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ON SOLVABILITY OF A CLASS OF HIGHER-ORDER NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH A NONCOMPACT INTEGRAL OPERATOR OF THE HAMMERSTEIN TYPE

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Abstract. In the present paper we investigate the question of solvability of one class of Hammerstein type Nth order nonlinear integro-differential equations with a noncompact integral operator on the semi-axis in the Sobolev space $W_\infty^N(0,+\infty)$. The existence of a positive solution in $W_\infty^N(0,+\infty)$ is proved, and the limit of this solution at infinity is found. The obtained results are generalized for nonlinear equations with sum-difference kernels.

Keywords: Factorization, polynomial, limit of iteration, Sobolev space. invariants.

1. Problem statement and introduction

The following class of nonlinear integro-differential equations in considered in the present paper with respect to the unknown real function $f(x)$:

$$\frac{d^N f}{dx^N}+\sum_{j=1}^{N} a_j\frac{d^{N-j} f}{dx^{N-j}}+\lambda(x)\int\limits_0^\infty K(x-t)G(f(t))dt=0,\quad x\in\mathbb{R}^+,\ N\geq 2. \tag{1}$$

Here $a_j (j=1,2,3,\ldots,N)$ are real coefficients, and

$$a_N<0, \tag{2}$$

and the corresponding polynomial $P(x)\equiv x^N+\sum\limits_{j=1}^{N} a_j x^{N-j}$ has only real rots.

In (1), $\lambda(x)$ and $K(x)$ are measurable functions on sets $(0,+\infty)$ and $(-\infty,+\infty)$, respectively and

$$\begin{array}{ll} a) & \lambda(x)\uparrow \text{ over } x \text{ in } (0,+\infty),\quad 0\leqslant\lambda(x)\leqslant 1,\quad x\in(0,+\infty),\\ & 1-\lambda\in L_1(0,+\infty),\quad \lambda\in W_\infty^N(0,+\infty)\end{array} \tag{3}$$

$$\begin{gathered} b)\quad K(x)\geq 0,\quad x\in(-\infty,+\infty),\quad a_N+\int\limits_{-\infty}^{+\infty}K(\tau)d\tau=0,\\ K\in C(\mathbb{R}),\quad \int\limits_{-\infty}^{+\infty}|\tau|K(\tau)d\tau<+\infty.\end{gathered} \tag{4}$$

The function $G(x)$ is measurable and is defined on $(-\infty,+\infty)$. At the same time it is assumed that there is a number $\eta>0$, such that

$$\begin{array}{ll} c) & G\in C[0,\eta],\quad G\uparrow \text{ over } x \text{ in the interval } [0,\eta],\\ & G(x)\geq x,\quad x\in[0,\eta],\quad G(\eta)=\eta.\end{array} \tag{5}$$

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The unknown solution to Equation (1) satisfies the following boundary-value conditions:

$$f(0)=0,\quad f\in W^N_\infty(0,+\infty)=\{\varphi(x):\varphi^{(k)}\in L_\infty(0,+\infty),\quad k=0,1,\ldots,N\},\tag{6}$$

where $\varphi^{(k)}$ denotes the k-th derivative of the function $\varphi(x)$.

First results in the study of nonlinear integral equations with compact Urysohn and Hammerstein operators were obtained in works of M.A. Krasnoselskii and his disciples ([1-5]). The works provide various necessary and sufficient conditions ensuring complete continuity of the Urysohn and Hammerstein operators. In [6-8], existence theorems for solution with assumption of complete continuity of the corresponding nonlinear integral operator were proved.

Equation (1) was investigated in the particular case when $G(x)\equiv x,\quad \lambda(x)\equiv 1,\quad N=2$ earlier in [9]. Recently, I proved structural theorems of existence for the case when $N=2$, $\lambda(x)\equiv 1$, and $G(x)$ satisfies the conditions (5) (see [10,12]).

