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SCENARIO OF STABLE TRANSITION FROM DIFFEOMORPHISM OF TORUS ISOTOPIC TO IDENTITY ONE TO SKEW PRODUCT OF ROUGH TRANSFORMATIONS OF CIRCLE

D.A. BARANOV, E.V. NOZDRINOVA, O.V. POCHINKA

Abstract. In this paper, we consider gradient-like diffeomorphisms of a two-dimensional torus isotopic to the identical one. The isotopicity of diffeomorphisms f_0, f_1 on an n-manifold M^n means the existence of some arc $\{f_t : M^n \to M^n, t \in [0,1]\}$ connecting them in the space of diffeomorphisms. If isotopic diffeomorphisms are structurally stable (qualitatively not changing their properties with small perturbations), then it is natural to expect the existence of a stable arc (qualitatively not changing its properties under small perturbations) connecting them. In this case, one says that the isotopic diffeomorphisms f_0, f_1 are stably isotopic or belong to the same class of stable isotopic connectivity. The simplest structurally stable diffeomorphisms on surfaces are gradient-like transformations having a finite hyperbolic non-wandering set, stable and unstable manifolds of various saddle points of which do not intersect. However, even on a two-dimensional sphere, where all orientation-preserving diffeomorphisms are isotopic, gradient-like diffeomorphisms are generally not stably isotopic. The countable number of pairwise different classes of stable isotopic connectivity is constructed on the base of a rough transformation of the circle $\phi_{\frac{k}{m}}$ with exactly two periodic orbits of the period m and the rotation number $\frac{k}{m}$, which can be continued to a diffeomorphism $F_{\frac{k}{m}} : \mathbb{S}^2 \to \mathbb{S}^2$ with two fixed sources at the North and South poles. On the torus $\mathbb{T}^2$, the model representative in the considered class is the skew products of rough transformations of a circle. We show that any isotopic gradient-like diffeomorphism of a torus is connected by a stable arc with some model transformation.

Keywords: diffeomorphisms, torus, stable arcs.

Mathematics Subject Classification: 37B35, 37C20, 37G10

1. Introduction and result

Throughout the work we deal with closed connected oriented n-manifolds M^n and homeomorphisms or diffeomorphisms defined on them, which preserve the orientation. The isotopy of diffeomorphisms f_0, $f_1 : M^n \to M^n$ means the existence of some curve $\{f_t : M^n \to M^n, t \in [0,1]\}$ connecting them in the space of diffeomorphisms. If isotopic diffeomorphisms are *structurally stable* (qualitatively not changing its properties under small perturbations), then it is natural to expect the existence of a *stable arc* (qualitatively not changing its properties under small perturbations) connecting them; for an exact definition see Section 2.3. In this case one says

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that the diffeomorphisms f_0, $f_1 : M^n \to M^n$ are *stably isotopical* or belong to the same class of a *stable isotopic connectivity*.

The simplest structurally stable diffeomorphisms are *gradient-like* transforms possessing a finite hyperbolic non-wandering set, the stable and unstable manifolds of various saddle points of which either are disjoint or intersect transversally by a set of a positive dimension, for an exact definition see Section 2.1. However, even gradient-like isotopic diffeomorphisms in the general case are not stably isotopic. Already on the circle, where all diffeomorphisms are mutually isotopic, there appears a countable set of classes of stable isotopic connectivity of rough transformations of the circle, each of which is uniquely determined by the rotation number $\frac{k}{m}$, $k \in (\mathbb{N} \cup 0)$, $m \in \mathbb{N}$, $k < m$, $(k, m) = 1$, where (k, m) is the greatest common divisor of the numbers k, m [1].

A similar situation occurs for gradient-like diffeomorphisms of 2-sphere. Namely, we consider $\mathbb{S}^1$ as an equator of the sphere $\mathbb{S}^2$. Then the diffeomorphisms of the circle $\phi_{\frac{k}{m}}$ with exactly two periodic orbits of period m and rotation number $\frac{k}{m}$ can be continued to a diffeomorphism $F_{\frac{k}{m}} : \mathbb{S}^2 \to \mathbb{S}^2$ with two fixed sources at the North and South poles. We denote by $F_0 : \mathbb{S}^2 \to \mathbb{S}^2$ a diffeomorphism "source-sink". It was shown in work [2] that each gradient-like diffeomorphism of 2-sphere is connected by a stable arc with exactly one of diffeomorphisms F_0, $F_{\frac{k}{m}}$, $F^{-1}_{\frac{k}{m}}$, $m \geqslant 3$, $k < m/2$, stable isotopic classes of which are pairwise disjoint.

The obtained result is closely related with the Nielsen-Thurston theory of homeomorphisms of surfaces, in particular, with the classification of periodic homeomorphisms of two-dimensional sphere obtained by Kerekjarto [3]. A class of topological conjugacy of *periodic transformation* of 2-sphere with period m (a homeomorphism, the mth degree of which is the identical mapping) is also determined by the rotation number $\frac{k}{m}$ about the axis North pole–South pole. This relation is not accidental and is due to the fact that each gradient-like diffeomorphism of a surface is conjugate to the composition of a periodic homeomorphism with a translation by the time unit of a gradient-like flow [4]. Moreover, the dynamics of a gradient-like diffeomorphism on a non-wandering set coincides with the dynamics of a periodic homeomorphism.

For instance, it follows from the results of work [2] that a stable isotopic class of a gradient-like diffeomorphism of 2-sphere without fixed sinks is completely determined by the topological conjugacy class of its periodic component.

In the present work we show that on the two-dimensional torus $\mathbb{T}^2$ the picture is fundamentally different.

