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ON THE GROWTH OF THE MAXIMUM MODULUS OF AN ENTIRE FUNCTION DEPENDING ON THE GROWTH OF ITS CENTRAL INDEX

P.V. FILEVYCH

Abstract. Let h be a positive function continuous on $(0,+\infty)$, $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be an entire function, and $M_f(r)=\max\{|f(z)|:|z|=r\}$, $\mu_f(r)=\max\{|a_n|r^n:n\geq 0\}$, and $\nu_f(r)=\max\{n\geq 0:|a_n|r^n=\mu_f(r)\}$ be the maximum modulus, the maximal term, and the central index of the function f, respectively. We establish necessary and sufficient conditions for the growth of $\nu_f(r)$ under which $M_f(r)=O(\mu_f(r)h(\ln\mu_f(r)))$, $r\to+\infty$.

Keywords: entire function, maximum modulus, maximal term, central index, order, lower order.

1. Introduction

Let us assume that I is a class of functions that are continuous on the right, nondecreasing, unbounded from above in $(0,+\infty)$, and L is a subclass of functions from I, that are continuous in $(0,+\infty)$, and C_+ is a class of continuous functions positive in $(0,+\infty)$.

Denote by $\mathcal{A}$ a class of transcendent entire functions

$$f(z)=\sum_{n=0}^{\infty}a_nz^n. \tag{1}$$

Determine the maximum of the modulus $M_f(r)=\max\{|f(z)|:|z|=r\}$, maximum term $\mu_f(r)=\max\{|a_n|r^n:n\geq 0\}$, and the central index of the function f $\nu_f(r)=\max\{n\geq 0:|a_n|r^n=\mu_f(r)\}$ for the entire function (1) and any $r>0$. Let

$$\rho_f:=\varlimsup_{r\to+\infty}\frac{\ln\ln M_f(r)}{\ln r},\qquad \lambda_f:=\varliminf_{r\to+\infty}\frac{\ln\ln M_f(r)}{\ln r}$$

be the order and the lower order of the function f, respectively. As it is known (see, e.g., [1], chapter IV), one has $\mu_f(r)\leq M_f(r)$, $\nu_f\in I$, $\nu_f(r)=r(\ln\mu_f(r))'_+$ for any function $f\in\mathcal{A}$, and in the definition of both orders $\ln M_f(r)$ can be substituted by $\ln\mu_f(r)$ or $\nu_f(r)$. Suppose that $\mathcal{A}(\alpha)=\{f\in\mathcal{A}:\ \nu_f(r)\sim\alpha(r),\ r\to+\infty\}$, if $\alpha\in I$.

According to the classical Borel theorem, the following relation holds for any $f\in\mathcal{A}$ such that $\rho_f<+\infty$:

$$\ln M_f(r)\sim\ln\mu_f(r),\quad r\to+\infty. \tag{2}$$

Using the Cauchy-Hadamard formula

$$\rho_f=\varlimsup_{n\to\infty}\frac{n\ln n}{-\ln|a_n|}$$

for calculating the order of the entire function f via coefficients of its expansion into power series, one can reformulate the Borel theorem as follows. If $\psi\in L$, and

$$\ln x=O(\psi(x)),\quad x\to+\infty, \tag{3}$$

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then the relation (2) holds for any function $f \in \mathcal{A}$ of the form (1) such that

$$|a_n| \le \exp\{-n\psi(n)\}, \quad n \ge n_0(f). \tag{4}$$

It follows from results of the work [2] by M.N. Sheremeta, that the condition (3) in the latter statement is also a necessary one: if $\psi \in L$ and (3) are not satisfied, then there is an entire function $f \in \mathcal{A}$ of the form (1), satisfying (4), for which the relation (2) does not hold. Thus, there are entire functions of an infinite order with coefficients tending to zero as rapidly as desired (i. e. functions, growing as slowly as desired) for which (2) does not hold.

Nevertheless, the class of entire functions for which (2) holds, is much broader than the class of entire functions of a finite order. J. Clunie [3] proved that for any function $l(r)$, convex with respect to $\ln r$ on $(0, +\infty)$ such that $\ln r = o(l(r))$, $r \to +\infty$, there exists an entire function $f \in \mathcal{A}$, for which $\ln M_f(r) \sim \ln \mu_f(r) \sim l(r)$, $r \to +\infty$. Hence, there are entire functions f with an arbitrary prescribed growth for $\ln M_f(r)$, satisfying (2). In this connection, the problem of finding conditions of growth for an entire function, more flexible than the condition $\rho_f < +\infty$, that guarantee validity of the relation (2) is considered in [4].