In the present paper, methods of the classical theory of functions of a complex variable, and some results from the theory of linear integral equations of convolution type make it possible to prove existence of a nontrivial solution to the problem (1),(6) and to describe its structure. Moreover, the limit of the constructed solution at infinity is calculated. The results obtained are generalized for the following class of nonlinear integro-differential equations:

$$\frac{d^Nf}{dx^N}+\sum_{j=1}^N a_j\frac{d^{N-j}f}{dx^{N-j}}+\lambda(x)\int_0^\infty K(x-t)G(f(t))dt+\int_0^\infty \overset{\circ}{K}(x+t)G_0(f(t))dt=0,\ N\geq 2,\tag{7}$$

where

$$0\leqslant \overset{\circ}{K}\in L_1(0,+\infty)\text{ and }\int_x^\infty \overset{\circ}{K}(\tau)d\tau\leqslant\int_x^\infty K(\tau)d\tau,\quad x\in\mathbb{R}^+,\tag{8}$$

and G_0 is a measurable function defined in $(-\infty,+\infty)$ and

$$\begin{gathered}G_0\in C[0,\eta],\quad G_0(x)\geq 0,\quad x\in[0,\eta],\quad G_0\uparrow\text{ over }x\text{ in }[0,\eta]\\ G_0(\eta)=\eta.\end{gathered}\tag{8'}$$

2. Factorization problem. Reduction to the basic nonlinear integral equation

Let us introduce the class of nonlinear integral operators $\mathcal{K}_G\in\Omega$, if there exists a measurable function $K^*(x,t),\quad ((x,t)\in\mathbb{R}^+\times\mathbb{R}^+)$, such that

$$\begin{gathered}(\mathcal{K}_Gf)(x)=\int_0^\infty K^*(x,t)G(f(t))dt,\quad f\in L_\infty(\mathbb{R}^+),\\ G:L_\infty(\mathbb{R}^+)\to L_\infty(\mathbb{R}^+),\end{gathered}\tag{9}$$

and the kernel $K^*(x,t)\geq 0$ satisfies the following estimates. There are functions $0\leqslant K\in L_1(\mathbb{R}),\quad 0\leqslant\lambda(x)\leqslant 1$, such that

$$K^*(x,t)\geq\lambda(x)K(x-t),\quad (x,t)\in\mathbb{R}^+\times\mathbb{R}^+.\tag{9'}$$

$$\sup_{x>0}\int_0^\infty K^*(x,t)dt<+\infty.\tag{9''}$$

The conditions $(9),(9'')$ provide $\mathcal{K}_G:L_\infty(\mathbb{R}^+)\to L_\infty(\mathbb{R}^+)$. Let $\alpha_1,\alpha_2,\ldots,\alpha_n\quad(n\in N)$ be positive roots of the polynomial $P(x)$. The condition (2), in view of the Vieta theorem, entails that n is an odd number. Denote by $-\beta_1,-\beta_2,\ldots,-\beta_m$ negative roots $P(x)\quad(m+n=N)$.

Let us introduce the differential operators

$$P(D)=D^N+a_1D^{N-1}+a_2D^{N-2}+\ldots+a_{N-1}D+a_NI,$$

acting from the space $W_\infty^N(0,+\infty)$ in $L_\infty(0,+\infty)$, where D is a differential operator, and I is a unit operator.

It follows from the above that

$$P(D) = (D-\alpha_1 I)(D-\alpha_2 I)\dots(D-\alpha_n I)(D+\beta_1 I)(D+\beta_2 I)\dots(D+\beta_m I). \tag{10}$$

Denote by $\mathcal{J}_{\alpha_j}$ $(j=1,2,\dots,n)$ the operators inverse to the operators $(\alpha_j I - D)$ in the space $W_\infty^1(0,+\infty)$.

Then, one can readily verify that

$$(\mathcal{J}_{\alpha_j} f)(x) = \int\limits_x^\infty e^{-\alpha_j(t-x)} f(t)dt, \quad f\in L_\infty(\mathbb{R}^+), \quad j=1,2,\dots,n. \tag{11}$$

The following lemma holds.

Lemma 1. *If $\mathcal{K}_G \in \Omega$ and $\lambda(x)\uparrow$ with respect to x, then $\mathcal{J}_{\alpha_j}\mathcal{K}_G\in\Omega$, $\quad j=1,2,\dots,n$.*

Proof. Let $f\in L_\infty(\mathbb{R}^+)$ be an arbitrary function. Then, one has

$$(J_{\alpha_j}\mathcal{K}_G f)(x) = \int\limits_x^\infty e^{-\alpha_j(t-x)}\int\limits_0^\infty K^*(t,\tau)G(f(\tau))d\tau dt =$$

$$= \int\limits_0^\infty\left(\int\limits_x^\infty e^{-\alpha_j(t-x)}K^*(t,\tau)dt\right)G(f(\tau))dt = \int\limits_0^\infty T^*(x,\tau)G(f(\tau))d\tau,$$

where $T^*(x,\tau) = \int\limits_0^\infty e^{-\alpha_j z}K^*(x+z,\tau)dz$.