We consider a class G of gradient-like diffeomorphisms of 2-torus $\mathbb{T}^2$ isotopic to the identical one. The simplest examples of such diffeomorphisms are skew products of the diffeomorphisms $\phi_{\frac{k_1}{m_1}}$, $\phi_{\frac{k_2}{m_2}}$ (see the construction of skew products in Section 3) and by GM we denote the class of skew products. It is straightforward to find that the periodic component of the skew product possesses the period $m = \mathrm{SCM}(m_1, m_2)$ and therefore, all periodic points of such diffeomorphism possess the same period. At the same time, all periodic transformations of the torus isotopic to the identical one and not equalling to the identical diffeomorphism are topologically conjugate to the rotation about the parallel with the rotation number $\frac{1}{m}$ [5]. Thus, the class of topological conjugacy of a periodic transform isotopic to the identical one is completely determined by the period of the homeomorphism.

The main result of the present work is the following theorem.

Theorem 1.1. *Each diffeomorphism $f \in G$ is connected by a stable arc with one of the diffeomorphisms $F_f \in GM$.*

2. Auxiliary facts

2.1. Morse–Smale diffeomorphisms. Let a diffeomorphism $f : M^n \to M^n$ be given on a smooth closed (compact and without edge) n-manifold ($n \geqslant 1$) M^n with a metric d.

Two diffeomorphisms $f, f' : M^n \to M^n$ are called *topologically conjugate* if there exists a homeomorphism such that $h : M^n \to M^n$ such that $fh = hf'$.

A point $x \in M^n$ is called *wandering* for f if there exists an open neighbourhood U_x of the point x such that $f^n(U_x) \cap U_x = \emptyset$ for $n \in \mathbb{N}$. Otherwise the point x is called *non-wandering*. The set of non-wandering points of a diffeomorphism f is called *non-wandering set* and is denoted by Ω_f.

For instance, all limiting points of a diffeomorphism are non-wandering. We recall that a point $y \in M^n$ is called *ω-limiting* point for a point $x \in M^n$ if there exists a sequence $t_k \to +\infty$, $t_k \in \mathbb{Z}$, such that $\lim\limits_{t_k \to +\infty} d(f^{t_k}(x), y) = 0$. A set $\omega(x)$ of all ω-limiting points for the point x is called its *ω-limiting set*. Replacing $+\infty$ by $-\infty$, we define an *α-limiting set* $\alpha(x)$ of the point x. The set $L_f = \mathrm{cl}\,(\bigcup\limits_{x \in M^n} \omega(x) \cup \alpha(x))$ is called a *limiting set* of the diffeomorphism f.

If the set Ω_f is finite, then each point $p \in \Omega_f$ is periodic and by $m_p \in \mathbb{N}$ we denote the period of a periodic point p. To each periodic point p we associate *stable* and *unstable* manifolds defined as follows:

$$W^s_p = \{x \in M^n : \lim_{k \to +\infty} d(f^{km_p}(x), p) = 0\}, \qquad W^u_p = \{x \in M^n : \lim_{k \to +\infty} d(f^{-km_p}(x), p) = 0\}.$$

One says that periodic orbits $\mathcal{O}_1, \dots, \mathcal{O}_k$ form a *cycle* if $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_{i+1}} \neq \emptyset$ for $i \in \{1, \dots, k\}$ and $\mathcal{O}_{k+1} = \mathcal{O}_1$.

A periodic point $p \in \Omega_f$ is called *hyperbolic* if the absolute values of all eigenvalues of the Jacobi matrix $\left(\frac{\partial f^{m_p}}{\partial x}\right)|_p$ are not equal to one. If the absolute values of all eigenvalues are greater (less) than one, then p is called *sink* (*source*) point. Sink and source points are called *node* points. If a hyperbolic periodic point is not *node*, then it is called a *saddle point*.

It follows from the hyperbolic structure of a periodic point p that its stable W^s_p and unstable W^u_p manifolds are the images for injective immersions of the spaces $\mathbb{R}^{q_p}$ and $\mathbb{R}^{n-q_p}$, where $q_p \in \{0, \dots, n\}$ is the number of the eigenvalues of the Jacobi matrix taken counting multiplicities, the absolute values of which exceed one. The number ν_p, which is equal to $+1(-1)$ if the mapping $f^{m_p}|_{W^u_p}$ preserves (changes) the orientation of W^u_p is called an *orientation type* of the point p. A component of linear connectivity of the set $W^u_p \setminus p$ ($W^s_p \setminus p$) is called an *unstable (stable) separatrix* of the point p. A smallest natural number m such that $f^m(\ell) = \ell$ is called a *period of separatrix* ℓ.

A closed f-invariant set $A \subset M^n$ is called an *attractor* of a discrete dynamical system f if it possesses a compact neighbourhood U_A such that $f(U_A) \subset \mathrm{int}\, U_A$ and $A = \bigcap\limits_{k \geqslant 0} f^k(U_A)$. The neighbourhood U_A is called either trapping or isolating. A *repeller* is defined as an attractor for f^{-1}. A complement to a trapping neighbourhood of an attractor is a trapping neighbourhood of a repeller; such attractor and repeller are called *conjugate*.

A diffeomorphism $f : M^n \to M^n$ is called a *Morse-Smale diffeomorphism* if

1) a non-wandering set Ω_f consists of finitely many hyperbolic orbits;

2) the manifolds W^s_p, W^u_q intersect transversally for all non-wandering points p, q.

A Morse-Smale diffeomorphism is called *gradient-like* if the condition $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$ for different points $\sigma_1, \sigma_2 \in \Omega_f$ implies that $\dim W^u_{\sigma_1} < \dim W^u_{\sigma_2}$.

2.2. Periodic homeomorphisms and their relation with gradient-like diffeomorphisms of surfaces. A homeomorphism $\varphi : M^2 \to M^2$ is called *periodic* if there exists $n \in \mathbb{N}$

such that $\varphi^n = id$. The smallest among such n is called a period of φ. A point x_0 is called a *point of a smaller period* $n_0 < n$ of a homeomorphism φ if $\varphi^{n_0}(x_0) = x_0$.

