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AN ANALOGUE OF THE PALEY-WIENER THEOREM AND ITS APPLICATIONS TO THE OPTIMAL RECOVERY OF ENTIRE FUNCTIONS

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Abstract. Full analogues of the Paley-Wiener theorem and, in a multidimensional case, of the Plancherel-Pólya theorem on the structure of the Fourier transform of any entire function $f \in W^2$ are found in a fundamentally new form in terms of the language of distributions. The analogue of the Paley-Wiener theorem is formulated for the Wiener class W^p of entire functions of the exponential type in $\mathbb{C}^n$ with traces in a real subspace $\mathbb{R}^n$ belonging to the space $L^p(\mathbb{R}^n)$, where $1 < p < \infty$. The results are applied to the problem of the best analytic continuation from a finite set of functions of the Wiener class. Of special interest is the description of the existence conditions for constructive algebraic formulae of characteristics for the optimal recovery of linear functionals.

Key words: Wiener class of entire functions, Fourier transform, distributions, optimal linear algorithm, Chebyshev polynomial.

1. Introduction

The main object of scientific inquiry in the given paper is the Wiener class of entire functions in $\mathbb{C}^n$ of the exponential type with traces on a real subspace belonging to $L^p(\mathbb{R}^n)$, where $1 < p < \infty$. In a one-dimensional case, it is the class W^p_σ of such functions of the type $\leqslant \sigma$,where $\sigma > 0$. The Paley-Wiener theorem describing the Fourier transform for functions of the class W^2_σ [1], and its multi-dimensional version by M. Plancherel and D. Pólya [2] are well-known results of the theory of Fourier integrals and have numerous applications. The given paper provides an analogue of the Paley-Wiener theorem for functions of W^p_σ class ($1 < p < \infty$) admitting a multidimensional generalization. Moreover, basic characteristics for the optimal recovery from a finite set of functions of the Wiener class are investigated in applications. The paper is closely related to the works [3], where similar issues are considered for the class W^p_σ, $1 < p < 2$, and [4], where estimate of the best analytic continuation for the case $p = 2$, $n > 1$ is investigated. It is a revised version of the preprint [5] by the same authors.

2. Analogue of the Paley-Wiener theorem

1°. Preliminaries. Let $\sigma > 0$. The classical Paley-Wiener theorem states that a space W^2_σ is equivalent to a space $\mathcal{F}^{-1}L^2[-\sigma,\sigma]$, where $L^2[-\sigma,\sigma]$ is considered as a subspace $L^2(\mathbb{R})$, consisting of all functions with a support in $[-\sigma,\sigma]$, and $\mathcal{F}^{-1}$ is an inverse Fourier transform

$$f(x) := \mathcal{F}^{-1}F\,(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{\imath x\xi} F(\xi) d\xi, \quad x \in \mathbb{R}; \quad F \in L^2[-\sigma,\sigma].$$

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The element $F = \hat{f} := \mathfrak{F}f$ is uniquely defined by the function f, with analytic continuation in $\mathbb{C}$ belonging to W^2_σ. Moreover, due to the Plancherel theorem, the norm f in W^2_σ, i.e. in $L^2(\mathbb{R})$, coincides with the norm F in $L^2[-\sigma,\sigma]$.

According to the Hausdorff-Young theorem, the Fourier transform of the function $f \in L^p(\mathbb{R})$ belongs to $L^q(\mathbb{R})$, if $1 \leqslant p \leqslant 2$. Here $1/p + 1/q = 1$. Moreover,

$$\|\hat{f}\|_{L^q(\mathbb{R})} \leqslant (2\pi)^{1/q-1/2}\|f\|_{L^p(\mathbb{R})}.$$

When $p > 2$, the Fourier transform $\hat{f}$ of the function f in $L^p(\mathbb{R})$ can appear to be a distribution of a positive order. In particular, if $p > 2$ and the integer o satisfies the inequality $o > 1/2 - 1/p$, then $\hat{f}$ is a distribution in $\mathbb{R}$ of the order $\leqslant o$ (see [6], Theorem 7.6.6).

Thus, we conclude that any function $f \in W^p_\sigma$, $1 < p < \infty$ admits the integral representation

$$f = \mathcal{F}^{-1}\hat{f}, \tag{1}$$

where $\hat{f} \in \mathcal{E}'_{[-\sigma,\sigma]}$ is a Fourier transform of f, and $\mathcal{E}'_{[-\sigma,\sigma]}$ is a standard space of distributions with supports in $[-\sigma,\sigma]$. However, the set $\mathcal{F}[W^p_\sigma]$, or the space of Fourier transforms for functions of the Wiener class W^p_σ, is characterized by the space $\mathcal{E}'_{[-\sigma,\sigma]}$ only partially. As it has been demonstrated in [3], $\mathcal{F}[W^p_\sigma]$, $1 < p < 2$, is a proper subspace $L^q[-\sigma,\sigma]$ if $1/p + 1/q = 1$.

2°. Distributions associated with elements $l^p(\mathbb{Z})$. In order to characterize the space $\mathcal{F}[W^p_\sigma]$, $p > 1$, consider the Banach space $l^p(\mathbb{Z})$, i.e. the set of all bilateral sequences $c = (c_n)_{n\in\mathbb{Z}}$ of complex numbers such that

$$\|c\|_{l^p(\mathbb{Z})} := \Big(\sum_{n\in\mathbb{Z}} |c_n|^p\Big)^{1/p} < \infty.$$

One can assume without loss of generality that $\sigma = \pi$, because $f(z) \in W^p_\sigma$ if and only if $f(\pi z/\sigma) \in W^p_\pi$. The following lemma is necessary to motivate a definition of the Fourier transform for Wiener class functions, other than the traditional one.

Lemma 1. *Let f be an entire function of an exponential type $\leqslant \pi$ such that $\hat{f} = \mathcal{F}f \in L^q[-\pi,\pi]$, where $1 < q < \infty$. Then,*

$$\langle \mathcal{F}f, \varphi\rangle := \int_{-\pi}^{\pi} \mathcal{F}f(t)\varphi(t)dt = \sum_{k\in\mathbb{Z}} f(k)\hat{\varphi}(k),\ \ \hat{\varphi}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-\imath tx}\varphi(t)dt,\ \ x \in \mathbb{R} \tag{2}$$

holds for any smooth function $\varphi \in C^\infty(\mathbb{R})$.

◀ Using notation of the above lemma, one obtains

$$\langle \mathcal{F}f, \varphi\rangle = \sum_{k\in\mathbb{Z}} c_k\hat{\varphi}(k) \tag{3}$$

from the generalized Parseval identity (See, e.g., [7], p. 255). Here $c := \{c_k = \sqrt{2\pi}d_{-k},\ k \in \mathbb{Z}\}$, $\{d_k,\ k \in \mathbb{Z}\}$ are Fourier coefficients for the function $\hat{f}$. In order to prove (2), one has to apply the inversion formula of the Fourier transform. ▶

Remark. 1. A result close to Lemma 1 can be found in [8, p. 115].

