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ON INVERTIBILITY OF DUHAMEL OPERATOR IN SPACES OF ULTRADIFFERENTIABLE FUNCTIONS

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Abstract. Let Δ be a non-point segment or an (open) interval on the real line containing the point 0. In the space of entire functions realized by the Fourier-Laplace transform of the dual space to the space of ultradifferentiable or of all infinitely differentiable functions on Δ, we study the operators from the commutator subgroup of the one-dimensional perturbation of the backward shift operator. We prove a criterion of their invertibility. In this case, the Riesz-Schauder theory is applied, the use of which in such a situation goes back to the works by V.A. Tkachenko. In the topological dual space to the original space, the multiplication $\circledast$ is introduced and we show that its dual space endowed with a strong topology is a topological algebra. Using the mapping associated with Fourier-Laplace transform, the introduced multiplication $\circledast$ is implemented as a generalized Duhamel product in the corresponding space of ultradifferentiable or infinitely differentiable functions on Δ. We prove a criterion for the invertibility of the Duhamel operator in this space. The multiplication $\circledast$ is used to extend the Duhamel's formula to classes of ultradifferentiable functions. It represents the solution of an inhomogeneous differential equation of finite order with constant coefficients, satisfying zero initial conditions at the point 0, in the form of Duhamel's product of the right-hand side and a solution to this equation with the right-hand side identically equalling to 1. The obtained results cover both the non-quasianalytic and quasianalytic cases.

Keywords: backward shift operator, entire function, Duhamel product, ultradifferentiable function.

Mathematics Subject Classification: 46E10, 47B38

1. Introduction

Let Δ be a non-point segment or an (open) interval on the real line containing the point 0. In this paper, we study a one-dimensional perturbation $D_{0,u}$ of the backward shift operator D_0 acting in the space $A_\omega(\Delta)$ of entire functions in $\mathbb{C}$ of exponential type; this space realizes the adjoint to the space of ultradifferentiable functions of Beurling type or infinitely differentiable functions on Δ by the Fourier-Laplace transform. The operator $D_{0,u}$ is defined by an entire function u such that $u(0) = 1$. It was first defined by V.A. Tkachenko [14] using the function $u = e^P$, where P is some polynomial such that $P(0) = 1$. In [14], [15] the operator afjoint to $D_{0,u}$, called the generalized integration operator, was studied. Moreover, $D_{0,u}$ acts in the countable inductive limit of weighted Banach spaces defined by some ρ-trigonometrically convex function. We note that a general approach to the study of spaces of ultradifferentiable functions was proposed in work by R.W. Braun, R. Miese, B.A. Taylor [19]; in this paper the non-quasianalytic case is studied in detail. Recently, there appeared many works devoted to ultradifferentiable functions, in which, in particular, the approach to this topic proposed in [19] is generalized and expanded, see, for example, book by A.V. Abanin [1], paper by A. Rainer, G. Schindl [25] and the references in these works. In [19] the case of a non-quasianalytic weight function was considered, but many of the results from [19] are also valid for the quasianalytic situation. Therefore, in some cases we refer to [19] for the general situation.

The main result of this paper, Theorem 3.1, provides a criterion for the invertibility of the operator B_μ from the commutant $D_{0,u}$ in the algebra of all linear continuous operators in $A_\omega(\Delta)$. It covers

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both the non-quasianalytic and quasianalytic case. The proof of the sufficiency of the invertibility condition uses Riesz-Schauder theory for Banach spaces by considering the corresponding operators on the Banach steps forming the inductive limit. The use of this method goes back to work by V.A. Tkachenko [15]. The proof of the injectivity of the operator B_μ essentially employs the results of [21]. This criterion was previously proven by the authors in [5, Thm. 2] in the case of $u \equiv 1$ for the space $C^\infty(\Omega)$, where Ω is an interval in $\mathbb{R}$ containing the point 0. Moreover, the proof of the injectivity of the corresponding operator in [5] was different and was based on singular integrals.

The duality theory allows us to apply the above results to the implementation of the adjoint of the operator B_μ called here the generalized Duhamel operator. The title of the paper reflects precisely this part of the work. In the strong dual space $A_\omega(\Delta)'$ of $A_\omega(\Delta)$ we introduce the multiplication $\circledast$ and we prove that $A_\omega(\Delta)'$ with it is a topological algebra. By means of an adjoint mapping to the Fourier-Laplace transform, the introduced operation $\circledast$ is implemented as a generalized Duhamel product in $\mathcal{E}_\omega(\Delta)$. If we fix one factor in it, then we obtain the corresponding Duhamel operator. This is an operator from the commutant of the realization of the generalized integration operator adjoint to $D_{0,u}$. Here we establish a criterion for the invertibility of the Duhamel operator in the space $\mathcal{E}_\omega(\Delta)$. Previously, such criterion was obtained by R. Tapdigoglu and B.T. Torebek [26] for the space $C^\infty[0,1]$ in the case of $u \equiv 1$.

In the final part of the work, we apply the product $\circledast$ to a new proof of the well-known Duhamel formula for solving a differential equation of finite order with constant coefficients satisfying zero initial conditions at point 0. It expresses this solution in the form of the Duhamel product of the right-hand side and such a solution with the right-hand side part identically equalling to 1. In particular, the mentioned formula was obtained for classes of ultradifferentiable functions that were not previously considered in this regard. The proof is based on the possibility of dividing linear continuous functionals on $A_\omega(\Delta)$ by a nonzero polynomial so that the resulting quotient vanishes on monomials the degrees of which are less than the degree of this polynomial.

2. Basic spaces and operators

Following [19], a continuous non-decreasing function $\omega : [0,+\infty) \to [0,+\infty)$ is called a *weight function* if it satisfies the following conditions:

(α) $\omega(2t) = O(\omega(t)), \quad t \to +\infty$;
(β) $\omega(t) = O(t), \quad t \to +\infty$;
(γ) $\log t = o(\omega(t)), \quad t \to +\infty$;
(δ) The function $\varphi = \omega \circ \exp$ is convex on $\mathbb{R}$.

By [19, Lm. 1.2] the weight function ω satisfies the following condition:
(α_1) There exists a constant $C \geqslant 1$ such that

$$\omega(x+y) \leqslant C(\omega(x) + \omega(y) + 1) \quad \text{for all} \quad x, y \in [0,+\infty).$$

The weight function ω is called non-quasianalytic if

$$\int_1^\infty \frac{\omega(t)}{t^2} dt < +\infty,$$

and it is quasianalytic if

$$\int_1^\infty \frac{\omega(t)}{t^2} dt = +\infty.$$

By φ^* we denote a Young adjoint to φ functions, that is, $\varphi^*(x) := \sup\limits_{y \geqslant 0}(xy - \varphi(y))$, $x \geqslant 0$.