Theorem A [4]**.** *If the condition*

$$\ln \nu_f(r) = o(\ln \mu_f(r)), \quad r \to +\infty \tag{5}$$

holds for an entire function $f \in \mathcal{A}$, then the relation (2) *holds for it as well.*

Note that one has $\ln \nu_f(r) < 2\rho_f \ln r = o(\ln \mu_f(r))$, $r \to +\infty$ for an entire function $f \in \mathcal{A}$ of a finite order. On the other hand, if, e.g., $\nu_f(r) \sim re^r$, $r \to +\infty$ then f is of an infinite order, but we have (5) and hence, (2) holds as well since $\ln \mu_f(r) \sim e^r$, $r \to +\infty$ due to the L'Hospital rule.

Let us assume that $\alpha \in I$, and $\hat{\alpha}$ is any fixed function for which $\alpha(r) = r\hat{\alpha}'_+(r)$ (i. e. $\hat{\alpha}(r)$ is $\alpha(r)$ transformed with respect to $\ln r$). If $\nu_f(r) \sim \alpha(r)$, $r \to +\infty$, then $\ln \mu_f(r) \sim \hat{\alpha}(r)$, $r \to +\infty$. The following theorem demonstrates that the condition (5) of Theorem A is unimprovable in a sense.

Theorem B [4]**.** *Let $\alpha \in L$. If $\ln \alpha(r) \ne o(\hat{\alpha}(r))$, $r \to +\infty$, then there is an entire function $f \in \mathcal{A}(h)$, for which the relation* (2) *does not hold.*

The aim of the present work is finding conditions of growth for the central index of an entire function such that relations more general than (2) hold for the function.

Theorem 1. *Let $\alpha \in I$, $h \in C_+$. The following relations are equivalent:*
a) $\exists \delta \in (0, 1)$: $\alpha(r) = O(h(\delta\hat{\alpha}(r)))$, $r \to +\infty$;
b) $\exists K_0 > 0\ \forall f \in \mathcal{A}(\alpha)$: $M_f(r) \le K_0 \mu_f(r) h(\ln \mu_f(r))$, $r \ge r_0(f)$;
c) $\forall f \in \mathcal{A}(\alpha)$: $M_f(r) = O(\mu_f(r) h(\ln \mu_f(r)))$, $r \to +\infty$.

Remark 1. Let $\alpha \in I$, $l \in L$, $h \in C_+$ and $\tilde{h}(x) := \inf\{h(t) : t \ge x\}$. The inequality

$$\alpha(r) \le h(l(r)), \quad r \ge r_0, \tag{6}$$

holds if and only if

$$\alpha(r) \le \tilde{h}(l(r)), \quad r \ge r_0. \tag{7}$$

Indeed, it is manifest that (6) follows from (7), because $\tilde{h}(x) \le h(x)$. If (6) holds, then $h(x) \to +\infty$, $x \to +\infty$ and therefore, one can substitute inf by min in definition of $\tilde{h}$. Whence, due to continuity and monotony of the function l, it follows that for any $r \ge r_0$ there exists $r' \ge r$ such that $\tilde{h}(l(r)) = h(l(r'))$. Hence, $\alpha(r) \le \alpha(r') \le h(l(r')) = \tilde{h}(l(r))$, i. e. (7) holds.

Remark 2. If $\beta, \alpha \in I$ and $\beta(r) \sim \alpha(r)$, $r \to +\infty$ then, according to the L'Hospital rule, $\hat{\beta}(r) \sim \hat{\alpha}(r)$, $r \to +\infty$. In view of this and Remark 1, one can readily demonstrate that the statement a) of Theorem 1 remains valid if α is substituted by β in it.

The following theorems are corollaries of Theorem 1. They indicate conditions for realization of the generalized Borel relation

$$\varphi(\ln M_f(r)) \sim \varphi(\ln \mu_f(r)), \quad r \to +\infty. \tag{8}$$

Theorem 2. *Let $\alpha \in I$, $\varphi \in L$, and*

$$\varphi(x+1) \sim \varphi(x), \quad x \to +\infty. \tag{9}$$

For the relation (8) *to hold for any entire function $f \in \mathcal{A}(\alpha)$, it is necessary and sufficient that the following condition be fulfilled:*

$$\forall \varepsilon > 0 \ \exists \delta \in (0,1): \quad \varphi(\delta\hat{\alpha}(r) + \ln \alpha(r)) \le (1+\varepsilon)\varphi(\delta\hat{\alpha}(r)), \quad r \ge r_0(\varepsilon). \tag{10}$$

Theorem 3. *Let $\varphi \in L$. For the function $\alpha \in I$ to be such that the relation* (8) *is valid for any entire function $f \in \mathcal{A}(\alpha)$, it is necessary and sufficient that the condition* (9) *hold.*

Remark 3. Theorem 3 indicates the the condition (9) is essential in Theorem 2.