2. In particular, equalities (2), (3) hold for $f \in W^p_\pi$ provided that $1/p + 1/q = 1$, $1 < p < 2$. In this case, $c \in l^p(\mathbb{Z})$ (see [3]).

Every sequence $c \in l^p(\mathbb{Z}), 1 < p < \infty$ determines a linear functional T_c in the space $C^\infty(\mathbb{R})$ by the following formula in the same notation as in (3):

$$\langle T_c, \varphi\rangle := \sum_{k\in\mathbb{Z}} c_k\hat{\varphi}(k), \quad \varphi \in C^\infty(\mathbb{R}). \tag{4}$$

In the meaning of the theory of distributions, Lemma 1 indicates that the Fourier transform of the trace of an entire function $f \in W^p_\pi$ in $\mathbb{R}$ with $1 < p < 2$ coincides with the functional $T_{c(f)}$, where $c(f) := \{f(k),\ k \in \mathbb{Z}\}$.

Lemma 2. *Let $1 < p < \infty$. The functional T_c, determined by the formula (3), is a distribution on a real axis with a support in the interval $[-\pi, \pi]$ of the order $\leqslant 1$.*[1]

◀ Invoking that $\varphi \in C^\infty(\mathbb{R})$, and integrating by parts, one obtains

$$\hat{\varphi}(k) = \frac{1}{-\imath k\sqrt{2\pi}}\Big[(-1)^k\left(\varphi(\pi) - \varphi(-\pi)\right) - \int_{-\pi}^{\pi} e^{-\imath kt}\varphi'(t)dx.\Big]$$

Therefore,

$$|\hat{\varphi}(k)| \leqslant [4\pi \sup\{|\varphi'(t)|,\ t \in [-\pi,\pi]\}]/(\sqrt{2\pi}|k|),\ k \in \mathbb{Z}\backslash\{0\}.$$

The latter, together with (3) and the Hölder inequality, leads to estimation of the module of the functional T_c:

$$\begin{aligned}\sqrt{2\pi}|\langle T_c, \varphi\rangle| &\leqslant |c_0|\Big(2\pi \sup_{t\in[-\pi,\pi]} |\varphi(t)|\Big) + \sum_{k\neq 0}\frac{|c_k|}{|k|}\Big(4\pi \sup_{t\in[-\pi,\pi]}|\varphi'(t)|\Big)\\ &\leqslant \|c\|_{l^p(\mathbb{Z})}\Big(2\pi \sup_{t\in[-\pi,\pi]}|\varphi(t)| + \Big(\sum_{k\neq 0}\frac{1}{|k|^q}\Big)^{1/q} 4\pi \sup_{t\in[-\pi,\pi]}|\varphi'(t)|\Big).\end{aligned}$$

This inequality implies that T_c is a distribution of the order $\leqslant 1$ in $\mathbb{R}$ with the support in $[-\pi, \pi]$. ▶

3°. Analogue of the Paley-Wiener theorem. Let us prove the analogue of the Paley-Wiener theorem for the Wiener class W^p_σ. In contrast to the work [3], we introduce another support in the space $\mathcal{F}[W^p_\sigma]$ of Fourier transforms of functions of this class. This approach allows one to obtain the desired result not only in a one-dimensional case with any $p \in (1, \infty)$, it also gives a possibility to find its complete analogue in a multi-dimensional case, i.e. the analogue of the Plancherel-Pólya theorem for W^2_σ in $\mathbb{C}^n$, $n > 1$ (see [2]).

Let us indicate the space of all distributions of the form T_c in $\mathbb{R}$, where $c \in l^p(\mathbb{Z})$, and $1 < p < \infty$ by $\mathcal{E}'^p_{[-\pi,\pi]}$ (see (3)). The classical Plancherel-Pólya theorem [8], [9, p. 152] is necessary for revealing the connection between $\mathcal{E}'^p_{[-\pi,\pi]}$ and the space $\mathcal{F}[W^p_\sigma]$.

Theorem 1. *Let $1 < p < \infty$.*

1) *For any sequence $\{c_k,\ \ k \in \mathbb{Z} \in l^p(\mathbb{Z})\}$, the series*

$$f(z) = \sum_{k=-\infty}^{\infty}(-1)^n c_k \frac{\sin \pi z}{\pi(z-k)}, \quad z \in \mathbb{C} \tag{5}$$

converges in the norm $L^p(\mathbb{R})$ (and uniformly on every compact in $\mathbb{C}$) to the function $f \in W^p_\pi$, which is the only solution of the interpolation problem $f(k) = c_k,\ k \in \mathbb{Z}$.

2) *Conversely, for any function $f \in W^p_\pi$ the sequence*

$$c(f) := \{f(k),\ k \in \mathbb{Z}\} \tag{6}$$

belongs to $l^p(\mathbb{Z})$.

3) *The norms $f \mapsto \|f\|_{L^p(\mathbb{R})}$ and $f \mapsto \|c(f)\|_{l^p(\mathbb{Z})}$, introduced in the space W^p_π, are equivalent.*

Lemma 3. *In notation of the formulae (3), (4), (6) the relation $\mathcal{F}f = T_c$, with $c = c(f)$, and $f \in W^p_\pi$, holds and is equivalent to the equation*

$$f = \mathcal{F}^{-1}T_{c(f)}. \tag{7}$$

[1] We use the terminology of [6, Definition 2.1.1].

◀ Lemma 2 states that $T_c \in \mathcal{E}'_{[-\pi,\pi]}$, therefore the inverse Fourier transform of the functional T_c is an element of $\mathcal{S}'(\mathbb{R})$. In fact, it is an entire function of the exponential type $\leqslant \pi$, defined by the formula

$$\begin{aligned} \mathcal{F}^{-1}T_c &= \frac{1}{2\pi}\langle T_c, e^{\imath z\xi}\rangle = \sum_{k\in\mathbb{Z}} c_k \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\imath(z-k)\xi} d\xi \\ &= \sum_{k\in\mathbb{Z}} c_k \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(z-k)\xi \, d\xi = \sum_{k\in\mathbb{Z}} c_k(-1)^k \frac{\sin\pi z}{\pi(z-k)}. \end{aligned}$$

This formula and Theorem 1 lead to (7) ▶

Theorem 2. *If $p > 1$, the Fourier transform determines the topological isomorphism of spaces W^p_π and $\mathcal{E}'^p_{[-\pi,\pi]}$.*

