ISSN 2304-0122 Ufa Mathematical Journal. Vol. 16. No 1 (2023). P. 23-33.

doi:10.13108/2023-16-1-23

RANDOM WALKS ON A LINE AND ALGEBRAIC CURVES

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Abstract. This work is devoted to the studying the generating function of the first hitting time of the positive semi-axis under the homogeneous discrete integer random walk on a line. In the first part of the work the increments are supposed to be independent. Recurrent relations for the probabilities allow us to write the system of equations for the sought generating function. Applying the resultants technique, we succeed to reduce this system to a single equation. Then we can study it by calculating the genus of the correposponding plane algrebraic curve via analyzing its singularities. In the work we write the sought equations for some random walks and we show that if the increments take all integer values from -2 to 2, or from -1 to 3 with equal probabilities or they take equally probable values -1 and 4, then the curve is rational, while this is not true in the first case.

In the second part of the work we consider a symmetric process, the increments take the values -1, 0, 1, but then we suppose a non-zero correlation of each next increment with the previous one. For such process the equation for the generating function defines an elliptic curve depending on the square of the correlation coefficient for neighbouring increments if all increments are non-zero and it defines a hyperelliptic curve of genus 2. The degeneration criterion of the latter is the presence of multiple roots of a sixth order polynomial under general symmetrically distributed conditional probabilities.

Keywords: random walks, algebraic curves.

Mathematics Subject Classification: 60J10

1. Introduction

In this work we study a homogeneous discrete random walk. We recall the definition.

Definition 1.1. *A homogeneous discrete random walk is a disrete random process with a fixed set of increments $a_1, \dots, a_n$, each of which occurs with a certain probability, which can depend on previous increments but is invariant with respect to the discrete time.*

Random walks are used in biology, economimcs and other scientific fields and this is why they are studied quite actively. We continue studies initiated in work [1].

The method we use consists in obtaining and studying an equation for the generating function of the first hitting time of the positive semi-axis. First this method was applied by V.A. Malyshev (in 1970s) for random walks in a quadrant with increments at most 1 in each coordinate, see [2]. Vadim Alexandrovich's ideas were later developed by several scientists, who obtained the following result:

Proposition 1.1 ([3]). *The generating function $G(t,a,b) = \sum P(k,m,n)t^k a^m b^n$, where $P(k,m,n)$ is the probability of returning back to the origin in k steps with m intersections of horizontal boundary and n intersections of the vertical boundary for the aforementioned random walk in the quadrant, possesses the following property:*

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Submitted July 17, 2023.

1) If the set of steps possesses a horizontal (vertical) symmetry axis, then $G(t,a,1)$ *is D-finite, that is, it satisfies a linear differential equation with polynomial coefficients as a function of the variable* t *for each* a *(respectively,* $G(t,1,b)$ *possesses this property for each* b*).*

2) If the set of step possesses both a vertical and horizontal symmetry axes, then $G(t,a,b)$ *is equal to the quotient of two D-finite functions on* t *for each* a *and* b*, while if the set of steps consists of* 4 *elements* $(\pm 1,\pm 1)$ *(the signs are not related), then it is D-finite for all* a *and* b*.*

3) If the set of steps is symmetric with respect to the axis $x=-y$*, then* $G(t,a,a)$ *is equal to the quotient of two D-finite functions on* t *for all* a*, while for* $a=1$ *it is a D-finite function. An exception is the central-symmetric case of* 6 *steps* $(\pm 1,0)$*,* $(0,\pm 1)$*,* $(\pm 1,\pm 1)$ *(the signs are consistent), for which we only known that* $G(t,1,1)$ *satisfies, as well as the studied generating function, some algebraic relation. In the case of three steps* $(-1,0)$*,* $(0,1)$*,* $(1,-1)$ *it is also known that* $G(t,a,b)$ *is equal to the quotient of two D-finite functions of* t *for arbitrary not necessary coinciding* a *and* b*.*

4) If the set of three steps is symmetric with respect to the straight line $x=y$*, then for each* a *the function* $G(t,a,a)$ *satisfies some algebraic relation. удовлетворяет некоторому алгебраическому соотношению.*

5) In the case of 4 *steps* $(\pm 1,0)$*,* $(\pm 1,\mp 1)$ *(the signs are consistent) the same as in Statement 1 holds, while in the case of* 4 *steps* $(\pm 1,0)$*,* $(\pm 1,\pm 1)$ *(the signs are consistent) we only know that* $G(t,1,1)$ *satisfies some algebraic relation.*