Let ω be a weight function, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. As in [19], we define the spaces of ultradifferentiable functions of Beurling type defined with help of ω. For a segment $K \subset \mathbb{R}$ with a non-empty interior we introduce the space

$$\mathcal{E}_\omega(K) := \left\{ f \in C^\infty(K) \,\middle|\, |f|_{K,m} := \sup_{\alpha \in \mathbb{N}_0} \sup_{x \in K} \frac{|f^{(\alpha)}(x)|}{\exp(m\varphi^*(\alpha/m))} < +\infty \quad \text{for all} \quad m \in \mathbb{N} \right\}$$

and define its locally convex topology by the family of semi-norms $|\cdot|_{K,m}$, $m\in\mathbb{N}$. Then $\mathcal{E}_\omega(K)$ is an (FS)-space, that is, the Fréchet-Schwartz space, see [2, Sect. 1], [24, Sect. 25].

Let Ω be an interval in $\mathbb{R}$. We fix a sequence of segments K_n, $n\in\mathbb{N}$, such that $K_n\subset \operatorname{int} K_{n+1}$, $n\in\mathbb{N}$, $\operatorname{int} K_1\neq\emptyset$ and $\Omega=\bigcup\limits_{n\in\mathbb{N}} K_n$; here $\operatorname{int} Q$ is the interior of a ser $Q\subset\mathbb{R}$ in $\mathbb{R}$. We let

$$\mathcal{E}_\omega(\Omega):=\left\{f\in C^\infty(\Omega)\;\middle|\; |f|_{\Omega,m,n}:=\sup_{\alpha\in\mathbb{N}_0}\sup_{x\in K_n}\frac{|f^{(\alpha)}(x)|}{\exp(m\varphi^*(\alpha/m))}<+\infty \quad \text{for all}\quad m,n\in\mathbb{N}\right\}.$$

A locally convex topology in $\mathcal{E}_\omega(\Omega)$ is defined by the family of semi-norms $|\cdot|_{\Omega,m,n}$, $m,n\in\mathbb{N}$, which make $\mathcal{E}_\omega(\Omega)$ an (FS)-space. The space $\mathcal{E}_\omega(\Omega)$ is algebraically and topologically independent of the choice of the segments $(K_n)_{n\in\mathbb{N}}$, as above.

For $\omega(t)=\log(1+t)$, $t\in[0,+\infty)$, for a segment $K\subset\mathbb{R}$ with a non-empty interior and for an interval $\Omega\subset\mathbb{R}$ we suppose that

$$\mathcal{E}_\omega(K):=C^\infty(K),\qquad \mathcal{E}_\omega(\Omega):=C^\infty(\Omega).$$

The spaces $C^\infty(K)$ and $C^\infty(\Omega)$ are equipped with their standard topologies.

For a bounded set $Q\subset\mathbb{R}$ the symbol H_Q denotes the support function of Q, that is, $H_Q(y):=\sup\limits_{x\in Q}(xy)$, $y\in\mathbb{R}$. Let $e_\lambda(x):=e^{-i\lambda x}$, $x\in\mathbb{R}$, $\lambda\in\mathbb{C}$. For a locally convex space H by H' we denote a topologically dual to H. The space H' is equipped with the strong topology. The Fourier-Laplace transform of a functional φ from $\mathcal{E}_\omega(K)'$ or from $\mathcal{E}_\omega(\Omega)'$ is defined by the identity

$$\mathcal{F}(\varphi)(\lambda):=\varphi(e_\lambda),\qquad \lambda\in\mathbb{C}.$$

We continue the function ω on $\mathbb{C}$ by letting $\omega(z):=\omega(|z|)$, $z\in\mathbb{C}$. By $H(\mathbb{C})$ we denote the space of all entire in $\mathbb{C}$ functions. For a segment $K\subset\mathbb{R}$, $n\in\mathbb{N}$, we define a Banach space of entire functions

$$A_{\omega,n}(K):=\left\{f\in H(\mathbb{C})\;\middle|\; \|f\|_{K,n}:=\sup_{z\in\mathbb{C}}\frac{|f(z)|}{\exp(H_K(\operatorname{Im} z)+n\omega(z))}<+\infty\right\}$$

with the norm $\|\cdot\|_{K,n}$ and we let $A_\omega(K):=\operatorname*{ind}\limits_{n\to}A_{\omega,n}(K)$. If Ω is an interval in $\mathbb{R}$, then $A_\omega(\Omega):=\operatorname*{ind}\limits_{n\to}A_{\omega,n}(K_n)$. At the same time, an algebraic and topological identity $A_\omega(\Omega)=\operatorname*{ind}\limits_{n\to}A_\omega(K_n)$ holds true. The spaces $A_\omega(K)$ and $A_\omega(\Omega)$ are (DFS)-spaces, see [2, Sect. 2.10], [24, Thm. 25.20]. If $f\in A_\omega(K)$ or $f\in A_\omega(\Omega)$, then for each zero z of the function f, the function $\frac{f(t)}{t-z}$ also belongs to $A_\omega(K)$, respectively, to $A_\omega(\Omega)$. We note that the spaces $A_\omega(K)$ and $A_\omega(\Omega)$ contain also polynomials if $0\in K$ or $0\in\Omega$.

By the Paley-Wiener-Schwartz theorem for ultradistributions and quasianalytic functionals [19, Prop. 3.5, Thm. 7.4], [23, Prop. 3.6] and for usual distributions [16, Thm. 7.3.1] we have the following theorem.

Theorem 2.1. *Let ω be a weight function or $\omega(t):=\log(1+t)$, $t\in[0,+\infty)$. The Fourier-Laplace transform $\mathcal{F}$ is a topological isomorphism of $\mathcal{E}_\omega(K)'$ onto $A_\omega(K)$ and of $\mathcal{E}_\omega(\Omega)'$ onto $A_\omega(\Omega)$.*

We provide a few statements, which will be employed in what follows.

Lemma 2.1. (i) *Let K be a segment in $\mathbb{R}$. Then*

$$|H_K(t)-H_K(z)|\leqslant\alpha_K|t-z|,\qquad t,z\in\mathbb{C},$$

where $\alpha_K=\sup\limits_{|\xi|=1}H_K(\xi)<+\infty$.

(ii) *For all $t,z\in\mathbb{C}$ obeying $|t-z|\leqslant\frac12\log(1+|z|)$ the inequality*

$$\log(1+|z|)\leqslant\log(1+|t|)+\log 2$$

holds true.

(iii) *Let ω be a weight function. For all $z,\xi,t\in\mathbb{C}$ such that*

$$|\xi-z|\leqslant\frac12\log(1+|z|)\quad\text{and}\quad|t-z|\leqslant\frac12\log(1+|z|)$$

the inequality

$$\omega(\xi)\leqslant C(C+1)\omega(t)+C(C\omega(\log 2)+C+1)$$

holds, where C is a constant from condition (α_1).

Proof. The inequality in (i) is well-known; it is implied by the semi-additivity and positivity homogeneity of the support function H_K.