According to the results by J. Nielsen [6], for each orientation preserving periodic homeomorphism of an oriented surface M^2 of genus p the set B_φ of points of small period is finite, while the space of orbits of the action of φ on M^2 is a sphere with $g \leqslant p$ handles called *modular surface*. In the vicinity of the point x_0 of a smaller period n_0 the mapping f^{n_0} is conjugate to the rotation by some rational angle $2\pi\dfrac{\delta_0}{\lambda_0}$, where $\lambda_0 = \dfrac{n}{n_0}$ and δ_0 is an integer number coprime with n_0.

We denote by X_i, $i = 1, \dots, k$, the orbits of points of smaller period, while their periods are denoted by n_i and we let $\lambda_i = \dfrac{n}{n_i}$. We denote by $\dfrac{\delta_i}{\lambda_i}$ the corresponding rotation number and we define a number d_i by the condition $d_i\delta_i \equiv 1 \pmod{\lambda_i}$. The set of the parameters

$$(n, p, g, n_1, \dots, n_k, d_1, \dots, d_k)$$

of a periodic homeomorphism φ is called its *total characteristics*.

Statement 2.1 ([5]). *For an orientation preserving periodic homeomorphism $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ of period $n \in \mathbb{N}$ the following conditions are equivalent:*

1. *φ is homotopic to the identity mapping;*
2. *$B_\varphi = \emptyset$;*
3. *$g = 1$;*
4. *φ is topologically conjugate to the diffeomorphism $\Psi_n\left(e^{i2x\pi}, e^{i2y\pi}\right) = \left(e^{i2\pi\left(x+\frac{1}{n}\right)}, e^{i2y\pi}\right)$.*

Statement 2.2 ([4, Thms. 3.1, 3.3]). *Each orientation preserving gradient-like diffeomorphism $f : M^2 \to M^2$ is represented as a composition*

$$f = \varphi \circ \xi^1,$$

where ξ^1 is the translation by the time unit along the trajectories of the gradient flow ξ^t of some Morse function[1]*, while φ is a periodic homemorphism. At the same time,*

1. *$\Omega_f = \Omega_{\xi^1}$;*
2. *$f|_{\Omega_f} = \varphi|_{\Omega_f}$;*
3. *$B_\varphi \subset \Omega_f$;*
4. *the period of each separatrix of each saddle point of a diffeomorphism f coincides with the period of the homeomorphism φ.*

The next fact follows directly from Statements 2.1, 2.2.

Corollary 2.1. *All periodic points and separatrices of a gradient-like diffeomorphism of the torus, which is isotopic to the identical diffeomorphism, have the same period, which is equal to the period of the periodic component.*

2.3. Stability of arc of diffeomorphisms. We consider an one-parametric family of diffeomorphisms (*arc*) $\varphi_t : M^n \to M^n$, $t \in [0, 1]$. The arc φ_t is called *smooth* if the mapping $F : M^n \times [0, 1] \to M^n$ defined by the formula $F(x, t) = \varphi_t(x)$ is a *diffeotopy*, that is, a smooth mapping being a diffeomorphism for each fixed t. In the topological category such mapping is called *isotopy*.

[1] C^2-smooth function with non-degenerate critical points.

A smooth arc φ_t is called a *smooth product* of smooth arcs ϕ_t and ψ_t such that $\phi_1 = \psi_0$ if

$$\varphi_t = \begin{cases} \phi_{2\tau(t)}, & 0 \leqslant t \leqslant \dfrac{1}{2}, \\ \psi_{2\tau(t)-1}, & \dfrac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

where $\tau : [0,1] \to [0,1]$ is a smooth monotone mapping such that $\tau(t) = 0$ for $0 \leqslant t \leqslant \frac{1}{3}$ and $\tau(t) = 1$ for $\frac{2}{3} \leqslant t \leqslant 1$. We shall write

$$\varphi_t = \phi_t * \psi_t.$$

According to [7], arcs φ_t, φ'_t are called *conjugate* if there exist homeomorphisms $h : [0,1] \to [0,1]$, $H_t : M^n \to M^n$ such that $H_t\varphi_t = \varphi'_{h(t)}H_t$, $t \in [0,1]$, and H_t depends continuously on t. A smooth arc φ_t is called *stable* if it possesses an open neighbourhood in the space of diffeotopies such that each arc in this neighbourhood is conjugate to the arc φ_t.

By $\mathcal{Q}$ we denote the set of smooth arcs φ_t, $t \in [0,1]$, such that each arc in this set begins and ends at Morse-Smale diffeomorphism and each diffeomorphism φ_t has a finite limiting set.

It was also established in [7] that an arc $\varphi_t \in \mathcal{Q}$, where $t \in [0,1]$, is stable if and only if all its points are structurally stable diffeomorphisms except for finitely many bifurcation points φ_{b_i}, $i = 1, \dots, q$, such that

1) the limiting set of the diffeomorphism φ_{b_i} contains a unique non-hyperbolic periodic orbit, which is either saddle-node or flip;

2) the diffeomorphism φ_{b_i} has no cycles;

3) stable and unstable manifolds of all periodic points of the diffeomorphism φ_{b_i} intersect transversally;

4) φ_{b_i} has the unique non-hyperbolic periodic orbit, which is an orbit of a non-critical saddle-node or flip and bifurcates in a general way[1].