Note also that $\int\limits_0^\infty T^*(x,\tau)d\tau \leqslant \frac{1}{\alpha_j}\sup\limits_{x>0}\int\limits_0^\infty K^*(x,\tau)d\tau < +\infty$. The estimate $(9')$ and the monotonous function λ provide

$$T^*(x,\tau)\geq\int\limits_0^\infty e^{-\alpha_j z}\lambda(x+z)K(x+z-\tau)dz \geq \lambda(x)\int\limits_0^\infty e^{-\alpha_j z}K(x+z-\tau)dz \equiv \lambda(x)T(x-\tau),$$

while $T\in L_1(\mathbb{R})$ and $\int\limits_{-\infty}^{+\infty}T(x)dx \leqslant \frac{1}{\alpha_j}\int\limits_{-\infty}^{+\infty}K(x)dx$. The lemma is proved.

Let us write Equation (1) in the operator form

$$(P(D)+\mathcal{K}_G)f = 0, \tag{12}$$

where

$$(\mathcal{K}_G f)(x) = \lambda(x)\int\limits_0^\infty K(x-t)G(f(t))dt,$$

and the functions λ, K and G satisfy the conditions $a), b), c)$.

Consider the following factorization problem. Given operators $P(D)$ and $\mathcal{K}_G\in\Omega$. Find an integral operator $T_G\in\Omega$ such that

$$P(D)+\mathcal{K}_G = \prod_{j=1}^n(D-\alpha_j I)\left(\prod_{j=1}^m(D+\beta_j I) - T_G\right). \tag{13}$$

In order to solve the problem, first consider the product of the operators $\mathcal{J}_{\alpha_1}$ and $\mathcal{K}_G : T_{1,G}\equiv\mathcal{J}_{\alpha_1}\mathcal{K}_G$.

One has

$$(T_{1,G}f)(x) = \int\limits_x^\infty e^{-\alpha_1(t-x)}\lambda(t) \int\limits_0^\infty K(t-\tau)G(f(\tau))d\tau dt =$$
$$= \int\limits_0^\infty G(f(\tau)) \int\limits_0^\infty e^{-\alpha_1 z_1} K(x+z_1-\tau)\lambda(z_1+x)dz_1 d\tau = \int\limits_0^\infty T_1(x,\tau)G(f(\tau))d\tau, \text{ where} \tag{14}$$

$$T_1(x,\tau) \equiv \int\limits_0^\infty e^{-\alpha_1 z_1} K(x+z_1-\tau)\lambda(x+z_1)dz_1 \geq$$
$$\geq \lambda(x)\int\limits_0^\infty e^{-\alpha_1 z_1} K(x+z_1-\tau)dz_1 \equiv \lambda(x)T_1^*(x-\tau), \tag{15}$$

$$T_1^*(x) = \int\limits_0^\infty K(x+z_1)e^{-\alpha_1 z_1}dz_1. \tag{16}$$

Lemma 1 provides also $T_{2,G} \equiv \mathcal{J}_{\alpha_2}\mathcal{K}_G \in \Omega$. Construct its kernel similarly:

$$(T_{2,G}f)(x) = \int\limits_x^\infty e^{-\alpha_2(t-x)} \int\limits_0^\infty T_1(t,\tau)G(f(\tau))d\tau dt =$$

$$= \int\limits_0^\infty G(f(\tau)) \int\limits_x^\infty e^{-\alpha_2(t-x)} T_1(t,\tau)dtd\tau = \int\limits_0^\infty T_2(x,\tau)G(f(\tau))d\tau, \text{ where}$$

$$T_2(x,\tau) = \int\limits_0^\infty e^{-\alpha_2 z_2} \int\limits_0^\infty e^{-\alpha_1 z_1} K(x+z_1+z_2-\tau)\lambda(x+z_1+z_2)dz_1 dz_2 \geq \lambda(x)T_2^*(x-\tau),$$

where

$$T_2^*(x) = \int\limits_0^\infty e^{-\alpha_2 z_2} \int\limits_0^\infty e^{-\alpha_1 z_1} K(x+z_1+z_2)dz_1 dz_2.$$