Already in our century there were obtained some particular results about random walks on the line with independent increments. In [4] walks with increments -1, 0, 1 were considered; it turned out that the probability of the first hitting time of the point $x>0$ in n steps is equal

$$P(x,n)=p_1^x p_0^{n-x}\frac{(n-1)!}{(n-x)!(x-1)!}F\left(\frac{x-n}{2},\frac{x-n+1}{2};x+1;\frac{4p_{-1}p_1}{p_0^2}\right), \tag{1.1}$$

where $F(a,b;c,d)$ is the Gaussian hypergeometric function, while the associated generating function is expressed by the formula

$$\left(\frac{z^{-1}-p_0}{2p_1}-\sqrt{\left(\frac{z^{-1}-p_0}{2p_1})^2-\frac{p_{-1}}{p_1}\right)}\right)^x. \tag{1.2}$$

Work [5] is devoted to walks with increments -2, -1, 1. In this work, a two-dimensional Galton-Watson process was considered, which is a one of kinds of branching random walk. This process was related in a certain way with the initial walk and from such consideration the equation

$$w=z(p_1+p_{-1}w^2+p_{-2}w^3),$$

was derived, where $w=f(z)$ is the value of the generating function of the first hitting time of the positive semi-axis at the point z. It was proved that this value is equal to the smallest real root of this equation. Thus, algebraic properties of generating functions for random walks are rather remarkable and are widely studied.

During the talk of the author on the Second All-Russian Conference of Mathematical Centers, A.V. Shklyaev proposed a gentle arguing for random walks $\xi_0=0$, $\xi_n=\xi_{n-1}+\delta\xi$ with increments $\delta\xi$ being equal to $1=a_1>\ldots>a_k$ with probabilities $p_1,\ldots,p_k$. Let $g(z)=\sum_{i=1}^k p_i t^{a_i}$ be a generating function of the increment. Then the quotient $\frac{t^{\xi_n}}{g^n(t)}$ is a martingale. As a stopping moment η we choose the hitting time. Since $\xi_\eta=1$, by the Wald identity from [6] we have $E(\frac{t}{g^\eta(t)})=1$ for sufficiently small t. Since by the definition the sought generating function is equal to $w(z) = E(z^\eta)$, then for $z = \frac{1}{g(t)}$ we obtain

$$w(\frac{1}{g(t)}) = \frac{1}{t} \quad \text{or} \quad w(z) = \frac{1}{g^{-1}(\frac{1}{z})}.$$

This gives the equation

$$\frac{1}{z} = \sum_{i=1}^{k} p_i w^{-a_i}, \tag{1.3}$$

which was obtained in [1] in another way: by means of recurrent relations on probability. Here we also employ the latter way.

We note that the point $w = z = 1$ lies on the studied curve for each random walk. This is related with the fact that the value of the generating function at the point 1 is equal to the sum of probabilities of all finite values η. This sum is equal to 1 in the case a non-negative mathematical expectation of the increment and this condition defines a half-space in the space of the parameters of walk, on which the generating function depends algebraically. Not only this point, but also (one of) the tangential(s) to the curve for non-negative mathematical expectation of the increment possesses a certain probabilistic meaning: its slope is equal to the mathematical expectation of η, which is finite for a positive mathematical expectation of the increment and is infinite for the zero expectation; in the latter case the tangential is vertical. This point is not singular only in the case of the maximal increment 1. The author does not know how to explain this fact from the point of view of the probability theory.

This work is organized as follows: in Section 2 we describe a special approach, which is used in Section 3 for proving theorems on generating function of the first hitting time of the positive semi-axis.

2. Study of system of algebraic equations

2.1. Reduction of system to single equation. Given a system of algebraic equations $f_i(x_1, \ldots, x_k, y_1, \ldots, y_n) = 0$, $1 \leqslant i \leqslant m$, if $m > k$, we can obtain a system of $m - k$ equation for y's without x's as follows: we consider first two equations are ones for x_1, while other variables are regarded as parameters. Then we apply the following statement.