(ii): Since

$$|z| \leqslant |t| + |t - z| \leqslant |t| + \frac{1}{2}\log(1 + |z|) \leqslant |t| + \frac{|z|}{2},$$

then $|z| \leqslant 2|t|$ and hence,

$$\log(1 + |z|) \leqslant \log(1 + 2|t|) \leqslant \log(1 + |t|) + \log 2.$$

(iii): Using condition (α_1) and Statement (ii), we obtain:

$$\begin{aligned} w(\xi) &\leqslant C(\omega(t) + \omega(\xi - t) + 1) \leqslant C(\omega(t) + \omega(\log(1 + |z|)) + 1) \\ &\leqslant C(\omega(t) + \omega(\log(1 + |t|) + \log 2) + 1) \leqslant C(\omega(t) + C(\omega(\log(1 + |t|)) + \omega(\log 2) + 1) + 1) \\ &\leqslant C(\omega(t) + C\omega(t) + C\omega(\log 2) + C + 1) = C(C + 1)\omega(t) + C(C\omega(\log 2) + C + 1). \end{aligned}$$

The proof is complete. □

In what follows Δ is a non-point segment or interval in $\mathbb{R}$ containing the point 0. We fix a function $u \in A_\omega(\Delta)$ such that $u(0) = 1$. An a generalized backward shift operator $D_{0,u}$, linear and continuous in $A_\omega(\Delta)$, is defined by the identity $D_{0,u}(f)(t) := \frac{f(t)-u(t)f(0)}{t}$, $f \in A_\omega(\Delta)$, see [3, Sect. 1]. If $u \equiv 1$, then $D_0 := D_{0,u}$ is the usual backward shift operator. We mention the identities

$$D_{0,u}(f)(t) = \frac{f(t) - u(t)f(0)}{t} = \frac{f(t) - f(0)}{t} - f(0)\frac{u(t) - u(0)}{t} = D_0(f)(t) - f(0)D_0(u)(t). \quad (2.1)$$

They show that $D_{0,u}$ is an one-dimensional perturbation of the operator D_0. The operator $D_{0,u}$ in the form as in the right hand sides of identities (2.1) was studied by Yu.S. Linchuk in the space of functions holomorphic in a domain in $\mathbb{C}$ [22].

Following [15], [18], we introduce shift operators

$$T_z(f)(t) := \frac{tf(t)u(z) - zf(z)u(t)}{t - z}$$

for the operator $D_{0,u}$ and Pommiez operators

$$D_z(f)(t) := \frac{f(t)u(z) - f(z)u(t)}{t - z}, \qquad f \in A_\omega(\Delta).$$

All of them linearly and continuously act in $A_\omega(\Delta)$.

Remark 2.1. *For all functions $f \in A_\omega(K)$, $z \in \mathbb{C}$ and a zero a of a function u, the function $u_a(t) := \frac{u(t)}{t-a}$ is an eigenvector of the operator T_z:*

$$T_z(u) = u(z)u, \qquad T_z(u_a) = -au_a(z)u_a.$$

These identities can be confirmed by straightforward calculations.

Lemma 2.2. *Let K be a non-point segment in $\mathbb{R}$ containing the point 0.*
(i) For each $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and a constant $c_1 \geqslant 0$ such that for each function $f \in A_{\omega,n}(K)$ the inequality

$$|f'(t)| \leqslant c_1\|f\|_{K,n} \exp\left(H_K(\operatorname{Im} t) + m(\omega(t))\right), \qquad t \in \mathbb{C}.$$

holds true.
(ii) For each $n \in \mathbb{N}$ there exist $s \in \mathbb{N}$ and a constant $c_2 \geqslant 0$ such that for each function $f \in A_{\omega,n}(K)$ the inequality

$$|T_z(f)(t)| \leqslant c_2\|f\|_{K,n} \exp\left(H_K(\operatorname{Im} t) + H_K(\operatorname{Im} z) + s(\omega(t) + \omega(z))\right), \qquad t, z \in \mathbb{C},$$

holds true.

This statement is implied by the maximum modulus principle for holomorphic functions in view of Lemma 2.1, see also a general result in [4, Lm. 4 (i)].

For a functional $\mu \in A_\omega(\Delta)'$ we introduce an operator

$$B_\mu(f)(z) := \mu(T_z(f)), \qquad z \in \mathbb{C}, \qquad f \in A_\omega(\Delta),$$

which is linear and continuous in $A_\omega(\Delta)$. By [4, Thm. 15] the set $\{B_\mu \,|\, \mu \in A_\omega(\Delta)'\}$ coincides with the commutant of the operator $D_{0,u}$ in the algebra of all linear continuous operators in $A_\omega(\Delta)$. We also observe that due to Lemma 2.1 the sequences of functions $(H_K(\operatorname{Im} z)+n\omega(z))_{n\in\mathbb{N}}$ and $(H_{K_n}(\operatorname{Im} z)+n\omega(z))_{n\in\mathbb{N}}$ defining the space $A_\omega(\Delta)$ satisfy original assumptions (1.1) in [4].

The following identities are useful, for instance, in using Riesz-Shauder theory in the problem on invertibility of the operators B_μ in $A_\omega(\Delta)$.

Remark 2.2. *For $\mu\in A_\omega(\Delta)'$, $f\in A_\omega(\Delta)$, $z\in\mathbb{C}$ the identities*

$$B_\mu(f)(z)=\mu(u)f(z)+\mu_t\left(t\frac{f(t)u(z)-f(z)u(t)}{t-u}\right)=\mu(u)f(z)+\mu_t\left(tD_z(f)(t)\right)$$

hold; the subscript t means that the functional μ acts in the variable t.

3. Invertibility criterion of operator B_μ

For a non-point segment K in $\mathbb{R}$, for $\nu\in A_\omega(K)'$, $n\in\mathbb{N}$, we let

$$\|\nu\|^*_{K,n}:=\sup_{\|f\|_{K,n}\leqslant 1}|\nu(f)|.$$

By the symbol $S_n(K)$ we denote a closed unit ball in $A_{\omega,n}(K)$.

Lemma 3.1. *Let K be a non-point segment in $\mathbb{R}$ containing the point 0, $\mu\in A_\omega(K)'$, $u\in A_{\omega,m}(K)$, $m\in\mathbb{N}$. Then for each $n\geqslant m$ the operator*

$$C_\mu(f)(z):=\mu_t\left(t\frac{f(t)u(z)-f(z)u(t)}{t-z}\right)$$

is compact in $A_{\omega,n}(K)$.

Proof. We use some modification of the proving method by V.A. Tkachenko [15, Thm. 2]. We let $d(z):=\max\left(1;\frac{1}{2}\log(1+|z|)\right)$, $z\in\mathbb{C}$.

We fix $n\geqslant m$. Let $|t-z|\geqslant d(z)$. Then for each function $f\in S_n(K)$ we have

$$\begin{aligned}|tD_z(f)(t)|&\leqslant\frac{|t|(|f(t)||u(z)|+|f(z)||u(t)|)}{d(z)}\\&\leqslant\frac{|t|}{d(z)}\|u\|_{K,m}\left(e^{H_K(\operatorname{Im} t)+n\omega(t)}e^{H_K(\operatorname{Im} z)+m\omega(z)}+e^{H_K(\operatorname{Im} z)+n\omega(z)}e^{H_K(\operatorname{Im} t)+m\omega(t)}\right)\\&\leqslant\frac{2|t|}{d(z)}\|u\|_{K,m}e^{H_K(\operatorname{Im} z)+n\omega(z)}e^{H_K(\operatorname{Im} t)+n\omega(t)}.\end{aligned}$$