3. Skew products of rough transformations of circle

In this section we construct a skew product of rough transformations of a circle. In order to do this, we consider a pair of natural numbers m_1, m_2 and non-negative integer numbers k_1, k_2, which are coprime respectively with m_1, m_2 and are such that $\frac{k_1}{m_1} \leqslant \frac{k_2}{m_2}$. By μ we denote the greatest common factor of the numbers m_1, m_2 and we choose an integer number ν coprime with μ. We define a diffeomorphism $\bar{\phi}_{\frac{k_i}{m_i}} : \mathbb{R} \to \mathbb{R}$, $i \in \{1, 2\}$ by the formula

$$\bar{\phi}_{\frac{k_i}{m_i}}(x_i) = x_i + \frac{1}{4m_i\pi}\sin(2m_i\pi x_i) + \frac{k_i}{m_i}.$$

We let

$$\bar{F}_{\frac{k_1}{m_1}, \frac{k_2}{m_2}, \frac{0}{1}}(x_1, x_2) = \left(\bar{\phi}_{\frac{k_1}{m_1}}(x_1), \bar{\phi}_{\frac{k_2}{m_2}}(x_2)\right).$$

We define a diffeomorphism $\bar{h}_{\frac{\nu}{\mu}} : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$\bar{h}_{\frac{\nu}{\mu}}(x_1, x_2) = \left(x_1 + \frac{\nu}{\mu}x_2, x_2\right).$$

Let

$$\bar{F}_{\frac{k_1}{m_1}, \frac{k_2}{m_2}, \frac{\nu}{\mu}} = \bar{h}_{\frac{\nu}{\mu}} \circ \bar{F}_{\frac{k_1}{m_1}, \frac{k_2}{m_2}, \frac{0}{1}} \circ \bar{h}^{-1}_{\frac{\nu}{\mu}} : \mathbb{R}^2 \to \mathbb{R}^2.$$

[1]For exact definition of all these objects see, for instance, [1].

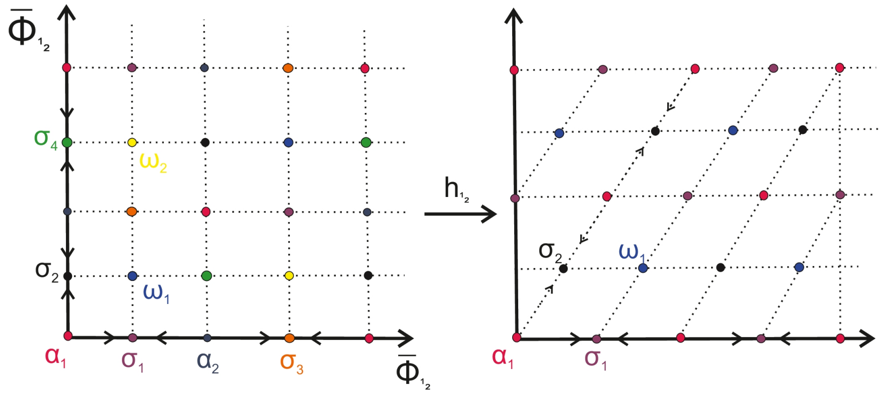

FIGURE 1. Skew product of diffeomorphisms $\phi_{\frac{1}{2}}, \phi_{\frac{1}{2}}$.

It is straightforward to confirm that the diffeomorphism $\bar{F}_{\frac{k_1}{m_1},\frac{k_2}{m_2},\frac{\nu}{\mu}}$ possesses the property

$$\bar{F}_{\frac{k_1}{m_1},\frac{k_2}{m_2},\frac{\nu}{\mu}}(x_1+1, x_2+1) = \bar{F}_{\frac{k_1}{m_1},\frac{k_2}{m_2},\frac{\nu}{\mu}}(x_1, x_2) + (1,1) \tag{3.1}$$

and hence, it is projected by means of the covering

$$\pi(x_1, x_2) = (e^{2\pi i x_1}, e^{2\pi i x_2}) : \mathbb{R}^2 \to \mathbb{T}^2,$$

into the diffeomorphism of the torus

$$F_{\frac{k_1}{m_1},\frac{k_2}{m_2},\frac{\nu}{\mu}} = \pi \bar{F}_{\frac{k_1}{m_1},\frac{k_2}{m_2},\frac{\nu}{\mu}} \pi^{-1} : \mathbb{T}^2 \to \mathbb{T}^2.$$

The constructed diffeomorphism has four periodic orbits of period $m = \frac{m_1 m_2}{\mu}$, one sink, one source and two saddle orbits, see Figure 1. Moreover, the diffeomorphism $F_{\frac{k_1}{m_1},\frac{k_2}{m_2},\frac{\nu}{\mu}}$ is isotopic to the identical one since $F_{\frac{k_1}{m_1},\frac{k_2}{m_2},\frac{\nu}{\mu}}(\nu,\mu) = (\nu,\mu)$ and $F_{\frac{k_1}{m_1},\frac{k_2}{m_2},\frac{\nu}{\mu}}(0,1) = (0,1)$.

We recall that an *algebraic automorphism* $\widehat{L} : \mathbb{T}^2 \to \mathbb{T}^2$ of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a diffeomorphism defined by the matrix $L = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ belonging to the set $GL(2,\mathbb{Z})$ of *unimodular integer matrices*, which are the matrices with the denominator equalling to ± 1. That is,

$$\widehat{L}(x,y) = (x,y) \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = (\alpha x + \beta y, \gamma x + \delta y) \pmod 1.$$

Let $J = \begin{pmatrix} \nu_1 & \nu_2 \\ \mu_1 & \mu_2 \end{pmatrix}$ be an integer matrix with the denominator $\mu \in \mathbb{N}$. Then there exists a unique integer matrix $S_J = \begin{pmatrix} \nu_1 & \gamma \\ \mu_1 & \delta \end{pmatrix}$ with the unit determinant such that $S_J^{-1}(\langle \nu_2, \mu_2 \rangle) = \langle \nu, \mu \rangle$, where $\nu \in [0, \mu)$, see, for instance, [8]. We let

$$F_J = \widehat{S_J} F_{\frac{k_1}{m_1},\frac{k_2}{m_2},\frac{\nu}{\mu}} \widehat{S_J}^{-1} : \mathbb{T}^2 \to \mathbb{T}^2.$$

We call the diffeomorphism F_J a *skew product of rough transformations of the circle* $\phi_{\frac{k_1}{m_1}}$, $\phi_{\frac{k_2}{m_2}}$, see Figure 1.