Proposition 2.1 ([7]). *Two equations of the variable x_1 with parameters $x_2, \ldots, x_n$ possess a joit root x_1 if and only if their resultant, as a polynomial of the parameters, vanishes.*

In the same way we "pair" the second equation with the third one and so forth up to "pairing" two latter equations. As a result we obtain a system not containing x_1 and consisting of $m - 1$ equations. Then we repeat the procedure with x_2, x_3 and so forth up to x_k. This gives the desired system. In our case we need to exclude all variables except for w and z and the total number of equations in the system is exactly so that finally a single equation remains.

2.2. Study of curve by method of analysis of singularities. We are interesting in the question whether a given curve is rational. A polynomial defining this curve possesses a greater degree even for a_i with not large absolute values. For studying such curves we need to apply a genus technique. An algebraic curve in $\mathbb{CP}^2$ as a real manifold is a compact oriented surface. Topologically each such surface is equivalent to a sphere with g handles. Now we can give a definition of the genus.

Definition 2.1 ([8]). *Then number of the handles is called a genus of a curve.*

The genus of the curve is equal to the dimension of the space of holomorphic differential 1-forms on it and this is a birational invariant.

Proposition 2.2 ([8]). *A plane algebraic curve is rational if and only if its genus is equal to zero.*

The study of an algebraic curve involves an analysis of its singularities. For instance, a delta-invariant of the singularity δ_s is the number of infinitesimally closed self-intersection points of the curve "contained" in a given singularity. This is the most important topological characteristic of the singularity and is calculated by the leading part of the curve in the vicinity of the singularity (the origin is moved to the singularity). For calculating the genus, the following statement is employed.

Proposition 2.3 ([8]). *If a plane curve has a degree d, then the genus of the curve is equal to*

$$g = \frac{(d-1)(d-2)}{2} - \sum_s \delta_s,$$

where the summation is made over all singular points on $\mathbb{CP}^2$.

Let us consider the most frequent singularities. For instance, if the terms of the minimal power k forms a homogeneous polynomial without multiple roots, which can be checked by means of the discriminant, the resultant of the polynomial and its derivative, then this is the leading part of the curve. The corresponding point is called an ordinary k-multiple point and its delta-invariant is equal $\frac{k(k-1)}{2}$. In particular, the infinity point $(0:1:0)$ of the graph of the rational function $y(x) = \frac{P(x)}{Q(x)}$, where P and Q has no multiple roots, an ordinary multiple point of multiplicity less by 1 than the degree of the curve, no other singularities, and this is why the genus of the curve calculated by the genus formula is zero, as it should be.

If a homogeneous polynomial of the minimal degree k possess multiple roots, then the leading part involves some terms of greater degree and at the same time, for calculating delta-invariant we apply a so-called procedure of singularity resolving by means of blow-up, which is a transformation $(x, y) \mapsto (xy, y)$, where the coordinate system is chosen so that $x = 0$ is a multiple root of a homogeneous polynomial of degree k and $y = 0$ is not a root. The blow-up splits off an ordinary k-multiple point from the singularity; algebraically this looks as moving out a comment factor y^k in the transformed leading part. This is why the following statement holds.

Proposition 2.4 ([7]). *The delta-invariant of the singularity is equal $\frac{k(k-1)}{2}$, k is the multiplicity of the point, plus the delta-invariant of the transformed singularity if the latter remains.*

As an example we consider a cusp (a, b), which is a singularity with the leading part of form $x^a + y^b$, where $a < b$. A sequence of bloats transforming the cusp (a, b) into the casp $(a, b-a)$ (Euclid algorithm implies that the process is finite) allows us to calculate its delta-invariant and in the case $gcd(a, b) \leqslant 2$ we can write out the result:

Proposition 2.5. *The delta-invariant of the cusp (a, b) is equal to* $\left[\frac{(a-1)(b-1)+1}{2}\right]$.

Доказательство. We argue by induction in the parameters of the cusp. The induction base is as follows: as $a = 1$ (if a and b are coprime), the delta-invariant is 0 (regular point) and as $a = b = 2$ (if $gcd(a, b) = 2$), the delta-invariant is 1 (ordinary double point). Let the statement holds for $a' < b' < b$; we are going to prove for a, b. Each blow-up splits off an ordinary a-multiple point from the cusp (a, b) and transforms it into the cusp $(a, b-a)$. For the latter we apply an induction assumption: the delta-invariant is equal to $\left[\frac{(a-1)(b-a-1)+1}{2}\right]$. Applying the previous proposition and adding the delta-invariant of the multiple point $\frac{a(a-1)}{2}$, we obtain a required result. □