Now let $|t-z|<d(z)$. By the maximum modulus principle of holomorphic functions for each function $f\in S_n(K)$ there exists a point $\xi\in\mathbb{C}$, for which $|\xi-z|=d(z)$ and

$$|tD_z(f)(t)|\leqslant|\xi D_z(f)(\xi)|\leqslant\frac{2|\xi|}{d(z)}\|u\|_{K,m}e^{H_K(\operatorname{Im} z)+n\omega(z)}e^{H_K(\operatorname{Im}\xi)+n\omega(\xi)}.$$

Owing to Lemma 2.1 and condition (γ) for the weight function ω there exist constants A_1, $A_2>0$, such that for all $t,z\in\mathbb{C}$, $f\in S_n(K)$ we have

$$\begin{aligned}|tD_z(f)(t)|&\leqslant\frac{|t|}{d(z)}\|u\|_{K,m}e^{H_K(\operatorname{Im} z)+n\omega(z)}e^{H_K(\operatorname{Im} t)+A_1\omega(t)+A_1}\\&\leqslant\frac{1}{d(z)}\|u\|_{K,m}e^{H_K(\operatorname{Im} z)+n\omega(z)}e^{H_K(\operatorname{Im} t)+A_2\omega(t)+A_2}.\end{aligned}$$

We take $s\in\mathbb{N}$ such that $s\geqslant A_2$. Then, for all $z\in\mathbb{C}$, $f\in S_n(K)$,

$$|C_\mu(f)(z)|\leqslant\|\mu\|^*_{K,s}\sup_{t\in\mathbb{C}}\frac{|tD_z(f)(t)|}{\exp(H_K(\operatorname{Im} t)+s\omega(t))}\leqslant e^{A_2}\frac{\|\mu\|^*_{K,s}\|u\|_{K,m}}{d(z)}e^{H_K(\operatorname{Im} z)+n\omega(z)}.$$

This is why

$$\lim_{|z|\to+\infty}\sup_{\|f\|_{K,n}\leqslant 1}\frac{|C_\mu(f)(z)|}{\exp(H_K(\operatorname{Im} z)+n\omega(z))}=0.$$

Hence, the set $C_\mu(S_n(K))$ is relatively compact in $A_{\omega,n}(K)$. The proof is complete. $\square$

Let $\mathbb{C}[z]$ and $\mathbb{C}[z]_n$, $n \in \mathbb{N}_0$, be the sets of all polynomials of one variable, respectively, of degree at most n over the field $\mathbb{C}$. For the sake for convenience we provide results from paper [21], which will be employed below; these are Lemmas 2, 4–6 from [21].

Lemma 3.2. *(i) Let $v, w \in H(\mathbb{C})$, $v(0) = w(0) = 1$. Then for all functions $h \in H(\mathbb{C})$, $j \in \mathbb{N}$ the identity $D^j_{0,vw}(vh) = vD^j_{0,w}(h)$ holds.*
(ii) If the polynomials $v, h \in \mathbb{C}$ be coprime, $v(0) = 1$, then the polynomials $D_{0,v}(h)$ and v are also coprime.
(iii) Let $v \in H(\mathbb{C})$, $v(0) = 1$. If a function $f \in H(\mathbb{C})$ satisfies the identity

$$\sum_{j=1}^{s} a_j D^j_{0,v}(f)(z) = 0, \quad z \in \mathbb{C}, \quad s \in \mathbb{N}, \quad a_j \in \mathbb{C}, \quad 1 \leqslant j \leqslant s, \quad a_s \neq 0,$$

then there exist polynomials $p, r \in \mathbb{C}[z]$ of degree at most $n-1$, for which $f = \frac{r}{p}v$.
(iv) Let $v, r \in \mathbb{C}[z]$, $v(0) = 1$. If the polynomials v, r are coprime, then the system

$$\{D^j_{0,v}(r) \,|\, 1 \leqslant j \leqslant \deg(v)\}$$

is linearly independent in $H(\mathbb{C})$.

By the symbol $\mathcal{N}(u)$ we denote the set of all zeros of the function u. We let $u_a(t) := \frac{u(t)}{t-a}$ for $a \in \mathcal{N}(u)$.

Theorem 3.1. *Let Δ be a non-point segment or interval in $\mathbb{R}$ and $0 \in \Delta$, while ω be a weight function or $\omega(t) = \log(1+t)$, $t \in [0, +\infty)$. For $\mu \in A_\omega(\Delta)'$ the following statements are equivalent:*
(i) B_μ is an isomorphism of $A_\omega(\Delta)$ onto itself;
(ii) $\mu(u) \neq 0$ and $\mu(u_a) \neq 0$ for each $a \in \mathcal{N}(u)$.

Proof. (i)⇒(ii): By Remark 2.1, $B_\mu(u) = \mu(u)u$ and if $\mathcal{N}(u) \neq \emptyset$, then $B_\mu(u_a) = -a\mu(u_a)u_a$, $a \in \mathcal{N}(u)$. Hence, $\mu(u) \neq 0$ and $\mu(u_a) \neq 0$ for each $a \in \mathcal{N}(u)$.

(ii)⇒(i): We first consider the case when Δ is the segment K. We choose $m \in \mathbb{N}$ such that $u \in A_{\omega,m}(K)$. We choose $n \geqslant m$. By Remark 2.2 and Lemma 3.1, the operator $D_{0,u}$ acts in $A_{\omega,n}(K)$. Moreover, by Lemma 3.1, due to the representation in Remark 2.2, the kernel $\operatorname{Ker} B_\mu$ of the operator $B_\mu : A_{\omega,n}(K) \to A_{\omega,n}(K)$ is finite-dimensional.

Let us show that the operator B_μ is injective in $A_{\omega,n}(K)$. Suppose that there exists a non-zero function $f \in A_{\omega,n}(K)$ such that $B_\mu(f) = 0$. We consider the case when f can not be represented in the form $f = \frac{r}{p}u$, where r, p are polynomials. According to Lemma 3.2, then f satisfies none of the equations $\sum\limits_{j=1}^{s} a_j D^j_{0,u}(f) = 0$, $s \in \mathbb{N}$, $a_s \neq 0$ in $H(\mathbb{C})$. Hence, the system $\{D^j_{0,u}(f) \,|\, j \in \mathbb{N}\}$ is linearly independent in $H(\mathbb{C})$. Since $B_\mu D_{0,u} = D_{0,u} B_\mu$ in $A_\omega(K)$ (and in $A_{\omega,n}(K)$), then $B_\mu(D^j_{0,u}(f)) = 0$ for each $j \in \mathbb{N}$. Hence, $\operatorname{Ker} B_\mu$ is infinite-dimensional and this is a contradiction. Thus, there exist coprime polynomials r and p such that $f = \frac{r}{p}u$. Then $\frac{u}{p} \in H(\mathbb{C})$ and without loss of generality we can suppose that $p(0) = 1$. By Lemma 3.2 for each $j \geqslant 0$ the identity holds:

$$D^j_{0,u}(f) = \frac{u}{p} D^j_{0,p}(r). \tag{3.1}$$

At the same time $D^j_{0,u}(f) \in \operatorname{Ker} B_\mu$ for each $j \geqslant 0$. We let $k := \deg(r)$, $l := \deg(p)$. If $l = 0$, then $p \equiv 1$, and the identities $\deg(D^k_{0,p}(r)) = 0$ and (3.1) as $j = k$ yield that $u \in \operatorname{Ker} B_\mu$. Therefore, $\mu(u)u = 0$ and this is a contradiction.