As for Theorem 1, let us make the following observations. Firstly, the theorem allows one to obtain necessary and sufficient conditions of growth of the central index $\nu_f(r)$ that provide the prescribed relation between $M_f(r)$ and $\mu_f(r)$, e.g., as in Theorem 2. Secondly, Theorem 1 provides exact relations between $M_f(r)$ and $\mu_f(r)$ in a class of all entire functions f, satisfying the conditions, when conditions of growth $\nu_f(r)$ are given. In particular, the following theorem holds for entire functions f such that $0 < \lambda_f \le \rho_f < +\infty$.

Theorem 4. *The following statements hold.*

a) For any entire function f, satisfying the conditions $0 < \lambda_f \le \rho_f < +\infty$, there is a function $\varepsilon \in C_+$ such that $\varepsilon(x) \to 0$, $x \to +\infty$, and

$$M_f(r) \le \mu_f(r)(\ln \mu_f(r))^{\frac{\rho_f}{\lambda_f} + \varepsilon(\ln \mu_f(r))}, \quad r \ge r_0. \tag{11}$$

b) If $0 < \lambda \le \rho < +\infty$, $\varepsilon \in C_+$, $\varliminf\limits_{x\to+\infty} \varepsilon(x) = 0$, then there exists an entire function f and a sequence $(r_n)_{n=0}^{\infty}$ increasing to $+\infty$ such that $\lambda_f = \lambda$, $\rho_f = \rho$ and

$$M_f(r_n) > \mu_f(r_n)(\ln \mu_f(r_n))^{\frac{\rho}{\lambda} + \varepsilon(\ln \mu_f(r_n))}, \quad n \ge 0. \tag{12}$$

2. Proof of Theorem 1

Let us prove that b) follows from a). Let $K_1 > 0$ and $\delta \in (0,1)$ be arbitrary constants such that $\alpha(r) \le K_1 h(\delta\hat{\alpha}(r))$, $r \ge r_1$ according to a). Manifestly, $h(x) \to +\infty$, $x \to +\infty$. Therefore, the function $\tilde{h}(x) = \min\{h(t) : t \ge x\}$ is defined for all $x \ge 0$, while (see Remark 1) $\tilde{h} \in L$ and

$$\alpha(r) \le K_1 \tilde{h}(\delta\hat{\alpha}(r)), \quad r \ge r_1. \tag{13}$$

Let $f \in \mathcal{A}(\alpha)$, and $(c_k)_{k=0}^{\infty}$ be an increasing sequence of all discontinuity points $\nu_f(r)$ in $(0, +\infty)$. Then, if $n_k = \nu_f(c_k - 0)$, one has $\nu_f(r) = n_0$ for $r \in (0, c_0)$, and $\nu_f(r) = n_{k+1}$ for $r \in [c_k, c_{k+1})$ and $k \ge 0$.

Since $\ln \mu_f(r) \sim \hat{\alpha}(r)$, $r \to +\infty$, then (13) provides

$$\nu_f(r) \le K_1 \tilde{h}(\delta' \ln \mu_f(r)), \quad r \ge r_2 \tag{14}$$

when $\delta' \in (\delta, 1)$ is fixed. Consider the sequence (d_k) such that $d_0 \in (0, c_0)$, $d_{k+1} \in (c_k, c_{k+1})$,

$$n_{k+1}(\ln c_k - \ln d_k) < \frac{1}{2^k}, \quad k \ge 0. \tag{15}$$

Assume that $\gamma(r) = \nu_f(r)$ for $r > 0$ if $r \notin \cup_{k=0}^{\infty}(d_k, c_k)$. If $r \in (d_k, c_k)$ for some $k \ge 0$, then assume that

$$\gamma(r) = \min\left\{K_1 \tilde{h}(\delta' \ln \mu_f(r)),\ n_{k+1} - (c_k - r)\frac{n_{k+1} - n_k}{c_k - d_k}\right\}.$$

Since $\gamma(d_k+0)=n_k=\nu_f(d_k)=\gamma(d_k)$, $\gamma(c_k-0)=n_{k+1}=\nu_f(c_k)=\gamma(c_k)$, then $\gamma\in L$. Moreover, it follows from (14) that

$$\nu_f(r)\le\gamma(r)\le K_1\tilde h(\delta'\ln\mu_f(r)),\quad r\ge r_2. \tag{16}$$