As a first example we provide a curve $x^a = y^b$, which for $2 \leqslant a \leqslant b-2$ and $gcd(a,b)=1$ possesses two cusps, at zero (a,b) and at infinity $(b-a,b)$, and has no other singularities. The calculation of its genus gives an expected value 0 since the curve is obviously rational. As the second example we observe that the only singularity of the degenerate hyperelliptic curve $y^2 = P_n(x)$, where $P_n(x)$ is a polynomial of degree n without multiple roots, is at infinity and this is a cusp $(n-2,n)$ with the delta-invariant $[\frac{(n-1)(n-3)+1}{2}]$ and this is why the genus is equal to $\left[\frac{n-1}{2}\right]$, which coincides with a known result obtained by a topological method.

3. Results

3.1. Random walks with independent increments. We are going to write the equations of curves and study their rationality for several cases in our problem.

Theorem 3.1. *1) An algebraic curve defined by an equation, which is satisfied by a generating function of the first hitting time of the positive semi-axis under a homogeneous discrete random walk without memory is rational in the following cases:*
a) The increment takes values from -1 *to* 3*, each is taken with probability* 0.2*;*
b) The increment takes values from -2 *to* 2*, each is taken with probability* 0.2*;*
c) The increment takes values -1 *and* 4*, each is taken with probability* 0.5*.*
2) If the increments from -2 *to* 2 *occurs with arbitrary probabilities, then in general the associated algebraic curve is not rational.*

Remark 3.1. *Statements 1 and 2 of Theorem 3.1 together with a result from* [1] *on random walks with increments from* -1 *to* 2 *suggest a conjecture that for the minimal increment* -1 *the curve is always rational. The author also expect that in the case of increments from* -2 *to* 2 *the curve is always rational under a symmetric with respect to* 0 *distribution of the increments since the degeneration of the singularity* $w = z = 1$ *in Statement 1b) is related with the appearance of a double vertical tangential at this point; however, this is also just a conjecture. The finding of the genus in the general case requires an independent study, probably, with applying algebraic geometry.*

In order to prove Theorem 3.1, we need the following auxiliary proposition.

Proposition 3.1. *For a random walk with increments from* m *to* M *and associated probabilities* $p_m, \ldots, p_M$ *the sought generating function satisfies the system of equations*

$$\begin{cases} w = w_1 + \ldots + w_M, \\ w_i = z(p_i + \sum\limits_{m \leqslant j \leqslant 0} (p_j \sum\limits_{1 \leqslant s_1, \ldots, s_n \leqslant M, s_n \geqslant i, s_1 + \ldots + s_n = i-j} w_{s_1} \ldots w_{s_n})) \end{cases},$$

where i *runs from* 1 *to* M*.*

Remark 3.2. *If in the general case we argue similarly to one from* [6] *and using Alexander Viktorovich's results, then we obtain* $E(\frac{t^{\xi_\eta}}{g^\eta(t)}) = 1$*. But the only fact we can obtain from this relation is a system of equations for* $w_1, \ldots, w_M$*, each of which contains an unknown function of* z *and these unknown functions are related just by the fact that their linear combination with unknown coefficients is equal to 1. The unknown functions are conditional mathematical expectations of the aforementioned variable for a fixed value* ξ_η*, while the coefficients defines a* *distribution of this variable, which is apriori unknown. This is why we need another method, namely, the method of recurrent relations.*

Доказательство. The proof almost literally reproduces the arguing from [1] used for derivations of similar systems as $M = 1$ and as $m = -1$, $M = 2$. Since the largest jump forward has a length M, the first positive value can be from 1 to M. This is why the generating function is split into the sum of M terms, each of which corresponds to a certain value i at the stopping moment. It can occur after the first step with the probability p_i or later if the first step turned out to be $j \leqslant 0$. In the latter case we mark all intermediate records of our walk, that is, the time moments k, when the value of the variable ξ_k describing the studied walk exceeds the previous maximum; the zeroth record is supposed to be j, while the last record is i.