◀ Theorem 1 entails that if $f \in W^p_\pi$, then the sequence $c(f)$ (see (6)) belongs to $l^p(\mathbb{Z})$, and the function f admits the representation (5). Whence, invoking Lemma 3, one concludes that $\mathcal{F}f = T_{c(f)}$ is an element of $\mathcal{E}'^p_{[-\pi,\pi]}$, i.e. the Fourier transform maps W^p_π into $\mathcal{E}'^p_{[-\pi,\pi]}$. Since $W^p_\pi \hookrightarrow \mathcal{S}'(\mathbb{R})$,[1] this mapping is injective. One has only to prove that it is surjective. Let us fix the mapping $T_c \in \mathcal{E}'^p_{[-\pi,\pi]}$, where $c \in l^p(\mathbb{Z})$. Let us determine f by the formula (5). Then, by virtue of Theorem 1, $f \in W^p_\pi$ and $c(f) = c$ (see (6)). Equation (7) provides $\mathcal{F}f = T_c$. Hence, there exists an algebraic isomorphism of the space W^p_π onto $\mathcal{E}'^p_{[-\pi,\pi]}$.

According to Theorem 1, two equivalent norms can be introduced into $W^p_\pi (p > 1)$; one of the norms is determined by injection into $L^p(\mathbb{R})$, the other is induced from $l^p(\mathbb{Z})$. They turn W^p_π into a Banach space. Due to the algebraic isomorphism $f \mapsto \mathcal{F}f = T_{c(f)}$, this is equivalent to the statement on topological isomorphism of spaces W^p_π and $\mathcal{E}'^p_{[-\pi,\pi]}$ ▶

Theorem 2 is close to the Paley-Wiener-Schwartz theorem (see [6], Theorem 7.3.1) and its generalizations [10].

4°. Counterexample. If $1 < p \leqslant 2$, then distributions $\mathcal{E}'^p_{[-\pi,\pi]}$ are functions of the class $L^q[-\pi,\pi]$ provided that $1/p + 1/q = 1$, and if $1 < p < 2$ they do not exhaust the space $L^q[-\pi,\pi]$ (see [3]). When $p > 2$, distributions $\mathcal{E}'^p_{[-\pi,\pi]}$ can be not only functionals and even not only measures given on $[-\pi,\pi]$. Let us demonstrate that the upper estimate of the order of the distribution T_c given in Lemma 2 is exact for $p > 2$.

Let us make use of the following well-known theorem (see, e.g., Theorem 6.4, p. 326 in [7]).

Theorem 3. *Let*

$$\sum_{k=1}^{\infty} a_k \cos(n_k t) + b_k \sin(n_k t), \quad t \in [-\pi,\pi],$$

be a lacunary trigonometric series, where $n_{k+1}/n_k \geqslant q > 1$. If it can be summed up by a linear method of summation [2] on the set of a positive measure, then $\sum_{k=1}^{\infty} a_k^2 + b_k^2 < \infty$.

Consider an entire function

$$f(z) = \sum_{k=1}^{\infty} (\sin\pi z)/[\pi\sqrt{k}(z-2^k)], \quad z \in \mathbb{C}.$$

By virtue of Theorem 3, $f \in W^p_\pi$ for every $p > 2$. It can be readily seen that the Fourier series of the Fourier transform $\hat{f}$ for f, given on $[-\pi,\pi]$, has the form

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{2\pi k}} \left(\cos(2^k t) - \imath \sin(2^k t)\right). \tag{8}$$

[1] $\mathcal{S}'(\mathbb{R})$ is a space of distributions of slow growth.

[2] For instance, the Abel-Poisson method [7, p.135].

Theorem 3 demonstrates that this series can be summed up by the Abel-Poisson method almost nowhere on $[-\pi, \pi]$. However, the Fourier series of the measure given in $[-\pi, \pi]$ can be summed up by this method almost everywhere (see, e.g., [11, p. 52]). Therefore, the series (5) can not be a Fourier series of any measure.

Thus, the sequence $c(f)$ belongs to all spaces $l^p(\mathbb{Z})$ when $p > 2$. However, the distribution $T_{c(f)}$ is not a measure.[1] Finally, due to Lemma 2, we conclude that the order of $T_{c(f)}$ equals to 1.

5°. Multidimensional generalizations. Let us assume that $\sigma_j > 0,\ j = 1, ..., n;\ W^p_\sigma = \{f\}$ is a space of entire functions of the exponential type satisfying the inequality

$$|f(z)| \leqslant C_f(\varepsilon)\ \exp\{\sum_{j=1}^{n}(\sigma_j + \varepsilon)|z_j|\} \quad \forall\ z \in \mathbb{C}^n \tag{9}$$

for any $\varepsilon > 0$, and that $f \in L^p(\mathbb{R}^n)$, where $1 < p < \infty$. Applying Theorem 1 with respect to every variable by the induction method, one obtains its complete analogue for the class W^p_σ in $\mathbb{C}^n$ (see [8]). Multidimensional variants of other results mentioned above are also true. We omit their evident formulations and proofs.

3. Optimal recovery from a finite set in the Wiener class W^p_σ, $1 < p < \infty$

1°. Scheme of the optimal recovery. Theorem 3 will be applied below to obtain characteristics of the best analytic continuation from a finite set of entire functions from W^p_σ when $p > 2$. This is a problem of recovery of delta functionals. The scheme for optimal recovery of a linear functional is known (see, e.e., [12]-[14]). Taking into account the specific character of the case when the information space is finite-dimensional, we consider a modification of the scheme used in [3] for optimal extrapolation in W^p_σ for $1 < p \leqslant 2$.

Let us assume that V is a vector space, $T : V \to B$ is an algebraic isomorphism of the space V into a Banach space B, while both V, and B are considered over one and the same field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let us determine the norm in V, assuming that $\|f\|_V := \|Tf\|_B$ for $f \in V$, and thus turning V into a normed space as well. One can easily verify that $\|L\|_{V'} = \|L \circ T^{-1}\|_{B'}$ for every continuous linear functional L on V. Here B' is a conjugate space with respect to the space B and it is a Banach space in a standard topology defined by a functional norm. Let $U = \{f \in V :\ \|f\|_V \leqslant R\}$; $L_1, \ldots, L_N$ be linearly independent linear functionals given on U. Consider the recovery problem for any fixed linear functional L on U proceeding from information on $L_1, \ldots, L_N$.