Induction over n readily verifies that $T_{n,G} \equiv \mathcal{J}_{\alpha_n} T_{n-1,G} = \prod\limits_{j=1}^n \mathcal{J}_{\alpha_j}\mathcal{K}_G \in \Omega$ and is given by the following formulae:

$$(T_{n,G}f)(x) = \int\limits_0^\infty T_n(x,\tau)G(f(\tau))d\tau, \quad f \in M(\mathbb{R}^+),$$
$$T_n(x,\tau) = \int\limits_0^\infty e^{-\alpha_n z_n} \int\limits_0^\infty e^{-\alpha_{n-1} z_{n-1}} \int\limits_0^\infty \ldots$$
$$\ldots \int\limits_0^\infty e^{-\alpha_1 z_1} K(x+z_1+\ldots+z_n-\tau)\lambda(x+z_1+\ldots+z_n)dz_1\ldots dz_n \geq \tag{17}$$
$$\geq \lambda(x)T_n^*(x-\tau),$$

where

$$T_n^*(x) = \int\limits_0^\infty e^{-\alpha_n z_n} \int\limits_0^\infty e^{-\alpha_{n-1} z_{n-1}} \int\limits_0^\infty \ldots \int\limits_0^\infty e^{-\alpha_1 z_1} K(x+z_1+\ldots+z_n)dz_1\ldots dz_n. \tag{18}$$

Therefore, since n is an odd number, then one can take $T_G \equiv T_{n,G}$ as the operator T_G. Thus, the following lemma is formulated.

Lemma 2. *Let the functions* λ, K *and* G *satisfy the conditions* $a), b), c)$ *and* $G : L_\infty(\mathbb{R}^+) \to L_\infty(\mathbb{R}^+)$. *Then, the operator* $P(D) + \mathcal{K}_G$ *admits factorization* (13), *where* $T_G \equiv T_{n,G} \in \Omega$ *is given by the formula* (17).

In view of the factorization (13), solution to the problem (1),(6) is equivalent to successive solution of the following connected equations:

$$\prod_{j=1}^{n}(D - \alpha_j I)\varphi = 0, \tag{19}$$

$$\left(\prod_{j=1}^{m}(D + \beta_j I) - T_G\right) f = \varphi. \tag{20}$$

Consider Equation (19):

$$(D - \alpha_1 I)(D - \alpha_2 I)\dots(D - \alpha_n I)\varphi = 0. \tag{21}$$

Since $f \in W^N_\infty(0, +\infty)$, then $\varphi \in W^n_\infty(0, +\infty)$. Hence, obviously, $\varphi(x) \equiv 0$. Thus, solution of the problem (1), (6) is reduced to the following integro-differential equation:

$$\prod_{j=1}^{m}(D + \beta_j I)f - \int_0^\infty T_n(x, \tau)G(f(\tau))d\tau = 0, \tag{22}$$

with the boundary-value conditions

$$f(0) = 0, \quad f \in W^N_\infty(0, +\infty). \tag{23}$$

Let use the notation

$$F(x) \equiv (D + \beta_m I)(D + \beta_{m-1} I)\dots(D + \beta_1 I)f(x).$$

Then, in view of (23), one can readily verify that

$$F \in W^n_\infty(0, +\infty)$$

and

$$f(x) = \int_0^x e^{-\beta_1(x-\tau_1)} \int_0^{\tau_1} e^{-\beta_2(\tau_1-\tau_2)} \dots \int_0^{\tau_{m-1}} e^{-\beta_m(\tau_{m-1}-\tau_m)} F(\tau_m)d\tau_m \dots d\tau_1. \tag{24}$$

Hence, the equation (22) takes the form

$$\begin{gathered} F(x) = \int_0^\infty T_n(x, \tau)G\bigg(\int_0^\tau e^{-\beta_1(\tau-\tau_1)} \int_0^{\tau_1} e^{-\beta_2(\tau_1-\tau_2)} \dots \\ \cdot \int_0^{\tau_{m-1}} e^{-\beta_m(\tau_{m-1}-\tau_m)} F(\tau_m)d\tau_m d\tau_{m-1} \dots d\tau_1\bigg)d\tau, \quad x \in \mathbb{R}^+. \end{gathered} \tag{25}$$

The following section is devoted to the questions of solvability of Equation (25). The limit of the function $F(x)$ at infinity is also calculated there.