The distance between neighbouring records can be $1 \leqslant s_k \leqslant M$, while the probability of record achieving time by the homogeneity of the process has the same distribution as the probability of first hitting time of the positive value s_k. By its meaning, the sum of all distances is equal to the length of the segment from j to i and the penultimate record necessarily does not exceed 0 otherwise the stopping would be on that record. It remains to write out the recurrent relations for the stopping probabilities at a given point in a given number of steps using the independence of disjoint pieces of the considered walk and to confirm that the aforementioned system of equations defines the same recurrent relations for the coefficients of the Taylor series of the corresponding functions. □

Before proving Theorem 3.1 we observe the following fact: the first equation in the system defines a hyperplane and moreover, this is the only equation containing w. This is why it is possible not to consider it in studying the curve. Moreover, in the considered cases (under minimal increment -1 or -2) we can express w_2 via w_1 from the second equation, express w_3 via w_2 and w_1, and hence, in view of the second equation, via w_1, and so forth. After that, by a straightforward substitution into the last equation we obtain a relation for w_1 and z. However the degree of the obtained relation, both general and separately in each variable is the same as in our consideration in terms of w and z since the complexity of calculating genus in two ways is same.

Proof of Theorem 3.1. 1a) Initial system of equations reads

$$\begin{cases} w = w_1 + w_2 + w_3, \\ 5w_1 = z(1 + w_1 + w_1^2 + w_2), \\ 5w_2 = z(1 + w_2 + w_1 w_2 + w_3), \\ 5w_3 = z(1 + w_3 + w_1 w_3). \end{cases}$$

We write out the obtained equation

$$\begin{aligned} w^4z^3 + w^3z^2(-15 + 6z) + w^2z(75 - 80z + 26z^2)& \\ + w(-125 + 225z - 170z^2 + 51z^3) &= z(-75 + 110z - 41z^2), \end{aligned}$$

or in coordinates $u = w - 1$, $x = z - 1$

$$\begin{aligned} P(u, x) :=&125ux^3 + u^2x(-25 + 25x + 50x^2) \\ &+ u^4(1 + 3x + 3x^2 + x^3) + u^3(-5 + 15x^2 + 10x^3) + 125x^3 = 0, \end{aligned}$$

that is $w = z = 1$ is an ordinary triple point (the discriminant of the cubic part is non-zero), the delta-invariant is $\frac{3(3-1)}{2} = 3$. Solving the system $P = \frac{\partial P}{\partial u} = \frac{\partial P}{\partial x} = 0$, we see that the curve possesses no other singular points in $\mathbb{C}^2$.

We are going to study singularities at infinities. The present curve has two infinite points $(x : u : t) = (1 : 0 : 0)$ and $(0 : 1 : 0)$ and both of them are singular. In the vicinity of the first we have the leading part $125ut^3 + 50u^2t^2 + u^4 + 10u^3t + 125t^4$, the discriminant is again non-zero and hence, this is an ordinary quadric point, the delta-invariant is $\frac{4(4-1)}{2} = 6$. In the vicinity of the second point the curve reads

$$(t + x)^3 + 5t(t + x)^2(2x - t) + 25xt^2(t + x)(2x - t) + 125x^3t^3 + 125x^3t^4$$

or

$$y^3 + 5ty^2(2y - 3t) + 25t^2y(y - t)(2y - 3t) + 125t^3(y - t)^3 + 125t^4(y - t)^3$$

in the coordinate system $(y = x + t, t)$. Here the leading part is

$$y^3 - 15y^2t^2 + 75yt^4 - 125t^6 - 125t^7 \quad \text{or} \quad (y - 5t^2)^3 - 125t^7.$$

The blow-up $(y, t) \mapsto (yt, t)$ maps the leading part of the singularity into $(y - 5t)^3 - 125t^4$, which corresponds to the cusp $(3, 4)$. Hence, the delta-invariant is $3 + \frac{(3-1)(4-1)}{2} = 6$. Hence, the genus of the curve of degree 7 is equal to $\frac{(7-1)(7-2)}{2} - 3 - 6 - 6 = 0$ and hence, the curve is rational.

1b) We write out the system

$$\begin{cases} w = w_1 + w_2, \\ 5w_1 = z(1 + w_1 + w_1^2 + w_2 + w_1^3 + 2w_1w_2), \\ 5w_2 = z(1 + w_2 + w_1w_2 + w_1^2w_2 + w_2^2). \end{cases}$$