Let $l \geqslant 1$. We apply a method used in the proof of Lemma 8 in [21]. By Lemma 3.2 the set $S := \{D^j_{0,p}(r) \,|\, 1 \leqslant j \leqslant l\}$ is linearly independent in $H(\mathbb{C})$. Suppose that $k < l$. Since $\deg(D^j_{0,p}(r)) < l$ for all j such that $1 \leqslant j \leqslant l$, then the system S is a basis in $\mathbb{C}[z]_{l-1}$. Let a be some root of p. Then a is also a root of the function u. Using (3.1), we obtain that $\frac{u}{p}\mathbb{C}[z]_{l-1} \subset \operatorname{Ker} B_\mu$ and this is why the function $\frac{u(t)}{t-a}$ also belongs to $\operatorname{Ker} B_\mu$, which is a contradiction.

Let $k = l$. Then the system S is a basis in $\mathbb{C}[z]_{l-1}$, and $S \cup \{r\}$ is a basis in $\mathbb{C}[z]_l$. This is why $\frac{u}{p}\mathbb{C}[z]_l \subset \operatorname{Ker} B_\mu$ and this leads us to a contradiction. Now let $k > l$. By Lemma 3.2 the polynomials

$D_{0,p}^{k-l}(r)$ and p are coprime and the set

$$\left\{D_{0,p}^{j}(r)=D_{0,p}^{j-k+l}(D_{0,p}^{k-l}(r))\,\Big|\,k-l+1\leqslant j\leqslant k\right\}$$

is linearly independent. Since $\deg(D_{0,p}^{k-l}(r))=l$, then this set is contained in $\mathbb{C}[z]_{l-1}$, and hence, it is a basis in $\mathbb{C}[z]_{l-1}$. This is again a contradiction.

Thus, the operator $B_\mu : A_{\omega,n}(K)\to A_{\omega,n}(K)$ is injective. Due to Remark 2.2 and Lemma 3.1, B_μ is an isomorphism of each space $A_{\omega,n}(K)$, $n\geqslant m$, onto itself. This is why B_μ is an isomorphism of $A_\omega(K)$ onto itself; the theorem on an open mapping implies that this is a topological isomorphism.

Now we consider the case, when Δ is an interval Ω containing the point 0. Then $A_\omega(\Omega)=\operatorname{ind}_{n\to} A_\omega(K_n)$, where K_n are segments such that $K_n\subset \operatorname{int}K_{n+1}$, $n\in\mathbb{N}$, $\operatorname{int}K_1\neq\emptyset$ and $\Omega=\bigcup_{n\in\mathbb{N}}K_n$. At the same time for each $n\in\mathbb{N}$ the restriction of μ on $A_\omega(K_n)$ is a linear continuous functional on $A_\omega(K_n)$. If $u\in A_\omega(K_s)$, then by the previous part of the proof the operator B_μ is an isomorphism of each space $A_\omega(K_n)$ as $n\geqslant s$. Hence, B_μ is an isomorphism of $A_\omega(\Omega)$ onto itself. The proof is complete. □

Remark 3.1. *(i) The criterion in Theorem 3.1 was proved in work* [5, Thm. 2] *in the case* $\omega(t)=\log(1+t)$, $u\equiv 1$.
(ii) In the proof of the previous theorem, the compactness of the operator $C_\mu(f)=\mu_t\left(t\frac{f(t)u(z)-f(z)u(t)}{t-z}\right)$ *in each space* $A_{\omega,n}(K)$ *for sufficiently large* n *has been essentially employed. Under certain conditions, this operator is not compact in* $A_\omega(K)$*, which means that in this case it is impossible to use the Riesz-Schauder theory for operators in locally convex spaces other than Banach ones, see, for instance,* [27], [13, Ch. VIII].

Let us show this fact for $\omega(t)=\log(1+t)$ *and* $u\equiv 1$. *Let* $\widehat{\mu}(x):=\mu_t(e^{-itx})$, $x\in\Delta$. *Then* $\widehat{\mu}\in\mathcal{E}_\omega(\Delta)$ *(see Section 4.2). Suppose that the function* $\widehat{\mu}-\widehat{\mu}(0)$ *is not flat at zero, i.e., that there exists* $k\in\mathbb{N}$ *such that* $0\neq\widehat{\mu}^{(k)}(0)=(-i)^k\mu_t(t^k)$ *and (for* $k\geqslant 2$*)* $0=\widehat{\mu}^{(j)}(0)=(-i)^j\mu_t(t^j)$ *if* $1\leqslant j<k$. *Let* $v_n(z):=(C_\mu)_t(t^n)(z)=\mu_t\left(t\frac{t^n-z^n}{t-z}\right)$, $z\in\mathbb{C}$, $n\in\mathbb{N}$. *Suppose that the operator* $C_\mu : A_\omega(K)\to A_\omega(K)$ *is compact, i.e., it maps some neighborhood of zero onto a subset of a compact set in* $A_\omega(K)$. *Since the countable inductive limit* $A_\omega(K)$ *is regular, that is, each bounded subset* $A_\omega(K)$ *is contained and bounded in some space* $A_{\omega,s}(K)$*, see* [2, Sect. 2, 2.9(c)], *then there exists* $s\in\mathbb{N}$ *such that all functions* v_n, $n\in\mathbb{N}$*, belong to* $A_{\omega,s}(K)$. *This means that for each* $n\in\mathbb{N}$

$$|v_n(z)|\leqslant\|v_n\|_{K,s}e^{H_K(\operatorname{Im}z)}(1+|z|)^s,\qquad z\in\mathbb{C},$$

where $\|v_n\|_{K,s}<+\infty$. *Therefore, for* $n=k+s+1$ *we get:*

$$\Big|\sum_{l=0}^{k+s}z^l\mu_t(t^{k+s+1-l})\Big|\leqslant\|v_n\|_{K,s}(1+|z|)^s,\qquad z\in\mathbb{C}.$$

But the last inequality does not hold for sufficiently large $|z|$ *and we get a contradiction.*

4. Results for Duhamel operator

Let Δ be still a non-point segment or interval $\mathbb{R}$, $0\in\Delta$, and ω be a weight function or $\omega(t)=\log(1+t)$, $t\in[0,+\infty)$.

4.1. $A_\omega(\Delta)'$ as topological algebra.

Following [4, Sect. 2.2], we introduce an operation $\circledast$:

$$(\varphi\circledast\psi)(f):=\varphi_z(\psi(T_z(f))),\quad \varphi,\psi\in A_\omega(\Delta)',\quad f\in A_\omega(\Delta).$$

According to [4, Sect. 2.2], $\circledast$ is an associative and commutative binary operation in $A_\omega(\Delta)'$. Let us show that $A_\omega(\Delta)'$ is a topological algebra with the multiplication $\circledast$. We use the following terminology. Algebra is a complex linear space $\mathcal{A}$ with a multiplication, i.e., a bilinear mapping from $\mathcal{A}\times\mathcal{A}$ to $\mathcal{A}$. It is called topological if $\mathcal{A}$ is a locally convex space and the multiplication is continuous from $\mathcal{A}\times\mathcal{A}$ into $\mathcal{A}$.