4. Connected characteristic space of diffeomorphism $f \in G$

We consider orientation preserving gradient-like diffeomorphism f defined on a smooth oriented closed surface M^2. We denote by Ω^0_f, Ω^1_f, Ω^2_f the set of sinks, saddles and sources of the diffeomorphism f. For each (possibly empty) f-invariant set $\Sigma\subset\Omega^1_f$ we let

$$A_\Sigma=\Omega^0_f\cup W^u_\Sigma,\qquad R_\Sigma=\Omega^2_f\cup W^s_{\Omega^1_f\setminus\Sigma}.$$

It follows from work [9] that A_Σ and R_Σ are the attractor and repeller of the diffeomorphism f, which are called *conjugate*. The set

$$V_\Sigma=M^2\setminus(A_\Sigma\cup R_\Sigma)$$

is called *characteristic space*. We denote by $\hat V_\Sigma$ the space of orbits of action of the group $F=\{f^k,\,k\in\mathbb{Z}\}$ on the characteristic space V_Σ and let $p_{_\Sigma}:V_\Sigma\to\hat V_\Sigma$ be its natural projection. According to work [10], each connected component of the manifold $\hat V_\Sigma$ is homeomorphic to the two-dimensional torus.

Statement 4.1 ([11, Thm. 1.1]). *For each orientation preserving gradient-like diffeomorphism $f:M^2\to M^2$ there exists a set Σ such that the space of orbits $\hat V_\Sigma$ is connected.*

For each diffeomorphism f and set Σ obeying the assumptions of Statement 4.1 we let

$$A_f=A_\Sigma,\qquad R_f=R_\Sigma,\qquad V_f=V_\Sigma,\qquad \hat V_f=\hat V_\Sigma,\qquad p_{_f}=p_{_\Sigma}.$$

Then, see, for instance, [4, Prop. 2.1], the set $\hat V_{_f}$ is connected and is homeomorphic to a torus, while the set V_f is not connected in the general case; by m we denote the number of connected components of the set V_f. Then the set V_f is homeomorphic to $(\mathbb{R}^2\setminus O)\times\mathbb{Z}_m$ and the restriction of the diffeomorphism f to V_f is topologically conjugate by means of some homeomorphism $h_f:V_f\to(\mathbb{R}^2\setminus O)\times\mathbb{Z}_m$ to a periodic contraction $a_m:(\mathbb{R}^2\setminus O)\times\mathbb{Z}_m\to(\mathbb{R}^2\setminus O)\times\mathbb{Z}_m$ given by the formula

$$a_m(x,y,i)=\begin{cases}(x,y,i+1), & i=0,\dots,m-2,\\ \left(\frac{x}{2},\frac{y}{2},0\right), & i=m-1.\end{cases}$$

For $i\in\{0,\dots,m-1\}$ we let

$$\begin{aligned} &W_i=h_f^{-1}((\mathbb{R}^2\setminus O)\times\{i\}), && c_i=h_f^{-1}(\mathbb{S}^1\times\{i\}),\\ &W_i^+=h_f^{-1}((\mathbb{D}^2\setminus O)\times\{i\}), && W_i^-=\operatorname{cl}(W_i\setminus W_i^+),\\ &W^\pm=W_0^\pm\cup\dots\cup W^\pm_{m-1}, && c=c_0\cup\dots\cup c_{m-1}.\end{aligned}$$

Then $U=W^+\cup A_f$, $V=W^-\cup R_f$ are trapping neighbourhoods of the attractor A_f and repeller R_f, respectively, that is,

$$f(U)\subset U,\qquad A_f=\bigcap_{j\in\mathbb{N}}f^j(U);\qquad f^{-1}(V)\subset V,\qquad R_f=\bigcap_{j\in\mathbb{N}}f^{-j}(V).$$

Lemma 4.1. *For each diffeomorphism $f\in G$ (up to considering the diffeomorphism f^{-1}) the following holds true:*

1. *The set U consists of $m\in\mathbb{N}$ mutually disjoint disks D, $f(D)$, …, $f^{m-1}(D)$ such that $f^m(\operatorname{cl}D)\subset\operatorname{int}D$;*
2. *The attractor A_f consists of m connected components A, $f(A)$, …, $f^{m-1}(A)$ such that $A=\bigcap\limits_{j\in\mathbb{N}}f^{jm}(D)$ and $f^m(A)=A$;*
3. *The repeller R_f is connected.*

Proof. By construction, the curves of the set c partition the ambient manifold $M^2\cong\mathbb{T}^2$ into two non-empty parts U, V, the boundaries of which are these curves. Since the diffeomorphism f is isotopic to the identity mapping, then all curves of the set c are pairwise homotopic. Suppose that these curves are not trivial, then $M^2\setminus c$ consists of m annuli. At the same time, $m\geqslant 2$

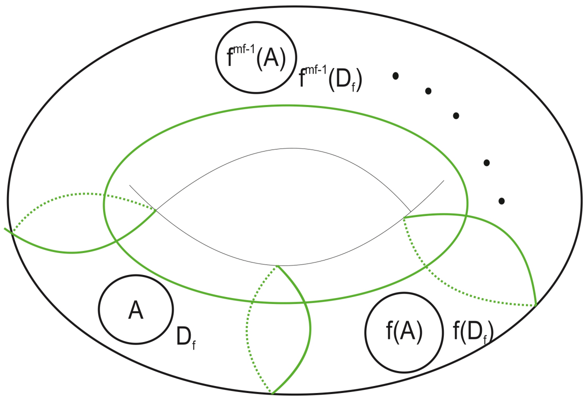

FIGURE 2. Illustration to the proof of Lemma 4.1.

since each curve in c is adjacent to two disjoint sets U, V. Let U_0 be one of the annuli enveloped by the curves c_0, $f^l(c_0)$, $(l,m)=1$. Since $f^m(U_0)\subset U_0$, then $f^m(c_0)$, $f^{m+l}(c_0)\subset U_0$, and this contradicts to the fact that the diffeomorphism f is isotopic to the identity mapping.