Let $\hat\gamma(r)=\int_1^r\frac{\gamma(t)}{t}dt$, $r>0$. According to (15)

$$0\le\int_1^r\frac{\gamma(t)}{t}dt-\int_1^r\frac{\nu_f(t)}{t}dt\le\sum_{k=0}^{\infty}\int_{c_k}^{d_k}\frac{\gamma(t)-\nu_f(t)}{t}dt\le$$
$$\le\sum_{k=0}^{\infty}(n_{k+1}-n_k)(\ln c_k-\ln d_k)\le\sum_{k=0}^{\infty}\frac{1}{2^k}=2$$

for $r\ge1$. Then,

$$\hat\gamma(r)\sim\ln\mu_f(r),\quad r\to+\infty. \tag{17}$$

Let us fix any $r\ge r_2$ such that $\gamma(r)>0$, and consider the function $\xi(x)=\gamma(r(1+x))$, $x>0$. Manifestly, ξ is a positive function from the class L. Therefore, the equation $\xi(x)=\frac1x$ has a unique solution $x=x(r)>0$, and

$$0\le\hat\gamma(r(1+x(r)))-\hat\gamma(r)=\int_r^{r(1+x(r))}\frac{\gamma(t)}{t}dt\le$$
$$\le\gamma(r(1+x(r)))\ln(1+x(r))\le\gamma(r(1+x(r)))x(r)=1.$$

Hence, $\hat\gamma(r(1+x(r)))\sim\hat\gamma(r)$, $r\to+\infty$. Invoking (16), (17) and the latter relation, one obtains

$$\gamma(r(1+x(r)))\le K_1\tilde h(\delta'\ln\mu_f(r(1+x(r))))\le K_1\tilde h(\ln\mu_f(r)),\quad r\ge r_3. \tag{18}$$

Now let us estimate the maximum of the module of the function f, considering it to be given in the form (1). Definitions $\nu_f(r)$ and $\mu_f(r)$ for any $r>0$ and $x>0$ yield

$$|a_n|(r(1+x))^n\le|a_{\nu_f(r(1+x))}|(r(1+x))^{\nu_f(r(1+x))}\le\mu_f(r)(1+x)^{\nu_f(r(1+x))}.$$

Whence, $|a_n|r^n\le\mu_f(r)(1+x)^{\nu_f(r(1+x))-n}$. Using the latter inequality with $x=x(r)$, as well as (16) and (18), one obtains

$$M_f(r)\le\sum_{n<\nu_f(r(1+x))}|a_n|r^n+\sum_{n\ge\nu_f(r(1+x))}|a_n|r^n\le$$
$$\le\mu_f(r)\nu_f(r(1+x))+\mu_f(r)\sum_{n\ge\nu_f(r(1+x))}(1+x)^{\nu_f(r(1+x))-n}=$$
$$=\mu_f(r)\left(\nu_f(r(1+x))+\frac1x+1\right)\le\mu_f(r)\left(\gamma(r(1+x))+\frac1x+1\right)\le$$
$$\le3\mu_f(r)\gamma(r(1+x))\le3K_1\mu_f(r)\tilde h(\ln\mu_f(r))\le3K_1\mu_f(r)h(\ln\mu_f(r))$$

for all $r\ge r_0$. Statement b) is proved.

The implication b) $\Rightarrow$ c) is manifest.

Let us prove that a) follows from b). To this end, suppose that a) does not hold, i. e.

$$\forall\delta\in(0,1):\qquad\varlimsup_{r\to+\infty}\frac{\alpha(r)}{h(\delta\hat\alpha(r))}=+\infty, \tag{19}$$

and prove that there is an entire function $f\in\mathcal{A}(\alpha)$ such that

$$\varlimsup_{r\to+\infty}\frac{M_f(r)}{\mu_f(r)h(\ln\mu_f(r))}=+\infty. \tag{20}$$

Without loss of generality, one can assume that $\alpha(r)=0$ for $r\in[0,1)$ and $\hat\alpha(r)=\int_0^r\frac{\alpha(t)}{t}dt$ for all $r\ge0$ (see Remark 2).

Let $\delta_p = 1 - \frac{1}{2^{p+1}}$, $p \geq 0$. Formula (19) yields

$$\varlimsup_{r\to+\infty} \frac{[\delta_{p+1}\alpha(r)] - [\delta_p\alpha(r)] - 2}{h(\delta_p\hat{\alpha}(r))} = +\infty,$$

when $p \geq 0$ is fixed. Therefore, there exists a sequence (d_p) increasing to $+\infty$ such that $d_0 = 1$, and

$$[\delta_{p+1}\alpha(d_p)] - [\delta_p\alpha(d_p)] - 2 > ph(\delta_p\hat{\alpha}(d_p)), \qquad \ln\frac{d_{p+1}}{d_p} > \hat{\alpha}(d_p) \tag{21}$$

for all $p \geq 0$. Note that if $\gamma \in I$, then $[\gamma] \in I$ as well, and $+\infty$ is the only condensation point for the set of discontinuity points of the function $[\gamma]$ in the interval $(0, +\infty)$.