Since the inductive limit $A_\omega(\Delta)$ is regular, then [24, Thm. 25.9] a strong dual space $A_\omega(\Delta)'$ is a Fréchet space with a fundamental sequence of continuous semi-norms

$$\|\varphi\|^*_{K,n}=\sup_{\|f\|_{K,n}\leqslant 1}|\varphi(f)|,\quad \varphi\in A_\omega(K)',\quad n\in\mathbb{N},$$

if Δ is a segment K and

$$\|\varphi\|^*_{K_n,n} = \sup_{\|f\|_{K_n,n}\leqslant 1} |\varphi(f), \quad \varphi \in A_\omega(\Omega)', \quad n \in \mathbb{N},$$

Δ is an interval Ω.

Theorem 4.1. *$(A_\omega(\Delta)', \circledast)$ is a topological algebra.*

Proof. We fix $n \in \mathbb{N}$ and let $Q := K$ if Δ is the segment K and $Q := K_n$ if Δ is the interval Ω. We choose s and c_2 by n and Q as in Lemma 2.2 (ii). Then

$$\begin{aligned}
\|\varphi \circledast \psi\|^*_{Q,n} = &\sup_{\|f\|_{Q,n}\leqslant 1} |(\varphi \circledast \psi)(f)|\,| = \sup_{\|f\|_{Q,n}\leqslant 1} |\varphi_z(\psi(T_z(f)))| \\
&\leqslant \|\varphi\|^*_{Q,s} \sup_{\|f\|_{Q,n}\leqslant 1} \sup_{z\in\mathbb{C}} \frac{|\psi(T_z(f))|}{\exp(H_Q(\operatorname{Im} z) + s\omega(z))} \\
&\leqslant \|\varphi\|^*_{Q,s}\|\psi\|^*_{Q,s} \sup_{\|f\|_{Q,n}\leqslant 1} \sup_{z\in\mathbb{C}} \sup_{t\in\mathbb{C}} \frac{|T_z(f)(t)|}{\exp(H_Q(\operatorname{Im} z) + H_Q(\operatorname{Im} t) + s(\omega(z)+\omega(t)))} \\
&\leqslant c_2\|\varphi\|^*_{Q,s}\|\psi\|^*_{Q,s}.
\end{aligned}$$

Hence, a bilinear mapping $(\varphi, \psi) \mapsto \varphi \circledast \psi$ is continuous from $A_\omega(\Delta)' \times A_\omega(\Delta)'$ into $A_\omega(\Delta)'$. The proof is complete. □

Remark 4.1. *For $\mu \in A_\omega(\Delta)'$ an adjoint for the operator $B_\mu : A_\omega(\Delta) \to A_\omega(\Delta)$ with respect to the dual pair $(A_\omega(\Delta), A_\omega(\Delta)')$ is an operator $B'_\mu : A_\omega(\Delta)' \to A_\omega(\Delta)'$ such that $B'_\mu(\varphi) = \varphi \circledast \mu$, $\varphi \in A_\omega(\Delta)'$. Indeed, for all $\varphi \in A_\omega(\Delta)'$, $f \in A_\omega(\Delta)$ we have*

$$B'_\mu(\varphi)(f) = \varphi(B_\mu(f)) = (\varphi \circledast \mu)(f).$$

Below in Section 4.3 we shall show that the operation $\circledast$ is realized in $\mathcal{E}_\omega(\Delta)$ as a generalized Duhamel product, while B'_μ is realized as a generalized Duhamel operator.

4.2. Operator of generalized integration. Let $\mathcal{F}' : A_\omega(\Delta)' \to \mathcal{E}_\omega(\Delta)$ be the adjoint mapping for the Fourier-Laplace transform $\mathcal{F} : \mathcal{E}_\omega(\Delta)' \to A_\omega(\Delta)$ with respect to dual pairs $(\mathcal{E}_\omega(\Delta)', \mathcal{E}_\omega(\Delta))$ and $(A_\omega(\Delta), A_\omega(\Omega)')$. Since the space $\mathcal{E}_\omega(\Delta)$ is reflexive and $\mathcal{F}$ is a topological isomorphism of $\mathcal{E}_\omega(\Delta)'$ onto $A_\omega(\Delta)$, see Theorem 2.1, then $\mathcal{F}'$ is a topological isomorphism of $A_\omega(\Delta)'$ onto $\mathcal{E}_\omega(\Delta)$. For $z \in \mathbb{C}$, and a function f defined at the point z, we let $\delta_z(f) := f(z)$. It is clear that $\delta_x \in \mathcal{E}_\omega(\Delta)'$ for $x \in \Delta$ and $\delta_z \in A_\omega(\Delta)'$ for $z \in \mathbb{C}$. Moreover, for all $\varphi \in A_\omega(\Delta)'$, $x \in \Delta$ we have

$$\mathcal{F}'(\varphi)(x) = \delta_x(\mathcal{F}'(\varphi)) = \varphi(\mathcal{F}(\delta_x)) = \varphi(e_x). \tag{4.1}$$

We let $\widehat{\varphi} := \mathcal{F}'(\varphi)$, $\varphi \in A_\omega(\Delta)'$. We observe that $\widehat{\delta_\alpha} = e_\alpha$ for each $\alpha \in \mathbb{C}$. Moreover, standard identities hold:

$$\varphi_t(t^j e^{-ixt}) = i^j \widehat{\varphi}^{(j)}(x), \quad \varphi \in A_\omega(\Delta)', \quad x \in \Delta, \quad j \in \mathbb{N}_0. \tag{4.2}$$

They are implied by the fact that for each function $f \in A_\omega(\Delta)$ there exists a limit $\lim\limits_{\eta\to 0} \frac{f(\cdot+\eta)-f}{\eta}$ in the space $A_\omega(\Delta)$, which is equal to f', see, for instance, [8, Lm. 2]; original assumptions (V1) and (V2) for considered in [8] spaces are satisfied in this case.

For a linear continuous operator $B : A_\omega(\Delta) \to A_\omega(\Delta)$ we denote by B' the operator in $A_\omega(\Delta)'$ adjoint to B with respect to the natural dual pair $(A_\omega(\Delta), A_\omega(\Delta)')$. Following V.A. Tkachenko [15], we call $D'_{0,u}$ a generalized integration operator. V.A. Tkachenko [15] introduced the generalized integration operator as well as the adjoint to the operator $D_{0,u}$ acting in the countable inductive limit of weighted Banach spaces of entire functions defined by a ρ-trigonometrically convex function. Moreover, in [15] $u = e^{\mathcal{P}}$, where $\mathcal{P}$ is a polynomial. Operators with this name were studied by R. Crownover, R. Hansen [20]. Due to (2.1), the identity

$$D'_{0,u}(\varphi) = D'_0(\varphi) - \varphi(D_0(u))\delta_0, \quad \varphi \in A_\omega(\Delta)' \tag{4.3}$$

holds.