The equation derived from this system reads as

$$\begin{aligned} &w^6z^3 + w^5z^2(-5 + 4z) + w^4z^2(-20 + 10z) + w^3z(50 - 45z + 25z^2) \\ &\qquad + w^2z(75 - 145z + 35z^2) + w(-125 + 200z - 95z^2 + 39z^3) = z(-50 + 65z - 11z^2), \end{aligned}$$

or in the coordinates $u = w - 1$, $x = z - 1$

$$\begin{aligned} &P(u, x) = u^2x^2(225 + 225x) + ux^2(125 + 250x) + u^4x(45 + 90x + 45x^2) \\ &\quad + u^3x(75 + 200x + 125x^2) + u^6(1 + 3x + 3x^2 + x^3) + u^5(5 + 20x + 25x^2 + 10x^3) + 125x^3 = 0. \end{aligned}$$

Thus, at the point $w = z = 1$, the leading part reads as $125x^2v + 75xv^3 + 5v^5 - 100x^4$, where $v = x + u$. After the blow-up $(x, v) \mapsto (xv, v)$ the leading part of the singularity reads as $125x^2 + 75xv + 5v^2$ and this corresponds to an ordinary double point and this is why the delta-invariant of the singularity is $\frac{3(3-1)}{2} + 1 = 4$.

Let us study other singular points. The solution of the system

$$P = \frac{\partial P}{\partial x} = \frac{\partial P}{\partial U} = 0$$

gives one more singularity in $\mathbb{C}^2$, which is $x = -5$, $u = -\frac{5}{2}$. Here the leading part is

$$-6250y^2 + 8000v^2y - 38400v^4,$$

where $v = u + \frac{5}{2}, y = x + 5$, that is, after the blow-up $(y, v) \mapsto (yv, v)$ it becomes

$$-6250y^2 + 8000vy - 38400v^2,$$

which corresponds to an ordinary double point and finally, the delta-invariant is $1 + 1 = 2$.

We proceed to studying the behavior at infinity. The leading part at the point $(1 : 0 : 0)$ is

$$225u^2t^4 + 250ut^5 + 45u^4t^2 + 125u^3t^3 + u^6 + 10u^5t + 125t^6,$$

the discriminant is zero, this is an ordinary sextus point, the delta-invariant is $\frac{6(6-1)}{2}=15$. The leading part at the point $(0:1:0)$ is

$$(t+x)^3+t(t+x)^2(10x+5t)+t^3x(x+t)(125x+75t)+x^2t^5(125t+250x),$$

or

$$y^3+y^2t(10y-5t)+yt^3(y-t)(125y-50t)+t^5(y-t)^2(250y-125t)$$

in coordinates $y=x+t$, t, in new coordinates we can omit unnecessary term and obtain $y^3-5y^2t^2+50yt^5-125t^8$ or $y^3-5t^2(y-5t^3)^2$. After two blow-up $(y,t)\mapsto(yt,t)$ we obtain the leading part in the form $y^3-5(y-5t)^2$ and this corresponds to the cusp $(2,3)$ and the delta-invariant is $1+3(3-1)=7$. The genus of the curve of degree 9 is equal to $\frac{(9-1)(9-2)}{2}-4-2-15-7=0$ and this curve turns out to be rational.

1c) We have the system

$$\begin{cases} w=w_1+w_2+w_3+w_4,\\ 2w_1=z(w_1^2+w_2),\\ 2w_2=z(w_1w_2+w_3),\\ 2w_3=z(w_1w_3+w_4),\\ 2w_4=z(1+w_1w_4). \end{cases}$$

In this case the equation of the curve reads as

$$\begin{aligned} w^5z^4-8w^4z^3&+w^2z(-32+32z-28z^2)+w^3z^2(24-12z+10z^2)\\ &+w(16-24z+32z^2-12z^3+5z^4)=z(8-8z+4z^2), \end{aligned}$$

and in the coordinates $w=1+u$, $z=1+x$

$$\begin{aligned} ux^3(24+40x)&+u^2x^2(8+48x+40x^2)+u^3x(-4+12x+36x^2+20x^3)\\ &+u^5(1+x)^4+u^4(-3-4x+6x^2+12x^3+5x^4)+16x^4=0. \end{aligned}$$

The leading part at the point $u=x=0$ is $24ux^3+8u^2x^2-4u^3x-3u^4+16x^4$, the discriminant is non-zero, this is an ordinary quadric point and the delta-invariant is 6. The curve possesses no other singular points.