Consider the function

$$\eta_p(x) = \delta_p\hat{\alpha}(d_p) + \int_{d_p}^{x} \frac{[\delta_{p+1}\alpha(t)]}{t}dt + \int_{x}^{d_{p+1}} \frac{[\delta_{p+1}\alpha(t)] + 2}{t}dt, \quad x > 0 \tag{22}$$

for any $p \geq 0$. Manifestly, the function η_p is continuous in $[d_p, d_{p+1}]$. Using the second inequality (21), one obtains

$$\eta_p(d_p) = \delta_p\hat{\alpha}(d_p) + \int_{d_p}^{d_{p+1}} \frac{[\delta_{p+1}\alpha(t)] + 2}{t}dt \geq \delta_p\hat{\alpha}(d_p) + \int_{d_p}^{d_{p+1}} \frac{\delta_{p+1}\alpha(t) + 1}{t}dt =$$

$$= \delta_{p+1}\hat{\alpha}(d_{p+1}) + \ln\frac{d_{p+1}}{d_p} - (\delta_{p+1} - \delta_p)\hat{\alpha}(d_p) > \delta_{p+1}\hat{\alpha}(d_{p+1}).$$

On the other hand,

$$\eta_p(d_{p+1}) \leq \delta_p\hat{\alpha}(d_p) + \int_{d_p}^{d_{p+1}} \frac{\delta_{p+1}\alpha(t)}{t}dt =$$

$$= \delta_{p+1}\hat{\alpha}(d_{p+1}) - (\delta_{p+1} - \delta_p)\hat{\alpha}(d_p) \leq \delta_{p+1}\hat{\alpha}(d_{p+1}).$$

Hence, there is a point x_p in the half-interval $(d_p, d_{p+1}]$ such that

$$\eta_p(x_p) = \delta_{p+1}\hat{\alpha}(d_{p+1}). \tag{23}$$

Consider the function β such that $\beta(r) = 0$ for $r \in (0, 1)$ and

$$\beta(r) = \begin{cases} [\delta_{p+1}\alpha(r)], & d_p \leq r < x_p; \\ [\delta_{p+1}\alpha(r)] + 2, & x_p \leq r < d_{p+1} \end{cases}$$

for all $p \geq 0$. Equations (22) and (23) provide

$$\delta_p\hat{\alpha}(d_p) + \int_{d_p}^{d_{p+1}} \frac{\beta(t)}{t}dt = \delta_{p+1}\hat{\alpha}(d_{p+1}). \tag{24}$$

Let us use the following lemma.

Lemma [5]. *Let (n_k) be an increasing sequence of nonnegative entire numbers and (c_k) be a positive sequence increasing to $+\infty$. If a complex sequence (a_n) is such that $a_0 = \ldots = a_{n_0-1} = 0$, $a_{n_0} \neq 0$ and*

$$|a_{n_{k+1}}| = |a_{n_0}| \prod_{j=0}^{k} \frac{1}{c_j^{n_{j+1}-n_j}}; \tag{25}$$

$$|a_n| = |a_{n_k}| c_k^{n_k - n}, \quad \text{if} \quad n \in (n_k, n_{k+1}) \tag{26}$$

for any $k \geq 0$, then the power series (1) with such coefficients a_n governs an entire function such that $\nu_f(r) = n_0$ when $r \in (0, c_0)$ and $\nu_f(r) = n_{k+1}$ when $r \in [c_k, c_{k+1})$ and $k \geq 0$.

Let (c_k) be an increasing sequence of all discontinuity points of the function β in the interval $(0, +\infty)$, and $n_k = \beta(c_k - 0)$. Then, $\beta(r) = n_0 = 0$ for $r \in (0, c_0)$ and $\beta(r) = n_{k+1}$ if $r \in [c_k, c_{k+1})$ and $k \geq 0$. Let us define a positive sequence (a_n) as follows. Assume that $a_0 = a_{n_0} = 1$ and find a_n, $n \geq 1$ proceeding from the equalities (25) and (26). According to the lemma, the power series (1) with such coefficients a_n assigns an entire function f for which $\nu_f(r) = \beta(r)$, $r > 0$. Since $\beta(r) \sim \alpha(r)$, $r \to +\infty$, then $f \in \mathcal{A}(\alpha)$. Let us demonstrate that the relation (20) holds for f.

Since $\ln \mu_f(0) = \ln a_0 = 0$, then $\ln \mu_f(r) = \int_0^r \frac{\nu_f(t)}{t} dt = \int_0^r \frac{\beta(t)}{t} dt$. By means of induction over p and using the latter equalities together with (24), one can easily demonstrate that $\ln \mu_f(d_p) = \delta_p \hat{\alpha}(d_p)$, $p \geq 0$. Therefore, according to the first inequality (21),

$$\begin{aligned} \beta(d_{p+1}) - \beta(d_{p+1} - 0) \geq [\delta_{p+2}\alpha(d_{p+1})] - [\delta_{p+1}\alpha(d_{p+1})] - 2 > \\ > (p+1)h(\delta_{p+1}\hat{\alpha}(d_{p+1})) = (p+1)h(\ln \mu_f(d_{p+1})) > 0. \end{aligned} \tag{27}$$

In particular, (27) provides existence of the sequence (k_p) such that $c_{k_p} = d_{p+1}$, $p \geq 0$.