3. Solution of Equation (25)

Consider the following iterations:

$$F_{(p+1)}(x)=\int_0^\infty T_n(x,\tau)G\Big(\int_0^\tau e^{-\beta_1(\tau-\tau_1)}\int_0^{\tau_1} e^{-\beta_2(\tau_1-\tau_2)}\dots$$
$$\cdot\int_0^{\tau_{m-1}} e^{-\beta_m(\tau_{m-1}-\tau_m)}F_{(p)}(\tau_m)d\tau_m d\tau_{m-1}\dots d\tau_1\Big)d\tau, \tag{26}$$
$$F_{(0)}(x)=\prod_{j=1}^m \beta_j\eta,\quad p=0,1,2,\dots,\quad x\in\mathbb{R}^+.$$

First, let us verify by means of induction over p that

$$F_{(p)}(x)\downarrow \text{ over } p \tag{27}$$

If $p=1$ one has

$$F_{(1)}(x)=\int_0^\infty T_n(x,\tau)G\left(\int_0^\tau e^{-\beta_1(\tau-\tau_1)}\int_0^{\tau_1} e^{-\beta_2(\tau_1-\tau_2)}\dots\int_0^{\tau_{m-1}} e^{-\beta_m(\tau_{m-1}-\tau_m)}\eta\prod_{j=1}^m\beta_j d\tau_m d\tau_{m-1}\dots d\tau_1\right)d\tau\leqslant$$
$$\leqslant\int_0^\infty T_n(x,\tau)G\left(\prod_{j=1}^{m-1}\beta_j\int_0^\tau e^{-\beta_1(\tau-\tau_1)}\int_0^{\tau_1} e^{-\beta_2(\tau_1-\tau_2)}\dots\int_0^{\tau_{m-2}} e^{-\beta_{m-1}(\tau_{m-2}-\tau_{m-1})}\eta d\tau_{m-1}d\tau_{m-2}\dots d\tau_1\right)d\tau\leqslant$$
$$\leqslant\int_0^\infty T_n(x,\tau)G(\eta)d\tau=\eta\int_0^\infty T_n(x,\tau)d\tau\leqslant\eta\int_{-\infty}^{+\infty}T_n^*(z)dz=-\frac{a_N\eta}{\prod\limits_{j=1}^n\alpha_j}\equiv æ \tag{28}$$

The Vieta theorem immediately provides that

$$(-1)^{N-m}\alpha_1\alpha_2\dots\alpha_n\beta_1\beta_2\dots\beta_m=a_N<0,$$

or

$$\prod_{j=1}^n\alpha_j\prod_{j=1}^m\beta_j=-a_N,$$

hence,

$$æ\equiv\eta\prod_{j=1}^m\beta_j\equiv F_{(0)}(x),\ \text{ i.e. } F_{(1)}(x)\leqslant F_{(0)}(x).$$

Further, assuming that

$$F_{(p)}(x)\leqslant F_{(p-1)}(x)$$

and using properties of the function G, one obtains

$$F_{(p+1)}(x)\leqslant F_{(p)}(x).$$

Thus, (27) is proved.

Consider the following linear integral equation together with Equation (25):

$$S(x)=\lambda(x)\int_0^\infty T_n^*(x-\tau)\int_0^\tau e^{-\beta_1(\tau-\tau_1)}\int_0^{\tau_1} e^{-\beta_2(\tau_1-\tau_2)}\dots$$
$$\dots\int_0^{\tau_{m-1}} e^{-\beta_m(\tau_{m-1}-\tau_m)}S(\tau_m)d\tau_m d\tau_{m-1}\dots d\tau_1 d\tau,\quad x\in\mathbb{R}^+. \tag{29}$$

This equation is readily reduced to the following integral equation:

$$S(x) = \lambda(x) \int_0^\infty W_m(x - \tau_m) S(\tau_m) d\tau_m, \quad x \in \mathbb{R}^+, \tag{30}$$

where

$$0 \leqslant W_m \in L_1(\mathbb{R}), \quad \int_{-\infty}^{+\infty} W_m(\tau) d\tau = \frac{1}{\prod_{j=1}^m \beta_j} \int_{-\infty}^{+\infty} T_n^*(z) dz = 1 \tag{31}$$

(see the chain of inequalities (28)).