Thus, each curve c_i in the set c envelopes a disk d_i; we let $D=d_0$. For the sake of definiteness, we suppose that the disk D is a connected component of the set U; otherwise this holds for the diffeomorphism f^{-1}. Since the restriction of the diffeomorphism f^m to $D\cap V_f$ is conjugate with the linear contraction, then $f^m(\operatorname{cl} D)\subset \operatorname{int} D$. Thus, the set $A=\bigcap\limits_{j\in\mathbb{N}} f^{jm}(D)$ is connected. Since $A_f=\bigcap\limits_{j\in\mathbb{N}} f^{jm}(U)$, then A is a connected component of the attractor A_f and $D=(D\cap V_f)\cup A$. In what follows we consider separately two cases: (1) $m=1$, (2) $m>1$.

(1) If $m=1$, then $A_f=A$, $R_f=\bigcap\limits_{j\in\mathbb{N}} f^{-j}(\mathbb{T}^2\setminus D)$ are connected attractors and repeller and this proves the lemma.

(2) If $m>1$, then $f(c)\cap(D\cap V_f)=\emptyset$ by the conjugacy to the periodic contraction and $f(c)\cap A=\emptyset$ since $f(c)\subset V_f$. Thus, $f(D)\cap D=\emptyset$ since $f(c)\cap D=\emptyset$. Therefore, the disk $f(D)$ contains the connected component $f(A)$ of the attractor A_f, which is disjoint with A, see Figure 2. Arguing in the same way, we obtain m different connected components $A, f(A),\dots,f^{m-1}(A)$ of the attractor A_f and this means that the attractor A_f consists of one orbit of period m.

Thus, the set U is the union of mutually disjoint disks D, $f(D)$, $\dots$, $f^{m-1}(D)$. This yields that the set $V=\mathbb{T}^2\setminus U$ is connected and this implies the connectedness of the repeller R_f. □

5. CONSTRUCTION OF A STABLE ARC FROM THE DIFFEOMORPHISM $f\in G$ TO THE DIFFEOMORPHISM $F_f\in GM$

5.1. Trivialization of the attractor A_f. In the present section we prove the following statement.

Lemma 5.1. *Each diffeomorphism $f\in G$ is connected by a stable arc with a diffeomorphism $g\in G$, which has the unique sink orbit.*

Proof. Let A_f be an attractor of a diffeomorphism f obeying the assumptions of the lemma. We let $\tilde f=f^m$. Then $\tilde f(D)\subset\operatorname{int} D$ and therefore, the diffeomorphism $\tilde f$ can be continued to a orientation-preserving diffeomorphism $\tilde f:\mathbb{S}^2\to\mathbb{S}^2$ such that $\Omega_{\tilde f}|_{\mathbb{S}^2\setminus D}=\alpha$, where α is a source, see Figure 3.

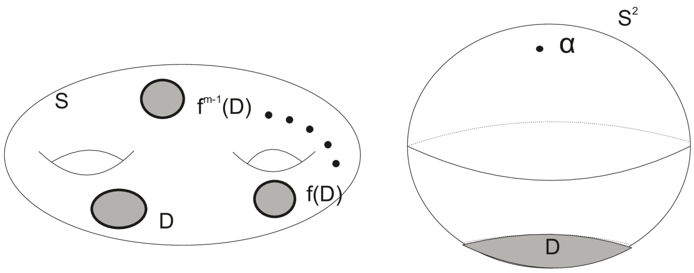

FIGURE 3. Illustration to the proof of Lemma 5.1.

Thus, $\tilde f$ is a gradient-like diffeomorphism on the sphere possessing a unique source in its non-wandering set. According to [2, Thm. 1.1], there exists a stable arc $\tilde f_t$ such that $\tilde f_0 = \tilde f$, $\tilde f_1$ is a diffeomorphism source-sink and $\tilde f_t|_{\mathbb{S}^2\setminus D} = \tilde f|_{\mathbb{S}^2\setminus D}$. Then the sought arc f_t coincides with the diffeomorphism f outside $f^{m-1}(D)$ and it is determied by the formula $f_t(f^{m-1}(d)) = \tilde f_t(d)$ for $d \in D$. $\square$

5.2. Trivialization of repeller R_f. By Lemma 5.1, without loss of generality, we can suppose that the diffeomorphism f has one sink orbit $\mathcal{O}_\omega$ of period m. In the present section we prove the following statement.

Lemma 5.2. *Each diffeomorphism $f \in G$ is connected by a stable arc with a diffeomorphism $g \in G$, which has a unique source orbit.*

Proof. Let $V_\omega = W^s_{\mathcal{O}_\omega} \setminus \mathcal{O}_\omega$. By $\hat V_\omega = V_\omega/f$ we denote the space of orbits of the action of the group $F = \{f^k,\, k \in \mathbb{Z}\}$ on V_ω, while $p_\omega : V_\omega \to \hat V_\omega$ stands for its natural projection. By [4, Prop. 2.5], the space $\hat V_\omega$ is diffeomorphic to the two-dimensional torus and the natural projection $p_\omega : V_\omega \to \hat V_\omega$ is a covering. Then by [4] unstable separatrices of the saddle points of the diffeomorphism f are projected into the nodes at the torus $\hat V_\omega$. By Corollary 2.1 the set $\hat W^u_\sigma = p_\omega(W^u_\sigma \setminus \sigma)$, $\sigma \in \Omega^1_f$ consists of a pair of essential nodes. Then, by [8], the set $\hat V_\omega \setminus \hat W^u_\sigma$ consists of two annuli. We denote by $Q_f \subset \Omega^1_f$ the set of saddle points σ, for which at least of one of the connected components, denoted by K_σ, does not intersect the set $p_\omega(W^u_{\Omega^1_f} \setminus \Omega^1_f)$, see Figure 4.