Recall the basic characteristics of optimal recovery. Any mapping of the form $A : U \to \mathbb{C}$ is termed as *algorithm*. We limit our consideration by linear algorithms, i. e. A type algorithms

$$A(f) = \ell(a; L_1, \ldots, L_N) := \sum_{k=1}^{N} a_k L_k(f), \quad f \in U, \quad a = (a_1, \ldots, a_N) \in \mathbb{K}^N. \tag{10}$$

The value

$$E(a; L, U) = \sup\{|L(f) - A(f)|,\ f \in U\}$$

is called an *error of the algorithm* A. The value

$$\varOmega(L, U) = \inf\{E(a; L, U),\ a \in \mathbb{K}^N\}^2$$

[1] This example is a small modification of the example considered in [8, p. 142-143].

[2] A more general definition can be found in § 4, 2°

is called an *optimal (unavoidable) recovery error* of the functional L. If $\Omega(L,U) = E(\alpha; L, U)$ when a certain $a = \alpha \in \mathbb{K}^N$, then the algorithm

$$A_0 = \ell(\alpha; L_1, \dots, L_N) \tag{11}$$

(see (10)) is called an *optimal linear recovery algorithm* of the functional L. An element $f_0 \in U$ is said to be *extremal,* if the following equality holds:

$$E(\alpha; L, U) = |L(f_0) - A_0(f_0)|.$$

We will need the funadamental result of the theory of approximation in normed spaces (see, e.g., [15], pp. 17–20).

Lemma 4. *Let $\{e_1, \dots, e_N\}$ be a linearly independent system in B'. Then, for every element $v \in B'$, there exists a vector $\alpha = (\alpha_1, \dots, \alpha_N)$ in $\mathbb{K}^N$ such that*

$$\Big\| v - \sum_{k=1}^{N} \alpha_k e_k \Big\| = \inf_{a \in \mathbb{K}^N} \Big\| v - \sum_{k=1}^{N} a_k e_k \Big\|.$$

If, in addition, B' is a strictly convex space,[1] *then, there exists only one vector $\alpha \in \mathbb{K}^N$ with the above mentioned property.*

An element $\ell(\alpha; e_1, \dots, e_N) = \sum\limits_{k=1}^{N} \alpha_k e_k$ is termed as the element of the linear envelope of the set $\{e_1, \dots, e_N\}$ of *least deviation* from v, or the *Chebyshev polynomial.* For the sake of simplicity, we denote it by $\ell(\alpha)$, provided that it is clear which element is meant.

Let us assume that $\mathcal{L}_0 \in B'$ is a nonzero functional. Consider the set

$$\partial \|\mathcal{L}_0\|_{B'} = \{F \in B : \|F\|_B = 1, \, \mathcal{L}_0(F) = \|\mathcal{L}_0\|_{B'}\},$$

with a simple geometric meaning. Namely, it is a bound of a closed unit ball S in B, belonging to its supporting hyperplane $\{F \in B : \mathcal{L}_0(F) = \|\mathcal{L}_0\|_{B'}\}$. The set $\partial\|\mathcal{L}_0\|_{B'}$ may be empty. It is impossible if, e.g., the ball $S \subset B$ is a weak compact. In particular, it happens if B is a reflexive Banach space (see, e.g., [16, p. 241]). In this case, the set $\partial\|\mathcal{L}_0\|_{B'}$ is termed as a subdifferential of the norm $\mathcal{L} \mapsto \|\mathcal{L}\|_{B'}$ in $\mathcal{L}_0$ [17]. Let G be a group of elements in $\mathbb{K}$ with every module equal to 1 in $\mathbb{K}$. In a general case, the set of all extremal elements determines the set

$$G\,\partial\|\mathcal{L}_0\|_{B'} = \{\lambda F \in B : \quad \lambda \in G, \; F \in \partial\|\mathcal{L}_0\|_{B'}\}$$

for a certain functional $\mathcal{L}_0$, depending on A_0.

Theorem 4. *Let us assume that in the previous notation*

$$\ell(\alpha) = \sum_{k=1}^{N} \alpha_k \, L_k \circ T^{-1} \tag{12}$$

is an element of the linear envelope of the set $\{L_1 \circ T^{-1}, \dots, L_N \circ T^{-1}\}$ of the least deviation from $L \circ T^{-1}$. Then,

1) *the optimal error $\Omega(L,U)$ of recovery of the functional L by means of information on functionals $L_1, \dots, L_N$ given on U equals to*

$$\Omega(L,U) = R\,\|L \circ T^{-1} - \ell(\alpha)\|_{B'}; \tag{13}$$

2) *the optimal linear algorithm for recovery of the functional L on U is defined by the formula* $A_0 = \ell(\alpha; L_1, \dots, L_N)$ (see (11))*, and it is unique provided that the space B' is strictly convex;*

3) *if $\mathcal{L}_0 = L \circ T^{-1} - \ell(\alpha)$ is a nonzero functional and $\partial\,\|\mathcal{L}_0\|_{B'} \neq \varnothing$, then $\{RT^{-1}F_0 : \; F_0 \in G\,\partial\|\mathcal{L}_0\|_{B'}\}$ is a set of all extremal elements .*

[1]Sometimes, such space is termed as strictly normed.

◀ Let us fix $a \in \mathbb{K}^N$. Consider the difference

$$\varDelta_a(f) = L(f) - \sum_{k=1}^{N} a_k L_k(f), \quad f \in V.$$

Applying the formula $f = RT^{-1}F$, where $F = T(f/R) \in B$, and using the definition of the norm of a continuous linear functional, we derive

$$E(a; L, U) = \sup_{f \in U} |\varDelta_a(f)| = R \sup_{\|F\|_B \leqslant 1} \left|\varDelta_a \circ T^{-1}(F)\right| = R \left\|\varDelta_a \circ T^{-1}\right\|_{B'},$$

taking into account that $f \in U$ if and only if $\|T(f/R)\|_B \leqslant 1$.

Whence,

$$\begin{aligned} \varOmega(L, U) &= R \inf_{a \in \mathbb{K}^N} \|L \circ T^{-1} - \ell(a; L_1 \circ T^{-1}, \dots, L_N \circ T^{-1})\|_{B'} \\ &= R \|L \circ T^{-1} - \ell(\alpha; L_1 \circ T^{-1}, \dots, L_N \circ T^{-1})\|_{B'}, \end{aligned} \tag{14}$$

since the lower bound is reached when $\alpha \in \mathbb{K}^N$ due to Lemma 4. Moreover, this lemma leads to a conclusion that an optimal linear algorithm A_0 is unique if B' is a strictly convex space.

One obtains from (14) that an optimal error $\varOmega(L, U)$ equals to 0 if and only if $\mathcal{L}_0 = L \circ T^{-1} - \ell(\alpha)$ is a zero functional. Then, $L = \alpha_1 L_1 + \ldots + \alpha_N L_N$ is the element of V' sought for. In this case, the coefficients $\alpha_1, \ldots, \alpha_N$ are defined uniquely by the functional L since $L_1, \ldots, L_N$ is a system of linearly independent elements V'.