We proceed to studying infinite singular points. The leading part at the point $(1:0:0)$ at

$$u^5+5u^4t+20u^3t^2+40u^2t^3+40ut^4+16t^5,$$

the discriminant is non-zero, this is an ordinary quintic point, the delta-invariant is 10. At the point $(0:1:0)$ we have

$$\begin{aligned} (t+x)^4+(5x-3t)(t+x)^3t+(20x-4t)x(1+x)^2t^2\\ +(40x+8t)x^2(1+x)t^3+x^3(24t+40x)t^4+16x^4t^5=0, \end{aligned}$$

the leading part in coordinates $(y=x+t,t)$ is of the form $y^4-8t^2y^3+24t^4y^2-32t^6y+16t^8+16t^9$, or $(y-2t^2)^4+16t^9$. After the blow-up $(y,t)\mapsto(yt,t)$ the leading part becomes $(y-2t)^4+16t^5$ and this corresponds to the cusp $(4,5)$ and this is why the delta-invariant is equal to $6+6=12$. Since the degree of the curve is 9, by the genus formula it is equal to $28-6-10-12=0$ and the curve is again rational.

2) Under the probabilities $p_{-2}=p_1=p_2=\frac{1}{3}$, when the system reads

$$\begin{cases} w=w_1+w_2,\\ 3w_1=z(1+w_1^3+2w_1w_2),\\ 3w_2=z(1+w_1^2w_2+w_2^2), \end{cases}$$

and the equation is

$$w^6z^3 - 3w^5z^2 + w^4z^2(2z-6) + w^2z(9-48z-3z^2) \\ + w^3z(18+9z^2) + w(19z^3+36z-27) + z(18-24z-z^2) = 0,$$

the point $w = z = 1$ is an ordinary triple point. Two singularities, in which one more finite singular point splits off, have irrational coordinates conjugate in $Q[\sqrt{6}]$. This is an obstacle in their direct studying but since nothing splits off from the point $w = z = 1$, then under the general values of the probabilities the delta-invariant of this singularity is equal 3. Concerning other singularities, the sum of the delta-invariants in the general case is not greater than in the above described example. Hence, the genus of the curve is not less than $28-3-(2+15+7) = 1$ and it is no longer rational. In the above described symmetric case the singularity $w = z = 1$, as the calculations show, degenerates. □

3.2. Symmetric random walk with memory. We recall that a discrete random process is a sequence ξ_n of random variables such that for each finite set of integer non-negative numbers $i_1, \dots, i_n$ for quantities $\xi_{i_1}, \dots, \xi_{i_n}$ the joint distribution is well-defined. Before we considered random processes with independent increments, that is, the quantities $\xi_n - \xi_{n-1}$, called nth increments, for different natural n we independent, in particular, they pairwise did not correlate. Such processes are Markov ones: the conditional distribution ξ_{n+1} for a fixed value ξ_n does not change under posing additional conditions for ξ_k as $k < n$. In this section we consider random walks, in which each increment from the second step correlates with the increment at the previous step. At the same, adding a condition for ξ_{n-1} influences the aforementioned conditional distribution and the process is no longer Markov one. Such processes are usually called processes with memory. However, this process can be considered as Markov one if into the notion "state" we include the information on the increment from the previous state.

For simplicity we suppose that walks are symmetric with steps ± 1. Then the first increment takes both admissible values with the probability $\frac{1}{2}$, and each next value coincides with the previous one with probability p and is opposite with the probability $q = 1-p$. In this walk the correlation coefficient between neighbouring increments is equal to $k = p - q$.

Proposition 3.2. *The generating function of the first hitting time of the positive semi-axis for the above described random walk as $0 < k < 1$ satisfies the equation*

$$2w = z[1 + w^2 - k^2(w-z)^2)], \tag{3.1}$$

which defines a regular elliptic curve, which is not rational.

Доказательство. By w_- and w_+ we denote generating functions for the first time of reaching $x+1$ from x under the conditions that at the previous step we respectively has a negative or positive increment. If at the first step the increment is positive (the probability of this even is $\frac{1}{2}$), then the first hitting time of the positive semi-axis is equal to 1. If at the first step the increment is negative, then we first need to return back to 0 and this corresponds to the definition of w_-. Then we need to reach 1 and this corresponds to w_+ since before this 0 there was a positive increment. This is why the sought function is calculated by the formula $w = \frac{z}{2}(1 + w_-w_+)$.