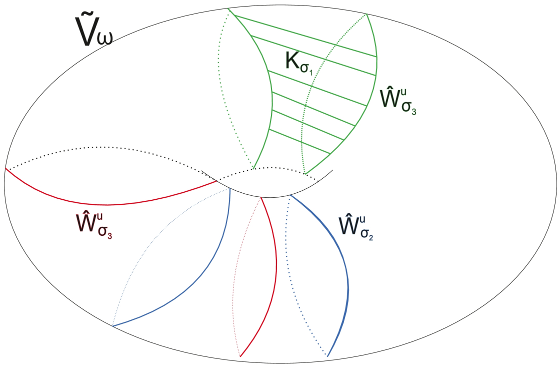

FIGURE 4. The saddle σ belongs to the set Q_f.

We are going to show that the diffeomorphism f is connected by a stable arc with a diffeomorphism $f_1 \in G$ having a unique sink orbit $\mathcal{O}_\omega$ and for which the set Q_{f_1} is empty.

Let a point σ lies in the set Q_f. We consider the attractor $A_\sigma = \mathcal{O}_\omega \cup W^s_{\mathcal{O}_\sigma}$. We let $\hat{V}_\sigma = (W^s_{\omega\cup\mathcal{O}_\sigma} \setminus A_\sigma)/f$. Then $\hat{V}_\sigma$ is diffeomorphic to two two-dimensional tori, and one of them contains the unique node, which is a projection of a stable separatrix γ^s_σ of the saddle σ. According to [2], the diffeomorphism f is connected by a stable arc having one of saddle-node bifurcation with a diffeomorphism g, for which the set Q_g contains one saddle orbit less than the set Q_f. Continuing the process, we obtain the sought diffeomorphism f_1.

Since f_1 is a gradient-like diffeomorphism of 2-torus, then the set $\Omega^1_{f_1}$ contains at least two saddle periodic orbits. Let us show that we can choose saddles $\sigma_1, \sigma_2 \in \Omega^1_{f_1}$ so that $\hat{W}^u_{\sigma_2}$ intersects both connected components $\hat{V}_\omega \setminus \hat{W}^u_{\sigma_1}$. We prove by contradiction.

We choose an arbitrary saddle $\sigma_0 \in \Omega^1_{f_1}$. Let K_0 be a connected component of the set $\hat{V}_\omega \setminus \hat{W}^u_{\sigma_0}$. Since the set Q_{f_1} is empty, then the annulus K_0 contains the projections of unstable manifolds of orbits of saddle points. For each such saddle σ, by κ_σ we denote a connected component of the set $\hat{V}_\omega \setminus \hat{W}^u_\sigma$ belonging to K_0, see Figure 5.

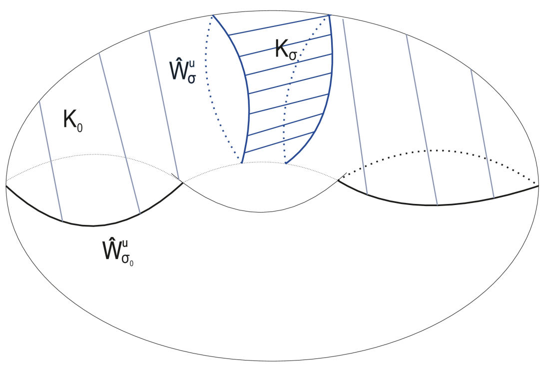

FIGURE 5. The annulus K_0.

Under our assumptions, each two such annuli κ_σ, $\kappa_{\sigma'}$ are either disjoint or one is a proper subset of the other. In view of the finiteness of the saddle orbits we find a saddle σ_*, for which the annulus κ_{σ_*} contains no other annuli κ_σ and this contradicts the emptyness of the set Q_{f_1}.

We choose the saddles $\sigma_1, \sigma_2 \in \Omega^1_{f_1}$ so that $\hat{W}^u_{\sigma_2}$ intersects both connected components $\hat{V}_\omega \setminus \hat{W}^u_{\sigma_1}$. Then $\mathrm{cl}(W^u_{\sigma_i}) \setminus W^u_{\sigma_i} = \omega \cup f_1^{l_i}(\omega)$, $l_i \leqslant m$. Let $M_i = \mathrm{SCM}(l_i, m)$ and $m_i = \frac{M_i}{l_i}$. Then the set

$$C_i = \bigcup_{i=0}^{m_i-1} f_1^{jk_1}(\mathrm{cl}(W^u_{\sigma_i}))$$

is homeomorphic to the circle and f^{l_i}-invariant. Since the mapping f^{l_i} induces an identity mapping in the fundamental group of the torus, then $f^{l_i}|_{C_i}$ preserves the orientation and therefore, the diffeomorphism $f^{l_i}|_{C_i}$ is topologically conjugate to a rough transformation of the circle $\phi_{\frac{k_i}{m_i}}$, $k_i l_i \equiv 1 \pmod{m_i}$. Then each point in the intersection $C_1 \cap C_2$ has the same index[1], which implies that the nodes C_1, C_2 are essential on the torus $\mathbb{T}^2$, see Figure 6.

[1]An *index* of a point x of the transversal intersection of oriented nodes C_1, C_2 on an oriented surface is the number $+1$ if an ordered pair of vectors tangent to the nodes defines the orientation of the surface and it is -1 otherwise. The sum of the indices of all intersection points is equal to the determinant of the matrix formed by the homotopical types of the nodes.