Thus, the statements 1) and 2) of the theorem are true. It remains only to consider the case, when $\mathcal{L}_0$ is a nonzero functional. If $\partial\|\mathcal{L}_0\|_{B'} \neq \varnothing$, then definition of a subdifferential of a norm provides $\varOmega(L, U) = R \|\mathcal{L}_0\|_{B'} = |\mathcal{L}_0(RF)|$ for all $F \in G\,\partial\|\mathcal{L}_0\|_{B'}$. Let us use the symbol $\mathcal{E}$ to designate the set of all elements $f \in U$ such that $f = RT^{-1}F$ for an element $F \in G\,\partial\|\mathcal{L}_0\|_{B'}$. Then, $\varOmega(L, U) = |L(f) - A_0 \circ I(f)|$, $f \in \mathcal{E}$, i.e. $\mathcal{E}$ is the set of all extremal elements belonging to U. Therefore, the statement 3) of the theorem is proved ▶

2°. An example of realization of the optimal recovery scheme. Theorem 4 plays the role of the scheme to be followed for finding characteristics of the optimal recovery of linear functionals. In this case, one comes across difficulties both in calculating coefficients of the Chebyshev polynomial with the least deviation from the desired linear functional and in describing the general form of such functionals, as well as in determining the structure of subdifferential of the considered norm. As it is demonstrated in the following example, these difficulties can be overcome to a great extent for a wide class of spaces of entire functions containing, in particular, the Wiener class W^p_σ, where $1 < p < 2$.

Let $B = B_q = L^q[-\sigma, \sigma]$, where $1 < q < \infty$ and $\sigma > 0$. Let us identify B_q with a subspace of the space $L^q(\mathbb{R})$, consisting of all functions with the support in $[-\sigma, \sigma]$. Let us designate a vector space of distributions of slow growth on $\mathbb{R}$ by $V = V_q$. Fourier transform of each distribution belongs to V_q. This vector space is equivalent to the space of all entire functions $\{f\}$ of the exponential type $\leqslant \sigma$ such that

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} e^{itz} F(t)dt, \quad z \in \mathbb{C}, \tag{15}$$

where F is an arbitrarily fixed element $L^q[-\sigma, \sigma]$. Note that here

$$F(t) = \hat{f}(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itx} f(x)dx, \quad t \in [-\sigma, \sigma] \tag{16}$$

is a Fourier transform of the function $f|_{\mathbb{R}}$. Therefore, the Fourier transform $T = \mathcal{F}$ of the trace $f|_{\mathbb{R}}$ of every function $f \in V_q$ on $\mathbb{R}$ defines an algebraic isomorphism V_q onto B_q. Let us introduce a topology in the space V_q by means of the norm $\|f\|_V := \|\tilde{f}\|_B$, where $\hat{f}$ is a Fourier transform of the function $f|_{\mathbb{R}}$. Then, V_q becomes a Banach space. According to the Hausdorff-Young theorem, the space V_q contains W^p_σ provided that $1/p + 1/q = 1$, $1 < p < 2$. The topology in W^p_σ, induced by the norm of the space V_q, is weaker than the topology induced by the injection W^p_σ into $L^p(\mathbb{R})$. On the contrary, the Hausdorff-Young theorem provides $W^p_\sigma \supset V_q$, if $2 < p < \infty$.

Let us turn to the problem of the best analytic continuation (or optimal extrapolation) of functions $f \in U$, where $U = \{f \in V_q : \|f\| \leqslant R\}$, from a finite set $S = \{z_1, \dots, z_N\} \subset \mathbb{C}$ into the point $z_0 \in \mathbb{C} \setminus S$. In the given case, linear functionals are delta-like

$$L_k(f) = f(z_k),\ f \in U,\ k = 0, 1, ..., N. \tag{17}$$

Invoking that the spaces $\{L^q(\mathbb{R}),\ 1 < q < \infty\}$ are strictly convex (see [15]), applying Theorem 4, in whose notation $L = L_0$, and using the plan for proving Theorem 5 from [3], we verify that all its statements are true for the space under consideration V_q with any $q \in (1, \infty)$ as well. Namely, the following theorem holds.

Theorem 5. *Let $q \in (1, \infty)$. Assume that in the notation mentioned above*

$$P_N(t; \alpha) = \sum_{k=1}^{N} \alpha_k e^{iz_k t}, \quad t \in [-\sigma, \sigma] \tag{18}$$

is an element of the linear envelope of the set $\{e^{iz_j t},\ j = 1, ..., N\}$, with the least deviation from $e^{iz_0 t}$ in the metric $L^p(-\sigma, \sigma)$ provided that $1/p + 1/q = 1$. Then,

1) *an optimal error of extrapolation of any function $f \in U$ into the point z_0 equals to* (see (13), (17), (18))

$$\Omega_N(z_0) := \Omega(L_0, U) = \frac{R}{\sqrt{2\pi}} ||e^{iz_0 t} - P_N(t; \alpha)||_p,$$

where $\|\cdot\|_p$ is a standard norm in $L^p(-\sigma, \sigma)$;

2) *there exists a unique optimal linear algorithm of analytic continuation of the set S into the point z_0, determined by the equality*

$$\omega(f) := [\ell(\alpha)](f) = \sum_{j=1}^{n} \alpha_j f(z_j), \quad f \in U \tag{19}$$

in notation of the formulae (12), (17);

3) *any extremal function $\{f_0\} \subset U$, i. e. a function satisfying the condition $\Omega_N(z_0) = |f_0(z_0) - w(f_0)|$* (see (19), (15), (16)), *has the following property: its Fourier transform $\hat{f}_0 \in L^q(-\sigma, \sigma)$ has the form*

$$\hat{f}_0(t) = \frac{Re^{i\theta}|h(t)|^{p-1}}{\exp\{i \arg h(t)\}||h||_p^{p-1}}, \quad t \in [-\sigma, \sigma],\ \theta \in \mathbb{R},\ h(t) = \frac{e^{iz_0 t} - P_N(t; \alpha)}{\sqrt{2\pi}}.$$

3°. Modification of Theorem 4 for the Wiener class W^p_σ, $1 < p < \infty$. Let us apply the scheme represented in Theorem 4 to the problem of optimal extrapolation from a finite set in the Wiener class W^p_σ, $1 < p < \infty$. It appears that it is much easier (see [3]) to solve the problem if we introduce a topology in the space by means of a norm $f \mapsto \|c(f)\|_{l^p(\mathbb{Z})}$, equivalent to the standard norm in W^p_σ (see the proof of Theorem 2). Meanwhile, it is possible to obtain the desired result not only for a one-dimensional case with any $p \in (1, \infty)$, but also to find its complete analogue in a multidimensional case (see (9), [4]). Taking into account considerations in § 2, 2°, 5°, for the sake of simplicity, we limit our consideration to the space W^p_σ in a one-dimensional case and when $\sigma = \pi$.

