \in G.$$

By condition of the Theorem, operator S realizes the automorphism of the space $B_2(G,\mu)$ and translates the family of reproducing kernels $\{K_H(z,t)\}_{t\in G}$ onto the family of Hilbert kernels $\{\frac{1}{(z-\tau)^2}\}_{\tau\in\mathbb{C}\backslash\overline{G}}$. Then,

$$K_H(z,t) = S_z^{-1}\tfrac{1}{(z-\rho(t))^2}, \quad t \in G$$

and

$$f(z) = \int_G (f(\tau), S_\tau^{-1}\tfrac{1}{(\tau-\rho(t))^2})_H S_z^{-1}\tfrac{1}{(z-\rho(t))^2}\, d\mu(t), \quad z \in G.$$

Substituting the variable in the latter integral $\xi = \rho(t)$, and invoking that $d\mu(\rho^{-1}(\xi)) = d\nu(\xi)$, one obtains

$$f(z) = \int_G (f(\tau), S_\tau^{-1}\tfrac{1}{(\tau-\xi)^2})_H S_z^{-1}\tfrac{1}{(z-\xi)^2}\, d\nu(\xi), \quad z \in G.$$

Whence, according to Theorem 2, one can introduce an equivalent integral norm of the form

$$\|\widetilde{f}\|_\nu = \sqrt{\int_{\widetilde{B}_2(G,\mu)} |\widetilde{f}(\xi)|^2\, d\nu(\xi)}$$

in the Hilbert space $\widetilde{B}_2(G,\mu)$. Theorem 4 is proved.

4. Example

Let us take the upper half-plane $U = \{z \in \mathbb{C} : \Im z > 0\}$ as the domain G and the planar Lebesgue measure v as the measure μ.

Consider the space $B_2(U,v)$, consisting of functions holomorphic in U and summable with a square of the module over the planar Lebesgue measure, i.e.

$$\|f\|^2_{B_2(U,v)} = \int_U |f(z)|^2\,dv(z) < \infty.$$

The system of functions $\frac{1}{(z-\xi)^2}$ is complete in the space $B_2(U,v)$ (see [7]). It is known (see, e.g., [16]), that if G is an arbitrary simply connected domain and $\varphi : G \to D$ is a conformal mapping of the domain G onto a unit circle D, then the reproducing kernel of the space $B_2(G,v)$ has the form

$$K_{B_2(G,v)}(z,\xi) = \frac{1}{\pi}\cdot\frac{\varphi'(z)\overline{\varphi'(\xi)}}{(1-\varphi(z)\overline{\varphi(\xi)})^2}, \quad z,\xi \in G.$$

The function $\varphi(z) = \frac{z-i}{z+i}$ maps the upper half-plane U onto a unit circle D conformally. Whence, one can readily demonstrate that

$$K_{B_2(U,v)}(z,\xi) = -\frac{1}{\pi}\cdot\frac{1}{(z-\overline{\xi})^2}, \quad z,\xi \in U. \tag{13}$$

By virtue of Theorem 1, any function $f \in B_2(U,v)$ can be represented in the form

$$f(z) = \int_G (f(\tau), K_{B_2(U,v)}(\tau,t))_{B_2(U,v)} K_{B_2(U,v)}(\xi,t)\,dv(t), \quad \xi \in U.$$

Consider the operator S

$$Sf(z) = -\pi\cdot f(z), \quad z \in U$$

on functions f from $B_2(U,v)$. Evidently, S is an automorphism of the space $B_2(U,v)$. Then, it follows from (13), that

$$SK_{B_2(U,v)}(z,\xi) = (-\pi)\cdot\left(-\frac{1}{\pi}\right)\cdot\frac{1}{(z-\overline{\xi})^2} = \frac{1}{(z-\overline{\xi})^2}, \quad z,\xi \in U.$$

If $\xi \in U$, then $\overline{\xi} \in \mathbb{C}\backslash\overline{U}$. Thus, the operator S satisfies the condition of Theorem 4; translates family of functions $\{K_{B_2(U,v)}(z,\xi)\}_{\xi\in U}$ onto a family of functions $\{\frac{1}{(z-\tau)^2}\}_{\tau\in\mathbb{C}\backslash\overline{U}}$. Obviously, $\rho(\xi) = \overline{\xi}$ (see formulation of Theorem 4). By Theorem 4, one can introduce an equivalent integral norm of the form

$$\|\widetilde{f}\| = \sqrt{\int_{\mathbb{C}\backslash U} |\widetilde{f}(\xi)|^2\,dv(\overline{\xi})} = \sqrt{\int_{\mathbb{C}\backslash U} |\widetilde{f}(\xi)|^2\,dv(\xi)}$$

in the space $B_2(U,v)$. The latter means that the spaces $B_2^*(U,v)$ and $B_2(\mathbb{C}\backslash\overline{U},v)$ are isomorphic.

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