The form of the kernels $W_j(x)$, $(j = 1, 2, \dots, m)$ is provided by the following recurrent relations with the use of Fubini's theorem:

$$\begin{gathered} W_j(x) = \int_0^\infty W_{j-1}(x - z_j) e^{-\beta_j z_j} dz_j, \ j = 2, 3, \dots, m, \\ W_1(x) = \int_0^\infty T_n^*(x - z_1) e^{-\beta_1 z_1} dz_1. \end{gathered} \tag{32}$$

Suppose that

$$\nu(W_m) \equiv \int_{-\infty}^{+\infty} \tau W_m(\tau) d\tau < 0. \tag{33}$$

Absolute convergence of the latter integral is provided by (4) and by Fubini's theorem.

It follows from results of the works [11, 12] that if conditions (31) and (33) are satisfied, Equation (30) has a monotone increasing nontrivial bounded solution $0 \leqslant S(x)$. Let us introduce the notation

$$S^*(x) = \frac{\prod_{j=1}^m \beta_j \eta S(x)}{c}, \quad c = \sup_{x>0} S(x). \tag{34}$$

Let us prove by induction that the sequence $\{F_{(p)}(x)\}_0^\infty$ satisfies the following estimate :

$$F_{(p)}(x) \geq S^*(x), \quad p = 0, 1, 2, \dots \tag{35}$$

In the case $p = 0$, it is obvious, since

$$F_{(0)}(x) = \eta \prod_{j=1}^m \beta_j = \sup_{x>0} S^*(x) \geq S^*(x) \geq 0. \tag{36}$$

Suppose that

$$F_{(p-1)}(x) \geq S^*(x).$$

In view of (29), (30), (34), (5), (17), the relations (26) provides:

$$F_{(p)}(x) \geq \int_0^\infty T_n(x, \tau) G\Bigg(\int_0^\tau e^{-\beta_1(\tau - \tau_1)} \int_0^{\tau_1} e^{-\beta_2(\tau_1 - \tau_2)} \dots$$

$$\dots \int_0^{\tau_{m-1}} e^{-\beta_m(\tau_{m-1} - \tau_m)} S^*(\tau_m) d\tau_m d\tau_{m-1} \dots d\tau_1 \Bigg) d\tau \geq$$

$$\geq \lambda(x) \int_0^\infty T_n^*(x - \tau) \int_0^\tau e^{-\beta_1(\tau - \tau_1)} \int_0^{\tau_1} e^{-\beta_2(\tau_1 - \tau_2)} \dots \int_0^{\tau_{m-1}} e^{-\beta_m(\tau_{m-1} - \tau_m)} S^*(\tau_m) d\tau_m d\tau_{m-1} \dots d\tau_1 d\tau =$$

$$= \lambda(x) \int_0^\infty W_m(x-\tau_m) S^*(\tau_m) d\tau_m = S^*(x).$$

Thus, (27) and (35) provide that the sequence of functions $\{F_{(p)}(x)\}_0^\infty$ has a pointwise limit:

$$\lim_{p\to\infty} F_{(p)}(x) \equiv F(x),$$

which satisfies Equation (25) due to B.Levy's theorem (see[13]). Now, prove that $F \in W_\infty^n(0,+\infty)$. Indeed, since $h_j(x,\tau) = \frac{\partial^j T_n(x,\tau)}{\partial x^j}$, $(j = 1,2,\ldots,n)$ is continuous in $(0,+\infty)\times(0,+\infty)$ and the summand function with every fixed $\tau \in (0,+\infty)$, and integrals

$$\int_0^\infty h_j(x,\tau) G\Bigg(\int_0^\tau e^{-\beta_1(\tau-\tau_1)} \int_0^{\tau_1} e^{-\beta_2(\tau_1-\tau_2)} \ldots$$

$$\ldots \int_0^{\tau_{m-1}} e^{-\beta_m(\tau_{m-1}-\tau_m)} F(\tau_m) d\tau_m d\tau_{m-1} \ldots d\tau_1\Bigg) d\tau, \quad j = 1,2,\ldots,n$$

converge uniformly, and $F \in M(\mathbb{R}^+)$ then, (25) provides that $\dfrac{d^j F}{dx^j} \in M(0,+\infty)$, $j = 0,1,2,\ldots,n$, i.e. $F \in W_\infty^n(0,+\infty)$ in view of the theorem on differentiation under the integral (see [14]).