Let us consider a finite set $S = \{z_1, \dots, z_N\}$ of pairwise different points in $\mathbb{C}$ and a point $z_0 \in \mathbb{C} \setminus S$. Using the terminology of 7°, consider the problem of the best analytic continuation of the function $f \in W^p_\pi$ from the set S into the point z_0. Such problem is equivalent to the question of optimal recovery of the functional $L_0(f) = f(z_0)$, $f \in W^p_\pi$, based on information about functionals $\{L_k(f) = f(z_k), f \in W^p_\pi;\ k = 1, ..., N\}$ (cp. (17)).

In order to proceed, we need the following statement known from the functional analysis (see, e.g., [16]).

Lemma 5. *Let $1 < p < \infty$. A linear continuous functional L on the space $l^p(\mathbb{Z})$ has the form*

$$L(c) = \sum_{n\in\mathbb{Z}} c_n\delta_n,\ \ c \in l^p(\mathbb{Z}),$$

where $\delta = \{\delta_n,\ n \in \mathbb{Z}\} \in l^q(\mathbb{Z})$ u $1/p + 1/q = 1$. Moreover, the inequality

$$|L(c)| \leqslant \|\delta\|_{l^q(\mathbb{Z})} \cdot \|c\|_{l^p(\mathbb{Z})} \quad \forall\ c \in l^p(\mathbb{Z}),$$

holds and equality is possible if and only if

$$c_n = \lambda e^{-\imath \arg \delta_n}\Big(\frac{|\delta_n|}{\|\delta\|_{l^q(\mathbb{Z})}}\Big)^{q-1} \quad \forall\ n \in \mathbb{Z},\ \lambda \in \mathbb{C},\ |\lambda| = 1.$$

Note that the lemma allows one to find subdifferentials of the norm $\|\cdot\|_{l^q(\mathbb{Z})}$ at the point δ. The following result generalizes Theorem 5.

Theorem 6. *Let $1 < p < \infty$. We assume that*

$$s_n(z) = (-1)^n\frac{\sin\pi z}{\pi(z-n)},\ \ n \in \mathbb{Z};\quad s(z_k) = \{s_n(z_k),\ n \in \mathbb{Z}\},\ \ k = 0, 1, ..., N; \tag{20}$$

$$\ell(\alpha) = \{\ell_n(\alpha),\ n \in \mathbb{Z}\} = \sum_{k=1}^{N} \alpha_k\, s(z_k)$$

is a sequence with the least deviation from the sequence $s(z_0)$ in the metric $l^q(\mathbb{Z})$, where $1/p + 1/q = 1$ (see Lemma 3 and the remark to it). *In notation of Theorem 5, the following statements hold for $U = \{f \in W^p_\pi :\ \|c(f)\|_{l^p(\mathbb{Z})} \leqslant R\}$ when $R > 0$:*

1) *an unavoidable error $\Omega_N(z_0)$ of optimal extrapolation of any functionf $\in U$ from a finite set $S = \{z_1, \dots, z_N\}$ into the point z_0 equals to*

$$\Omega_N(z_0) = R\,\|s(z_0) - \ell(\alpha)\|_{l^q(\mathbb{Z})};$$

2) *there exists the only optimal linear algorithm of analytic continuation of the function $f \in U$ from the set S into the point z_0, determined by the relations $\omega(f) = \sum\limits_{k=1}^{N} \alpha_k\, f(z_k)$;*

3) *any extremal function $f_0 \in U$ has the following property: its values in integral lattice points are determined by the formula*

$$f_0(n) = \lambda e^{-\imath \arg \delta_n}\Big(\frac{|\delta_n|}{\|\delta\|_{l^q(\mathbb{Z})}}\Big)^{q-1},\quad n \in \mathbb{Z},\ \delta_n = s_n(z_0) - \ell_n(\alpha),\ \lambda \in \mathbb{C},\ |\lambda| = R.$$

◄ Consider a linear operator (see (6)) $T : W^p_\pi \to l^p(\mathbb{Z})$, $Tf = c(f)$. Due to Theorems 1, 2, it defines a topological isomorphism. Therefore, the choice of the norm $f \mapsto \|c(f)\|_{l^p(\mathbb{Z})}$ in the space W^p_π indicates the transition from the standard topology in W^p_π to the equivalent one. Representation (5) for any function $f \in U$ allows to describe the general form of linear functionals under consideration:

$$L_k(f) = f(z_k) = \sum_{n\in\mathbb{Z}} c_n s_n(z_k),\ \ f \in U;\quad k = 0, 1, ..., N.$$

Indeed, by means of the Hölder inequality, one obtains from (5) the following estimate:

$$|f(z)| \leqslant \Big(\sum_{n\in\mathbb{Z}} |s_n(z)|^q\Big)^{1/q} \cdot \|c(f)\|_{l^p(\mathbb{Z})},\quad z \in \mathbb{C}.$$

Whence, together with Lemma 5, one concludes that the mentioned functionals are continuous. Thus, all conditions of Theorem 4 hold. Invoking that spaces$\{l^q(\mathbb{Z}),\ 1 < q < \infty\}$ are strictly convex (see [15]), one verifies the statements 1) and 2) of Theorem 6 directly from Theorem 4. The statement 3) follows from the statement 3) of Theorem 3 and Lemma 5 ►

4. Constructive development of optimal linear algorithm and Chebyshev polynomials

Theorem 6 (as well as Theorem 4) claims only that the optimal linear algorithm exists when extrapolation from a finite set in W^p_π takes place. As one can see from Theorem 1 in [4], the exception is the case $p = 2$. The latter provides quite simple algebraic formulae of all characteristics of the best analytic continuation of functions of the Wiener class W^2_σ. Let us demonstrate that such formulae are true for a certain class of subsets $\{U\}$ of a linear space V over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In particular, there is a *constructively* definable linear algorithm

$$A_0 = \alpha_1 L_1 + \ldots + \alpha_N L_N \tag{21}$$

among optimal algorithms. Here, $\{L_1, \ldots, L_N\}$ is a linearly independent system of linear functionals on V.

1°. Some algebraic properties of linear functionals $\{L_1, \ldots, L_N\}$. Assume that

$$Z := \ker L_1 \cap \ldots \cap \ker L_N. \tag{22}$$

It is a linear space of codimension N in V with a cofactor S such that $V = S \oplus Z$ (see, e.g., [18], pp. 16–17).

The following elementary fact of linear algebra will be of use.