We define a complex-valued bilinear form

$$\langle f, g\rangle := \mathcal{F}^{-1}(f)(g), \quad f \in A_\omega(\Delta), \quad g \in \mathcal{E}_\omega(\Delta).$$

It makes a duality between $A_\omega(\Delta)$ and $\mathcal{E}_\omega(\Delta)$. We note that

$$f(z) = \langle f, e_z\rangle, \quad f \in A_\omega(\Delta), \quad z \in \mathbb{C}, \quad \text{and} \quad h(x) = \langle e_x, h\rangle, \quad h \in \mathcal{E}_\omega(\Delta), \quad x \in \Delta.$$

For a linear continuous operator $B : A_\omega(\Delta) \to A_\omega(\Delta)$, by $\widetilde{B}$ we denote an operator acting in $\mathcal{E}_\omega(\Delta)$ and adjoint to B with respect to the dual pair $(A_\omega(\Delta), \mathcal{E}_\omega(\Delta))$. The identity $\widetilde{B} = \mathcal{F}'B'(\mathcal{F}')^{-1}$ holds. If $u \equiv 1$, then $\widetilde{D}_0 = \widetilde{D}_{0,u}$ is a Volterra operator:

$$\widetilde{D}_0(h)(x) = -i\int_0^x h(\xi)d\xi, \ x \in \Delta, \, h \in \mathcal{E}_\omega(\Delta). \tag{4.4}$$

The proof of this identity is standard, see, for instance, [7, Lm. 2]. By (2.1) the identity holds:

$$\widetilde{D}_{0,u}(h)(x) = \int_0^x h(\xi)d\xi - \langle D_0(u), h\rangle, \quad x \in \Delta, \quad h \in \mathcal{E}_\omega(\Delta). \tag{4.5}$$

Let us specify the latter representation in the case $u = Pe_\lambda$, where $P \in \mathbb{C}[z]$, $P(0) = 1$, $\lambda \in \Delta$. For a polynomial $w(z) = \sum\limits_{j=0}^n b_j z^j \in \mathbb{C}[z]$ we define a differential operator

$$w(d)(f) := \sum_{j=0}^n i^j b_j f^{(j)}.$$

We note that for all $w \in \mathbb{C}[z]$, $h \in \mathcal{E}_\omega(\Delta)$, $x \in \Delta$, we have

$$\langle we_x, h\rangle = w(d)(h)(x). \tag{4.6}$$

Since

$$\begin{aligned} D_0(Pe_\lambda)(t) &= \frac{P(t)e_\lambda(t) - 1}{t} = \frac{P(t) - 1}{t}e_\lambda(t) + \frac{e_\lambda(t) - 1}{t} \\ &= D_0(P)(t)e_\lambda(t) + D_0(e_\lambda)(t), \end{aligned}$$

then for $h \in \mathcal{E}_\omega(\Delta)$

$$\langle D_0(u), h\rangle = \langle D_0(P)e_\lambda, h\rangle + \langle D_0(e_\lambda), h\rangle = D_0(P)(d)(h)(\lambda) + \int_0^\lambda h(\xi)d\xi. \tag{4.7}$$

It follows from identities (4.4)–(4.7) that for all $h \in \mathcal{E}_\omega(\Delta)$, $x \in \Delta$ the idenity holds:

$$\widetilde{D}_{0,u}(h)(x) = \int_\lambda^x h(\xi)d\xi - D_0(P)(d)(h)(\lambda).$$

4.3. Generalized Duhamel product. We consider the case $u = Pe_\lambda$, where $P \in \mathbb{C}[z]$, $P(0) = 1$, $\lambda \in \Delta$. Let $P(z) = \sum\limits_{j=0}^m a_j z^j$, $m \in \mathbb{N}$ (at the same time $a_0 = 1$ and the case $a_m = 0$ is not excluded). We introduce polynomials p_j, $0 \leqslant j \leqslant m-1$, such that

$$\sum_{j=0}^{m-1}(-i)^j p_j(t)z^j = \frac{P(t) - P(z)}{t - z}$$

for all $t, z \in \mathbb{C}$. The identities hold:

$$p_j(t) = i^j \sum_{k=j}^{m-1} a_{k+1}t^{k-j}, \qquad 0 \leqslant j \leqslant m-1.$$

We let $\widetilde{p}_j(t) := tp_j(t)$, $0 \leqslant j \leqslant m-1$, $t \in \mathbb{C}$.

We define a generalized Duhamel product: for $g, h \in C^\infty(\Delta)$, $x \in \Delta$ we let

$$(g * h)(x) = P(d)(g)(\lambda)h(x) + \int_\lambda^x (P(d)(g))'(\xi)h(x + \lambda - \xi)d\xi - \sum_{j=0}^{m-1} \widetilde{p}_j(d)(g)(x)h^{(j)}(\lambda).$$

It is clear that $g*h \in C^\infty(\Delta)$ and a bilinear mapping $(g, h) \mapsto g*h$ is continuous from $C^\infty(\Delta)\times C^\infty(\Delta)$ into $C^\infty(\Delta)$, and hence, from $\mathcal{E}_\omega(\Delta) \times \mathcal{E}_\omega(\Delta)$ into $C^\infty(\Delta)$. If $P \equiv 1$, then

$$(g * h)(x) = g(\lambda)h(x) + \int_\lambda^x g'(\xi)h(x + \lambda - \xi)d\xi.$$

As $P \equiv 1$ and $\lambda = 0$, the product $g * h$ is the usual Duhamel product:

$$(g * h)(x) = g(0)h(x) + \int_0^x g'(\xi)h(x - \xi)d\xi.$$

Earlier, a generalized Duhamel product similar to the above introduced one was defined in the space of germs of all functions holomorphic on a convex locally closed subset $\mathbb{C}$ [21, Sect. 4] and in the space of entire functions of exponential type realizing by means of the Laplace transform the dual space for one of functions holomorphic in a simply-connected domain in $\mathbb{C}$ [6, Sect. 1.2]. In [9], M.T. Karasev considered a generalized Duhamel product as some discrete analogue of the Duhamel product.

Lemma 4.1. *The mapping $t \mapsto \delta_t$ is continuous from $\mathbb{C}$ into $A_\omega(\Delta)'$.*

Proof. We fix $n \in \mathbb{N}$. Let $Q := K$ if Δ is the segment K and $Q := K_n$ in the case when Δ is the interval Ω. We choose $m \in \mathbb{N}$, c_1 by n as in Lemma 2.2 (i). For a fixed $t_0 \in \mathbb{C}$ and for $t \in \mathbb{C}$ we obtain:

$$\begin{aligned}\|\delta_t - \delta_{t_0}\|^*_{Q,n} &= \sup_{\|f\|_{Q,n}\leqslant 1} |f(t) - f(t_0)| = \sup_{\|f\|_{Q,n}\leqslant 1} \Big| \int_{t_0}^t f'(\xi)d\xi \Big| \\ &\leqslant |t - t_0| \sup_{\|f\|_{Q,n}\leqslant 1} \sup_{\xi\in[t_0,t]} |f'(\xi)| \leqslant c_1|t - t_0| \sup_{\xi\in[t_0,t]} \exp((H_K(\operatorname{Im}\xi) + m\omega(\xi)).\end{aligned}$$