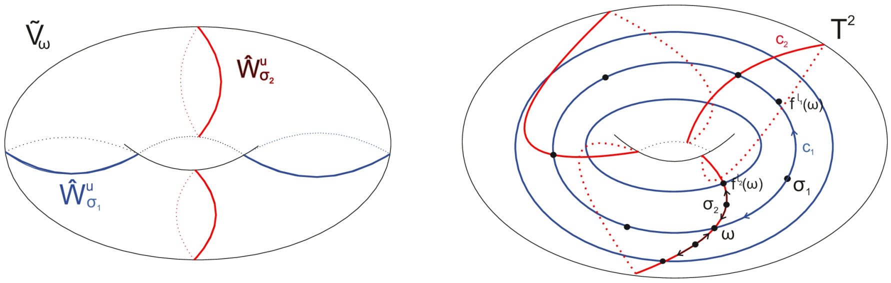

FIGURE 6. Saddles σ_1, σ_2.

We denote by μ the index of the intersection of these nodes. Then $m = \frac{m_1 m_2}{\mu}$. We let

$$C = \bigcup_{j=0}^{m_2-1} f^{l_2 j}(C_1) \cup \bigcup_{j=0}^{m_1-1} f^{l_1 j}(C_2).$$

Then each connected component Δ of the set $Q = \mathbb{T}^2 \setminus C$, see Figure 7, is an open two-dimensional disk, see, for instance, [12], such that

$$Q = \bigsqcup_{j=0}^{m-1} f^j(\Delta).$$

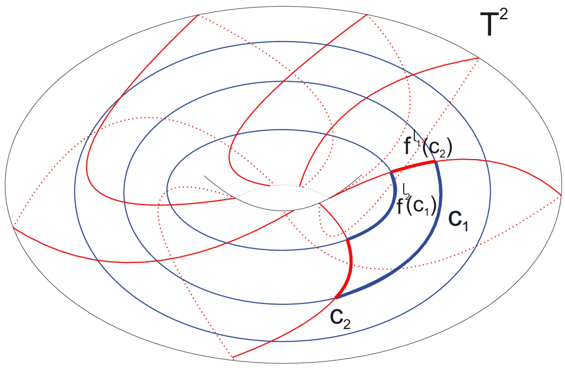

FIGURE 7. Set Δ.

According to [9], the set $C = \mathrm{cl}(W^u_{\Omega^1_{f_1}})$ is the attractor of the diffeomorphism f_1. Moreover, there exists its trapping neighbourhood U_C such that $D_C = \Delta \setminus \mathrm{int}(U_C)$ is homeomorphic to a two-dimensional disk such that $f^{-m}(D_C) \subset \mathrm{int}\, D_C$. Similarly to Lemma 5.1 one can prove that the diffeomorphism f_1 is connected by a stable arc with a diffeomorphism $g \in G$ having a unique source orbit in $\mathbb{T}^2 \setminus \mathrm{int}(U_C)$. $\square$

In Figure 8 we sketch the milestones of the proof of Theorem 2.1 by the example of diffeomorphism $\phi_{\frac{1}{2}} \times \phi_{\frac{1}{2}}$.

5.3. Straightening of curves C_1, C_2. In view of the results of two previous section, without loss of generality we can suppose that the diffeomorphism f possesses one sink orbit $\mathcal{O}_\omega$ and one source orbit $\mathcal{O}_\alpha$, that is, it is polar and the same is true for the skew product of the rough transformations of the circle. Moreover, the closure of unstable manifold of saddle point of the diffeomorphism f contain the nodes C_1, C_2 such that $f^{l_i}(C_i) = C_i$ and $f^{l_i}|_{C_i}$ is a rough

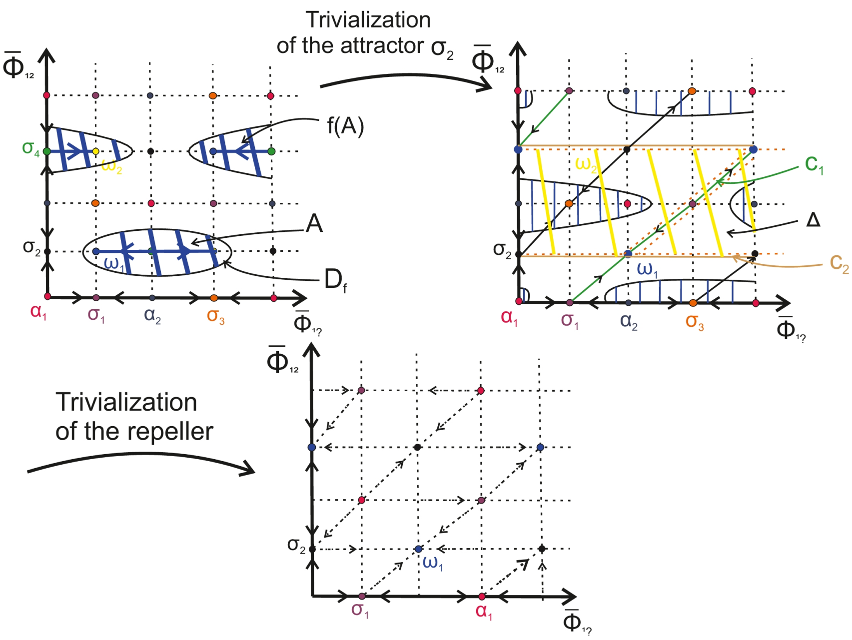

FIGURE 8. Illustration to the proof of Theorem 2.1.

transform of the circle with the rotation number $\frac{k_i}{m_i}$. Without loss of generality we can suppose that $\frac{k_1}{m_1} \leqslant \frac{k_2}{m_2}$ for the diffeomorphism f; otherwise we change the indexation of the curves C_1, C_2. By $\langle \nu_i, \mu_i \rangle$ we denote the homotopicy type of the curve C_i and we let $J = \begin{pmatrix} \nu_1 & \nu_2 \\ \mu_1 & \mu_2 \end{pmatrix}$. Then the diffeomorphism f is topologically conjugate to the diffeomorhpism F_J, see, for instance, [13, Thm. 1], by means of a diffeomorphism homotopic to the identity one. Similarly to [14, Lm. 5.1] we prove that there exists an arc without bifurcations connecting the diffeomorphism f with the diffeomorphism F_J.

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