Lemma 6. *Let $\{f_1, \ldots, f_N\}$ be a basis in S. The Gramian matrix*

$$G = \begin{pmatrix} L_1(f_1) & \ldots & L_N(f_1) \\ & \ldots & \\ L_1(f_N) & \ldots & L_N(f_N) \end{pmatrix}$$

is invertible, i.e. the determinant of the matrix G is other than 0.

The following formulae give us a possibility to make the expansion $V = C \oplus Z$ explicit. They are known for the case of Hilbert space [19, p.228].

Lemma 7. *Let $\{f_1, \ldots, f_N\}$ be a basis in S. Then, in notation of Lemma* 6, *any element $f \in V$ admits the only representation in the form*

$$f = \frac{-1}{\det G} \det \begin{pmatrix} 0 & L_1(f) & \ldots & L_N(f) \\ f_1 & & & \\ \ldots & & G & \\ f_N & & & \end{pmatrix} + \pi_Z(f), \tag{23}$$

where $\pi_Z(f) \in Z$, and the following equality holds:

$$\pi_Z(f) = \frac{1}{\det G} \det \begin{pmatrix} f & L_1(f) & \ldots & L_N(f) \\ f_1 & & & \\ \ldots & & G & \\ f_N & & & \end{pmatrix} \qquad \forall\ f \in V. \tag{24}$$

◄ Indeed, there is a unique set of constants $c_1, \ldots, c_N \in \mathbb{K}$ and an element $\pi_Z(f) \in Z$ such that $f = c_1 f_1 + \ldots + c_N f_N + \pi_Z(f)$. Applying the functionals $L_1, \ldots, L_N$ to this equality, we obtain a linear system of equations defining unknown coefficients $c_1, \ldots, c_N$. In particular,

$$L_j(f) = c_1 L_j(f_1) + \ldots + c_N L_j(f_N), \quad j = 1, ..., N.$$

Solving the system by the Cramer rule and using the expansion theorem for determinants, one obtains (23). Whence, one readily obtains (24) ►

2°. Development of optimal linear algorithms. By analogy to the works [12]-[13], [4], consider another approach to the problem of optimal recovery of a linear functional L on a subset $U \subset V$ according to information on $L_1, \dots, L_N$. Let us designate a set of all algorithms $A : \mathbb{K}^N \to \mathbb{K}$ recovering L on U via $L_1, \dots, L_N$ by the symbol $\mathcal{A}$. Error $E(A; L, U)$ of the algorithm A is a supremum of the functional $|L(f) - A \circ I(f)|$ for all $f \in U$, where

$$I(f) = (L_1(f), \dots, L_N(f)).$$

The value

$$\Omega(L, U) = \inf\{E(A; L, U),\ A \in \mathcal{A}\}$$

is an optimal recovery error for the functional L. As it is known [13], this measure coincides with a similar characteristics of optimal recovery. The latter has been considered, in particular, in §3, 1° for the case when U is a convex and *circular* set, i.e. it has the following property: the element $\lambda f \in U$ for all $\lambda \in \mathbb{K}$, $|\lambda| = 1$ for any $f \in U$. These quantities do not require a topology to be introduced in the space V. Such operation is necessary in application of methods of functional analysis.

The dual problem for the above mentioned problem of recovery of a fixed linear functional L on U is the problem of calculating the value (see (22)) of $\sup\{|L(f)|,\ f \in U \cap Z\}$ provided that L is limited on U. Let us investigate this problem for a set of circular sets invariant with respect to the mapping $\pi_Z : \ V \to Z$ (see (22)-(24)).[1] There is a *constructively* definable algorithm (Chebyshev polynomial) of the form (21) among optimal algorithms. A similar problem was considered in [12] with additional assumption about existence of an extremal element. Investigations on *existence* of such algorithms can be found in the monograph [13].

Theorem 7. *Let us assume that U is a circular set invariant with respect to the mapping π_Z, and G is the Gramian matrix $(L_j(f_i))_{\substack{i=1,\dots,N\\ j=1,\dots,N}}$, considered in Lemma* 6. *Then, the following statements hold:*

1) *an optimal recovery error $\Omega(L, U)$ for a linear functional L by means of linear functionals $L_1, \dots, L_N$ is determined by the formula*

$$\Omega(L, U) = \sup_{f \in U \cap Z} |L(f)| = \sup_{f \in U} |L(\pi_Z(f))|;$$

2) *a linear algorithm $A_0 = \ell(\alpha; L_1, \dots, L_N)$* (see (21)) *given by the equality*

$$A_0 \circ I(f) = \frac{-1}{\det G} \det \begin{pmatrix} 0 & L_1(f) & \dots & L_N(f) \\ L(f_1) & & & \\ \dots & & G & \\ L(f_N) & & & \end{pmatrix}, \qquad (25)$$

is optimal and the inequality

$$|L(f) - A_0 \circ I(f)| \leqslant \Omega(L, U) \quad \forall\ f \in U$$

holds.

◀ 1. First, let us prove that

$$M := \sup_{f \in U \cap Z} |L(f)| = \inf_{c \in \mathbb{K}} \sup_{f \in U \cap Z} |L(f) - c|. \qquad (26)$$

Designate the right-hand side of the equality by B. Obviously, $B \leqslant M$. In order to prove the inverse inequality, fix $f \in U \cap Z$ such that $L(f) \neq 0$ and the number $c \in \mathbb{K}$ is other than zero. Since $L_1, \dots, L_N$ are linear functionals, then the set $U \cap Z$ is also circular. Hence, the element

$$f_c = -f \, \exp \imath(\arg c - \arg L(f))$$

belongs to $U \cap Z$. Upon elementary transformations, one obtains

$$|L(f) - c| \leqslant |L(f)| + |c| = |L(f_c) - c|.$$

[1]I. e. $\pi_Z(f) \in U$ for all $f \in U$.

Whence,

$$\sup\{|L(f)-c|,\ f\in U\cap Z\} = \sup\{|L(f)|+|c|,\ f\in U\cap Z\}.$$

Therefore, $M \leqslant B$, i.e. the formula (26) holds.

2. If $A:\mathbb{K}^N\to\mathbb{K}$ is an algorithm recovering L on U according to information on $L_1,\ldots,L_N$, then

$$|L(f)-A\circ I(f)| = |L(f)-A(L_1(f),\ldots,L_N(f))| = |L(f)-A(0,\ldots,0)|$$

for all $f\in U\cap Z$. Whence, and from (26), one obtains

$$\Omega(L,U) \geqslant \inf_{A\in\mathcal{A}} \sup_{f\in U\cap Z} |L(f)-A(0,\ldots,0)| = \inf_{c\in\mathbb{K}} \sup_{f\in U\cap Z} |L(f)-c| = \sup_{f\in U\cap Z} |L(f)|,$$

where $\Omega(L,U)$ is an optimal recovery error (see § 3, 1°). The first equality here is explained by the fact that among all elements of $\mathcal{A}$ there exists an algorithm A such that $A(0,\ldots,0)=c$ for any given $c\in\mathbb{K}$.