Furthermore, according to relations (25), (26) and the lemma, one has

$$M_f(c_k) \geq \sum_{n=n_k}^{n_{k+1}} a_n c_k^n = \sum_{n=n_k}^{n_{k+1}} a_{n_{k+1}} c_k^{n_{k+1}} = (n_{k+1} - n_k + 1)\mu_f(c_k).$$

Therefore, using (27), one obtains

$$\begin{aligned} M_f(c_{k_p}) \geq \mu_f(c_{k_p})(n_{k_p+1} - n_{k_p}) = \mu_f(c_{k_p})(\beta(c_{k_p}) - \beta(c_{k_p} - 0)) = \\ = \mu_f(c_{k_p})(\beta(d_{p+1}) - \beta(d_{p+1} - 0)) > (p+1)\mu_f(c_{k_p})h(\ln \mu_f(c_{k_p})) \end{aligned}$$

for all $p \geq 0$, whence (20). Theorem 1 is proved.

3. Proof of Theorem 2

Without loss of generality, one can assume that the function φ is increasing in $(0, +\infty)$. Then, the function

$$h_\varepsilon(x) = \exp\{\varphi^{-1}((1+\varepsilon)\varphi(x)) - x\} \tag{28}$$

is defined when $\varepsilon > 0$ is fixed and $h_\varepsilon \in C_+$.

Sufficiency. Let us assume that the conditions (10) and (9) hold. Consider an entire function $f \in \mathcal{A}(\alpha)$ and demonstrate that the relation (8) holds for this function.

Let us fix any $\varepsilon > 0$. Then, according to (10), there is $\delta \in (0, 1)$ such that

$$\alpha(r) \leq \exp\{\varphi^{-1}((1+\varepsilon)\varphi(\delta\hat{\alpha}(r))) - \delta\hat{\alpha}(r)\} = h_\varepsilon(\delta\hat{\alpha}(r)), \quad r \geq r_0,$$

i. e. if $h = h_\varepsilon$, the statement a) of Theorem 1 holds. Therefore, according to the statement b) of Theorem 1,

$$M_f(r) \leq K_0 \mu_f(r) h_\varepsilon(\ln \mu_f(r)), \quad r \geq r_1, \tag{29}$$

where K_0 is a positive constant. Using (28), rewrite (29) in the form

$$\varphi(\ln M_f(r) - \ln K_0) \leq (1+\varepsilon)\varphi(\ln \mu_f(r)), \quad r \geq r_1.$$

Whence, in view of the Cauchy inequality $\mu_f(r) \leq M_f(r)$ and the relation (9), one obtains

$$1 \leq \varliminf_{r \to +\infty} \frac{\varphi(\ln M_f(r))}{\varphi(\ln \mu_f(r))} \leq \varlimsup_{r \to +\infty} \frac{\varphi(\ln M_f(r))}{\varphi(\ln \mu_f(r))} \leq 1 + \varepsilon. \tag{30}$$

Since ε is arbitrary, (30) provides (8). Sufficiency is proved.

Necessity. Let us assume that (9) holds and the relation (8) is true for any entire function $f \in \mathcal{A}(\alpha)$. Let us prove that in this case (10) holds.

Let us fix an arbitrary $\varepsilon > 0$ and let $\varepsilon_1 = \frac{\varepsilon}{2}$. Then,

$$\forall f \in \mathcal{A}(\alpha): \qquad \varphi(\ln M_f(r)) \leq (1+\varepsilon_1)\varphi(\ln \mu_f(r)), \quad r \geq r_0(f). \tag{31}$$

Using (28), one rewrites (31) in the form

$$\forall f\in\mathcal{A}(\alpha):\qquad M_f(r)\le \mu_f(r)h_{\varepsilon_1}(\ln\mu_f(r)),\quad r\ge r_0(f).$$

According to Theorem 1, there are constants $K_1>0$ and $\delta\in(0,1)$, for which

$$\alpha(r)\le K_1h_{\varepsilon_1}(\delta\hat\alpha(r)),\quad r\ge r_1. \tag{32}$$

The formula (32) provides $\varphi(\delta\hat\alpha(r)+\ln\alpha(r)-\ln K_1)\le(1+\varepsilon_1)\varphi(\delta\hat\alpha(r))$, $r\ge r_1$, whence, according to (9), $\varphi(\delta\hat\alpha(r)+\ln\alpha(r))\le(1+\varepsilon)\varphi(\delta\hat\alpha(r))$, $r\ge r_2$, which was to be proved. Theorem 2 is proved.