On the other hand, since $S^*(x) \uparrow \eta \prod_{j=1}^m \beta_j$ (see (34) and the work [12]), then it follows immediately from the following inequality $S^*(x) \leqslant F(x) \leqslant \eta \prod_{j=1}^m \beta_j$ that $\exists \lim_{x\to\infty} F(x) = \eta \prod_{j=1}^m \beta_j$.

In view of the known convolution operations (see [15]), (24) provides existence of the limit

$$\lim_{x\to\infty} f(x) = \eta.$$

Since $F \in W_\infty^n(0,+\infty)$, then (24) yields $f \in W_\infty^N(0,+\infty)$.

Thus, the following theorem is proved.

Theorem 1. *Let us assume that the polynomial $P(x)$ has only real roots and $a_N < 0$. Then, provided that the conditions* (3)-(5) *and $\nu(W_m) < 0$ hold, the problem* (1), (6) *has a nonvanishing, nonnegative solution with the limit η at infinity.*

Reasoning by analogy, one can verify the following theorem.

Theorem 2. *Let us assume that all conditions of Theorem 1 hold. Then, if functions K_0 and G_0 satisfy the conditions* (8) *and* (8′)*, the problem* (7), (6) *has a nonvanishing, nonnegative solution with the limit $\lim_{x\to\infty} f(x) = \eta$.*

4. Examples of functions G and G_0

In what follows, several examples of the function G are given.

a) $G(x) = x^\alpha, \quad \eta = 1, \quad 0 < \alpha < 1, \quad x \in \mathbb{R}^+,$

b) $G(x) = x + \sin x, \quad \eta = \pi, \quad x \in \mathbb{R}^+,$

c) $G(x) = \sqrt{xe^{x-1}}, \quad \eta = 1, \quad x \in \mathbb{R}^+.$

d) $G(x) = x + sin^2 x,\ x \in \mathbb{R}^+,\ \eta = \pi k,\ k = 1,2,3,\ldots.$

Prove that $G(x) = \sqrt{xe^{x-1}}$ satisfies all the requirements of Theorem 1. Indeed,

$$G(0) = 0, \quad G(1) = 1, \quad G(x) \uparrow \text{ over } x, \text{ since}$$

$$G'(x) = \frac{1}{2\sqrt{xe^{x-1}}}(e^{x-1} + xe^{x-1}) > 0, \quad x > 0.$$

On the other hand,

$$e^{x-1} \geq x, \quad x \in \mathbb{R}^+.$$

Hence,

$$G(x) \geq x.$$

Since conditions imposed on the function G_0 are weaker than conditions on G, then one can take the function G as G_0. However, consider several examples G_0 as well:

d) $G_0(x) = x^\alpha, \quad \alpha \neq 1, \quad \alpha > 0, \quad \eta = 1,$

e) $G_0 = \eta \sin x, \quad \eta = \dfrac{\pi}{2},$

f) $G_0(x) = \eta \ln(x+1), \quad \eta = e - 1.$

Remark . *In the linear case when* $G(x) \equiv x$*, one can choose any positive number as* G_0*, and due to Theorem 1 in this case (due to linearity) one obtains a one-parameter family of positive solutions* $f_\eta(x)$*,* $(\eta \in (0, +\infty))$ *with the limit* η *from the Sobolev space* $W_\infty^N(0, +\infty)$*,* $(N \geq 2)$*. If* η *is not uniquely defined from the condition imposed on the function* G *(e.g. if* $G(x) = x + \sin^2 x$*, then* $\eta = \pi k$*,* $k = 1, 2, 3, \ldots$*) then, in this nonlinear case we also obtain a one-parameter family of nonnegative, nonzero solutions and the limit of every function* $f_k(x)$ *from this family equals to the number* $\pi k \, (k = 1, 2, 3, \ldots)$ *correspondingly when* $x \to +\infty$*.*

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