3. Let us prove the inverse inequality. Let $\mathcal{A}_0$ be the set of all linear algorithms of the form $A=\ell(a;L_1,\ldots,L_N)$ (see (21)), where $a\in\mathbb{K}^N$. Let us estimate the deviation of the functional L from an arbitrarily fixed linear algorithm $A\in\mathcal{A}_0$. Representing any element $f\in U$ in the form (see (22), (24))

$$f = c_1f_1+\ldots+c_Nf_N+\pi_Z(f),$$

one obtains

$$\Big|L(f)-\sum_{k=1}^N a_kL_k(f)\Big| = \Big|\sum_{j=1}^N c_j\Big(L(f_j)-\sum_{k=1}^N a_kL_k(f_j)\Big)+L(\pi_Z(f))\Big|,$$

since $L_k(\pi_Z(f))=0$ for any $k=1,\ldots,N$. Whence,

$$\Big|L(f)-\sum_{k=1}^N a_kL_k(f)\Big| \leqslant \Big|\sum_{j=1}^N c_j\Big(L(f_j)-\sum_{k=1}^N a_kL_k(f_j)\Big)\Big| + |L(\pi_Z(f))|,\ f\in U. \tag{27}$$

In order to find the optimal linear algorithm, select $a_1,\ldots,a_N$ so that the effect of the element $f-\pi_Z(f)\in S$ on the estimate of the "residual"functional in (27) is eliminated. Namely, define $\alpha=(\alpha_1,\ldots,\alpha_N)\in\mathbb{K}^N$ from the system of equations

$$L(f_j)-\sum_{k=1}^N \alpha_kL_k(f_j)=0,\quad j=1,2,...,N. \tag{28}$$

According to Lemma 6, the system has the only solution $\alpha\in\mathbb{K}^N$. This element corresponds to a linear algorithm (see (21)) $A_0=\ell(\alpha;L_1,\ldots,L_N)$. Formulae (27) and (28) provide

$$\begin{aligned}\Omega(L,U) &\leqslant \inf_{A\in\mathcal{A}_0}\sup_{f\in U} |L(f)-A\circ I(f)| \leqslant E(A_0;L,U)\\ &= \sup_{f\in U}|L(f)-A_0\circ I(f)| \leqslant \sup_{f\in U} |L(\pi_Z(f))| \leqslant \sup_{f\in U\cap Z} |L(f)|,\end{aligned} \tag{29}$$

since $\pi_Z(f)\in U\cap Z$ for all $f\in U$. The inverse inequality is proved in paragraph 2 therefore, the statement 1) of Theorem 7 holds.

4. Due to inequality in paragraph 2, we conclude that

$$\Omega(L,U)=E(A_0;L,U), \tag{30}$$

i.e. the algorithm $A_0=\ell(\alpha;L_1,\ldots,L_N)$ (see (21)) is optimal. Its representation (see (25)) is written upon obtaining the vector $\alpha\in\mathbb{K}^N$ from (28) by the Cramer rule. Finally, the desired inequality follows from (30) ▶

Remarks. 1. Formula (25) of optimal linear algorithm is *independent* of the set U with the described properties.

2. If V is a Hilbert space and $V = S \oplus Z$ is an orthogonal expansion then, according to the Pythagorean theorem, any closed ball with the center in a null element $\Theta \in V$ is invariant with respect to the mapping π_Z. This property can be untrue if V is the Banach space. However, the linear space V can be given a topology by means of an equivalent norm where any closed ball with the centre in Θ is π_Z-invariant. To this end, the norm of any element $f = s+z \in V = S \oplus Z$ can be defined as $||f||_V = \|s\|_V + \|z\|_V$. This norm is equivalent to the original one according to the open mapping theorem by Banach. Therefore, the requirement that the set U is invariant with respect to the mapping π_Z in Theorem 7 is not too limiting in the given case.

Example. Let $V = W^p_\pi,\ 1 < p < \infty$. Assume that $L_k(f) = f(z_k),\quad f \in W^p_\pi,\quad k = 0, 1, ..., N$ in notation of Theorem 6. Let $Z := \ker L_1 \cap \ldots \cap \ker L_N$, S be its cofactor such that $V = S \oplus Z$. The set

$$f_k(t) = \frac{\sin \pi(\overline{z_k} - t)}{\pi(\overline{z_k} - t)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\tau(\overline{z_k}-t)} d\tau, \quad t \in \mathbb{C};\ k = 1, ..., N \tag{31}$$

is a linearly independent system of elements in W^p_π with any $p \in (1, \infty)$, since the final system of exponents with different coefficients is also a linearly independent system of elements. Direct verification demonstrates that $L_k(f_k) \neq 0,\ k = 1, ..., N$, therefore $F = \{f_k,\ k = 1, ..., N\} \subset S$. Whence, invoking that S is a linear subspace of the dimension N in V, we conclude that the system F is a basis in S. Formula (25) allows one to find an optimal linear algorithm for the set U, satisfying the conditions of Theorem 7.

3°. Construction of Chebyshev polynomials. One can develop the Cherbyshev polynomials constructively (in terms of §3, 1°) for a certain class of cases if the optimal linear algorithm and the Chebyshev polynomial connected with each other are *unique*. Theorem 7 provides the following addition to Theorem 4.

Theorem 8. *Let us use the notation of Theorems* 4 *and* 7*. Let $T : V \to B$ be an algebraic isomorphism of a vector space V into a normed space B. Both V, and B are considered over one and the same filed $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Moreover, assume that S is a cofactor of the set Z* (see (22)) *in the expansion $V = S \oplus Z$; $\{f_1, \ldots, f_N\}$ is a basis in S, and the ball $U = \{f \in V :\ \|f\|_V \leqslant R\}$ is a set, invariant with respect to the mapping π_Z* (see (23), (24))*. If the conjugate space B' is strictly convex, then there exists only one optimal linear algorithm A_0 of the form* (21)*, definable by the formula* (25)*, and $A_0 \circ T^{-1}$ is the only Chebyshev polynomial.*

Theorem 8 provides a criterion for finding constructive formulae of the optimal linear algorithm and the Chebyshev polynomial, specifically, for the spaces V_q, W^p_π, $1 < p < \infty$ (see Theorems 5, 6). In the first case, the basis is, e.g., the system of functions (31), and in the second case it is the system of bilateral interpolating sequences $c(f_k),\ k = 1, ..., N$ of the same functions (see (6)).

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