4. Proof of Theorem 3

Necessity. Similarly to the above, the function φ can be considered to be increasing in $(0,+\infty)$.

Let us suppose that there is a function $\alpha\in I$ such that (8) holds for any $f\in\mathcal{A}(\alpha)$, and prove that (9) is true.

Indeed, if the condition (9) is not met, then there is a number $\varepsilon>0$ and a sequence $(x_n)_{n=0}^{\infty}$ increasing to $+\infty$ such that $\varphi(x_n+1)>(1+\varepsilon)\varphi(x_n)$, $n\ge0$. However, in this case $e>h(x_n)$, $n\ge0$ according to (28). Hence, the statement a) of Theorem 1 is not valid and therefore, the statement b) of the same theorem is not true as well. Thus, there exist an entire function $f\in\mathcal{A}(\alpha)$ and a sequence $(r_n)_{n=0}^{\infty}$ increasing to $+\infty$ such that

$$M_f(r_n)>\mu_f(r_n)h_\varepsilon(\ln\mu_f(r_n)),\quad n\ge0.$$

Whence, using (28) one readily obtains

$$\varphi(\ln M_f(r_n))>(1+\varepsilon)\varphi(\ln\mu_f(r_n)),\quad n\ge0,$$

i. e. the relation (8) does not hold for f. This contradicts the assumption made above. Necessity is proved.

Sufficiency. Let us assume that the condition (9) holds. Then, as one can readily notice, there is a function $l\in L$ such that

$$\varphi(x+l(x))\sim\varphi(x),\quad x\to+\infty. \tag{33}$$

Consider a function $\alpha\in I$, for which $\alpha(r)\le\exp\{l(\ln r)\}$, $r\ge r_0$. Since $\ln r=o(\hat\alpha(r))$, $r\to+\infty$, then

$$\ln\alpha(r)\le l(\ln r)\le l\left(\frac12\hat\alpha(r)\right),\quad r\ge r_1. \tag{34}$$

Using (33) with $x=\frac12\hat\alpha(r)$ and (34), one obtains

$$\varphi\left(\frac12\hat\alpha(r)+\ln\alpha(r)\right)\sim\varphi\left(\frac12\hat\alpha(r)\right),\quad r\to+\infty.$$

Whence, it follows that the condition (10) of Theorem 2 holds for the function α. According to this theorem, the relation (8) holds for any $f\in\mathcal{A}(\alpha)$. Thus, Theorem 3 is proved completely.

5. Proof of Theorem 4

a) Let f be an entire function such that $0<\lambda_f\le\rho_f<+\infty$. Using the fact that one can substitute $\ln M_f(r)$ by $\ln\mu_f(r)$ or $\nu_f(r)$ in definition of the order $\rho=\rho_f$ and the lower order $\lambda=\lambda_f$ of the function f, one obtains

$$\nu_f(r)\le r^{\rho+\eta(\ln\mu_f(r))},\qquad \ln\mu_f(r)\ge r^{\lambda-\eta(\ln\mu_f(r))}$$

for some function $\eta\in C_+$ such that $\eta(x)\to0$, $x\to+\infty$ and for all $r\ge r_1$. Hence,

$$\nu_f(r)\le 2^{\frac{\rho+\eta(\ln\mu_f(r))}{\lambda-\eta(\ln\mu_f(r))}}\left(\frac12\ln\mu_f(r)\right)^{\frac{\rho+\eta(\ln\mu_f(r))}{\lambda-\eta(\ln\mu_f(r))}}\le 4^{\frac{\rho}{\lambda}}\left(\frac12\ln\mu_f(r)\right)^{\frac{\rho+\eta(\ln\mu_f(r))}{\lambda-\eta(\ln\mu_f(r))}}$$

for all $r \geq r_2$. Then, according to Theorem 1 (when $\alpha(r) = \nu_f(r)$), there is a constant $K_0 \geq 1$ such that

$$M_f(r) \leq K_0 \mu_f(r)(\ln \mu_f(r))^{\frac{\rho+\eta(\ln \mu_f(r))}{\lambda-\eta(\ln \mu_f(r))}} =$$
$$= \mu_f(r)(\ln \mu_f(r))^{\frac{\rho}{\lambda}+\frac{(\rho+\lambda)\eta(\ln \mu_f(r))}{\lambda(\lambda-\eta(\ln \mu_f(r)))}+\frac{\ln K_0}{\ln \ln \mu_f(r)}}, \quad r \geq r_3,$$

whence, choosing the function $\varepsilon \in C_+$ so that

$$\varepsilon(x) = \frac{(\rho+\lambda)\eta(x)}{\lambda(\lambda-\eta(x))} + \frac{\ln K_0}{\ln x}, \quad x \geq x_0,$$

one obtains directly (11) and readily verifies that $\varepsilon(x) \to 0$, $x \to +\infty$.