We argue in the same way for w_- and w_+ and we get the equations

$$\begin{cases} w_- = z(q + pw_-w_+), \\ w_+ = z(p + qw_-w_+). \end{cases}$$

Multiplying one equation by the other and denoting $u = w_-w_+$, we obtain

$$u = z^2(p+qu)(q+pu), \tag{3.2}$$

and since $w = \frac{z}{2}(1+u)$, we express $u = 2\frac{w}{z} - 1$ and after substituting into (3.2), we take into consideration that $p = \frac{1+k}{2}$ and $q = \frac{1-k}{2}$ and we obtain the relation (3.1).

Now we rewrite the obtained equation in the form

$$w^2 z(1-k^2) + 2w(k^2z^2 - 1) + z(1-k^2z^2) = 0. \tag{3.3}$$

We calculate the discriminant of this square equation with respect to w: it is equal to $(1-k^2z^2)(1-z^2)$ and, as $0 < k < 1$, it has for different roots and this is why it defines a regular elliptic curve. As it is known, the genus of this curve is 1 and therefore, it is rational. $\square$

In the case $k = 0$ equation (3.1), as it is expected, becomes a usual equation for generating function of the first hitting time of the positive semi-axis under a simple symmetric random walk. Another limiting case is interesting: $k = \pm 1$. In this case the equation becomes $(2w - z)(1-z^2) = 0$ and this implies $w = \frac{z}{2}$. This result is explained by the fact that in the case of a linear dependence of the next increment on the previous one the process either stops at the first step or never. It is curious to mention that if k differs from ± 1, then the curve intersect the straight lint $z = 1$ at a unique point $w = z = 1$ and this means that the process ends in a finite time with the probability 1 and this is not true in the limiting case.

In more complicated case we need to apply the methods described in the previous section. Consider, for instance, the following modification of the previous walk: besides the increments ± 1, the zero increment is also possible. In other words, at the first step the probabilities are equal $\frac{1}{3}$, while at the next steps they are equal to this number only once the previous increment is 0, while under the increment ± 1 the probabilities are equal to $p_{\pm 1} = p$, $p_{\mp 1} = q$, $p_0 = 1-p-q$ and self-correlation coefficient is equal to $k = \frac{(5-3p-3q)(p-q)}{2}$.

Let us write an equation for the generating function of such walk.

Proposition 3.3. *The generating function of the first hitting time of* 1 *for the aforementioned walk satisfies the equation*

$$\begin{aligned}(1 - 3p - 3q + 2p^2 + 5pq + 2q^2)w^2z^3 + (3p + 3q - 3p^2 - 12pq - 3q^2)w^2z^2 + 9pqw^2z \\ + (p-q)^2wz^3 + 3(p-q)^2wz^2 + wz - 3w - (p-q)^2z^3 + z = 0,\end{aligned} \tag{3.4}$$

which for general p, q defines a hyperelliptic curve of genus 2 *(as $p = q$ we obtain a process without the memory and the curve is rational, while for $p + q = 1$ we obtain the process from the previous proposition and the genus is* 1*.) It is likely that there are parameters, for which the curve is degenerate, but the discriminant of the polynomial of sixth degree with parameters can not be calculated in a reasonable time.*

Remark 3.3. *As in the case of walk without the memory, the projectivization of the curve has singularities. At the same time, if for walk without memory the compactification $\mathbb{CP}^1 \times \mathbb{CP}^1$ does not eliminate the singularities (and for symmetric walks with memory without zero increment it even added them), at the latter case the curve in the mentioned compactification possesses no singularities.*

Доказательство. Reproducing the arguing for the previous process with minor modifications, we obtain that the generating function for the first hitting time of the positive semi-axis satisfies the system

$$\begin{cases} w = \dfrac{z}{3}(1 + w + w_- w_+), \\ w_+ = z(p + p_0 w + q w_- w_+), \\ w_- = z(q + p_0 w + p w_- w_+). \end{cases}$$

Excluding w_+ and w_- from this system by the resultant method and taking into consideration that $p_0 = 1 - p - q$, we obtain equation (3.4). By a fiber-to-fiber transform this quintic is mapped into a hyperelliptic curve of form $w^2 = P_6(z)$ with some polynomial of sixth degree in the right hand side. The calculation of the discriminant of the polynomial, for instance, as $p + q = \frac{2}{3}$ and $|p - q| = \frac{1}{2}$ shows that in the general case it possesses no multiple roots. The genus of such curve is equal to 2. $\square$

4. Acknowledgments

The author thanks professor Grigory Gennadievich Amosov for a comprehensive help and support in the writing this work and also express his gratitude to Alexander Viktorovich Shklyaev for a provided elegant arguing.

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