b) Let $(\varepsilon_n)_{n=0}^{\infty}$ be a fixed sequence, decreasing to zero. The condition $\varliminf\limits_{x\to+\infty} \varepsilon(x) = 0$ provides the existence of an increasing sequence $(c_n)_{n=0}^{\infty}$ such that $c_0 > 1$ and

$$\varepsilon_n > \varepsilon(c_n^\lambda), \qquad c_n^{\lambda\varepsilon_n} > n, \qquad c_{n+1} > c_n^{2\left(\frac{\rho}{\lambda}+\varepsilon_n\right)} \tag{35}$$

for all $n \geq 0$.

Let $d_n = c_n^{\left(\frac{\rho}{\lambda}+2\varepsilon_n\right)}$, $n \geq 0$. Using the third inequality (35), one readily obtains $c_n < d_n < c_{n+1}$, $n \geq 0$. Assume that $\alpha(r) = 0$ for all $r \in [0, c_0)$ and let

$$\alpha(r) = \begin{cases} \lambda c_n^{\rho+2\lambda\varepsilon_n}, & c_n \leq r \leq d_n; \\ \lambda r^\lambda, & d_n \leq r < c_{n+1} \end{cases}$$

for every $n \geq 0$.

Manifestly, $\alpha \in I$, and

$$\varlimsup_{r\to+\infty} \frac{\ln \alpha(r)}{\ln r} = \rho, \qquad \varliminf_{r\to+\infty} \frac{\ln \alpha(r)}{\ln r} = \lambda. \tag{36}$$

Suppose that $h(x) = x^{\frac{\rho}{\lambda}+\varepsilon(x)}$. According to the first and the second inequalities (35),

$$\alpha(c_n) = \lambda c_n^{\rho+2\lambda\varepsilon_n} = \lambda c_n^{\lambda(2\varepsilon_n-\varepsilon(c_n^\lambda))} h(c_n^\lambda) \geq \lambda c_n^{\lambda\varepsilon_n} h(c_n^\lambda) > \lambda(n+1)h(c_n^\lambda), \quad n \geq 0. \tag{37}$$

Let us consider an antiderivative of $\alpha(r)$ with respect to $\ln r$:

$$\hat{\alpha}(r) = \int_0^r \frac{\alpha(t)}{t} dt.$$

One has

$$\hat{\alpha}(c_{n+1}) = \int_{c_0}^{d_n} \frac{\alpha(t)}{t} dt + \lambda \int_{d_n}^{c_{n+1}} t^{\lambda-1} dt \leq \alpha(d_n) \ln d_n + c_{n+1}^\lambda = (1+\eta_n) c_{n+1}^\lambda \tag{38}$$

for the function $\hat{\alpha}$ for any $n \geq 0$. Here, according to the third inequality (35),

$$\eta_n = \frac{\alpha(d_n) \ln d_n}{c_{n+1}^\lambda} = \frac{(\rho+2\lambda\varepsilon_n) c_n^{\rho+2\lambda\varepsilon_n} \ln c_n}{c_{n+1}^\lambda} \to 0, \quad n \to \infty. \tag{39}$$

Let us prove that the statement a) of Theorem 1 is not true for the functions h and α. Indeed, if the statement holds, then there are constants $\delta \in (0,1)$ and $K > 0$, for which

$$\alpha(r) \leq K h(\delta \hat{\alpha}(r)), \quad r \geq 0.$$

Whence, in view of Remark 1, (38) and (39), one obtains

$$\alpha(c_n) \leq K \tilde{h}(\delta \hat{\alpha}(c_n)) \leq K \tilde{h}(\delta(1+\eta_{n-1}) c_n^\rho) \leq K \tilde{h}(c_n^\rho) \leq K h(c_n^\rho)$$

for all $n \geq n_0$. This contradicts (37). Hence, the statement a) of Theorem 1 is not true. Consequently, the statement c) of the same theorem is not true as well. Therefore, the equality

$$\varlimsup_{r\to+\infty} \frac{M_f(r)}{\mu_f(r)h(\ln \mu_f(r))} = +\infty,$$

holds for an entire function $f \in \mathcal{A}(\alpha)$. Whence, one immediately concludes that the sequence $(r_n)_{n=0}^{\infty}$ is such that (12) holds for f. Moreover, according to (36), $\lambda_f = \lambda$, $\rho_f = \rho$. Thus, Theorem 4 is proved.

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Petr Vasil'evich Filevych,
Lviv Stepan Gzhytsky National University
of Veterinary Medicine and Biotechnologies,
Pekarskaya Str., 50,
29010, Lviv, Ukrain
E-mail: filevych@mail.ru

Translated from Russian by E.D. Avdonina.