This implies that $\|\delta_t - \delta_{t_0}\|^*_{Q,n} \to$ as $t \to t_0$. □

Theorem 4.2. *For all $\varphi, \psi \in A_\omega(\Delta)'$ the identity $\widehat{\varphi \circledast \psi} = \widehat{\varphi} * \widehat{\psi}$ holds.*

Proof. First let us show that for all $\alpha, \beta \in \mathbb{C}$, in $C^\infty(\Delta)$ the identity

$$\widehat{\delta_\alpha \circledast \delta_\beta} = e_\alpha * e_\beta \tag{4.8}$$

holds. Indeed, for $x \in \Delta$,

$$\begin{aligned}\widehat{\delta_\alpha \circledast \delta_\beta}(x) &= (\delta_\alpha \circledast \delta_\beta)(e_x) = (\delta_\alpha)_z \left((\delta_\beta)_t \left(\frac{te^{-ixt}P(z)e^{-i\lambda z} - ze^{-ixz}P(t)e^{-i\lambda t}}{t - z} \right)\right) \\ &= \frac{\alpha e^{-ix\alpha}P(\beta)e^{-i\lambda\beta} - \beta e^{-ix\beta}P(\alpha)e^{-i\lambda\alpha}}{\alpha - \beta}.\end{aligned}$$

A straightforward calculation shows that

$$(e_\alpha * e_\beta)(x) = \frac{\alpha e^{-ix\alpha}P(\beta)e^{-i\lambda\beta} - \beta e^{-ix\beta}P(\alpha)e^{-i\lambda\alpha}}{\alpha - \beta}.$$

We take $\varphi, \psi \in A_\omega(\Delta)'$. Since the set $\{\delta_t \,|\, t \in \mathbb{C}\}$ is dense in the Fréchet space $A_\omega(\Delta)'$, then there exist sequences of functionals φ_n, ψ_n, $n \in \mathbb{N}$, from the linear span of the set $\{\delta_t \,|\, t \in \mathbb{C}\}$ such that $\varphi_n \to \varphi$, $\psi_n \to \psi$ in $A_\omega(\Delta)'$. Due to (4.8), $\widehat{\varphi_n \circledast \psi_n} = \widehat{\varphi_n} * \widehat{\psi_n}$ for each $n \in \mathbb{N}$. Since the mappings $(\nu, \eta) \mapsto \nu \circledast \eta$ are continuous from $A_\omega(\Delta)' \times A_\omega(\Delta)'$ into $A_\omega(\Delta)'$, $\mathcal{F} : A_\omega(\Delta)' \to \mathcal{E}_\omega(\Delta)$, $(g, h) \mapsto g * h$ from $C^\infty(\Delta) \times C^\infty(\Delta)$ into $C^\infty(\Delta)$, $\mathcal{E}_\omega(\Delta)$ is continuously embedded into $C^\infty(\Delta)$, then passing to the limit in the later identity, we obtain $\widehat{\varphi \circledast \psi} = \widehat{\varphi} * \widehat{\psi}$ in $\mathcal{E}_\omega(\Delta)$. Simultaneously we have shown that $g * h \in \mathcal{E}_\omega(\Delta)$ for all functions $g, h \in \mathcal{E}_\omega(\Delta)$. The proof is complete. □

For $g \in \mathcal{E}_\omega(\Delta)$ we define the Duhamel operator $S_g(h) := g * h$, $h \in \mathcal{E}_\omega(\Delta)$, linear and continuous in $\mathcal{E}_\omega(\Delta)$. It follows from [4, Thm. 15], Remark 4.1, Theorem 4.2 that the set $\{S_g \,|\, g \in \mathcal{E}_\omega(\Delta)\}$ is the commutant of the realization $\widetilde{D}_{0,u}$ of the generalized integration operator in the algebra of all linear continuous in $\mathcal{E}_\omega(\Delta)$ operators. We note that in the space $C^\infty[0,1]$ the commutant of the Volterra operator

$$\widetilde{D}_0(h)(x) = \widetilde{D}_{0,u}(h)(x) = \int_0^x h(\xi)d\xi$$

corresponding to the case $u \equiv 1$ was described in work [26].

For a root a of a polynomial P we let $P_a(t) := \frac{P(t)}{t-a}$. By means of usual dual arguing, Theorems 3.1, 4.2 and identity (4.6) implies the following corollary.

Corollary 4.1. *The Duhamel operator S_g is an isomorphism of $\mathcal{E}_\omega(\Delta)$ onto itself if and only if $P(d)(g)(\lambda) \neq 0$ and $P_a(d)(g)(\lambda) \neq 0$ for each root a of the polynomial P.*

4.4. Proof of Duhamel formula for solution of differential equaiton with constant coefficients by means for multiplication $\circledast$. In the section we suppose that $u \equiv 1$. We apply the multiplication $\circledast$ for proving a formula expressing a solution $f \in \mathcal{E}_\omega(\Delta)$ to a differential equation with constant coefficients

$$\sum_{j=0}^n a_j f^{(j)} = g, \quad g \in \mathcal{E}_\omega(\Delta), \quad n \in \mathbb{N}, \quad a_n \neq 0, \tag{4.9}$$

satisfying zero initial conditions $f^{(j)}(0) = 0$, $0 \leqslant j \leqslant n-1$, via a similar solution for the right hand side identically equalling to 1. There are various approaches for justifying this formula for some classes of functions different from the spaces considered in this paper, see, for instance, monograph by M.A. Lavrentiev, B.V. Shabat [11, Ch. VI] and paper by I.L. Kogan [10].

For a polynomial $q \in \mathbb{C}[z]$, $\varphi \in A_\omega(\Delta)'$ we let

$$(q\varphi)(f) := \varphi(qf), \quad f \in A_\omega(\Delta).$$

Since the operator $M_q(f) := qf$ of multiplication by q is linear and continuous in $A_\omega(\Delta)$, then $q\varphi \in A_\omega(\Delta)$ for each functional $\varphi \in A_\omega(\Delta)'$. Let $m_j(z) := z^j$, $z \in \mathbb{C}$, $j \in \mathbb{N}_0$.

Lemma 4.2. *Let $L \in \mathbb{N}$. For all pairwise different numbers λ_l, $1 \leqslant l \leqslant L$ and all $k_l \in \mathbb{N}$, $1 \leqslant l \leqslant L$, $c_j \in \mathbb{C}$, $0 \leqslant j \leqslant n-1$, where $n := \sum\limits_{l=1}^L k_l$, the system of equations*

$$\sum_{l=1}^L \sum_{s=0}^{k_l-1} b_{l,s} m_j^{(s)}(\lambda) = c_j, \quad 0 \leqslant j \leqslant n-1, \tag{4.10}$$

has a unique equation $b_{l,s} \in \mathbb{C}$, $1 \leqslant l \leqslant L$, $0 \leqslant s \leqslant k_l - 1$.

Proof. With the family of complex numbers $c = (c_{l,s})_{1\leqslant l\leqslant L, 0\leqslant s\leqslant k_l-1}$ we associate a stretched vector

$$\sigma(c) := (c_{1,0}, \ldots, c_{1,k_1-1}, \ldots, c_{L,0}, \ldots, c_{L,k_L-1}) \in \mathbb{C}^n.$$

The mapping

$$\Phi(f) := \sigma\left(\left(f^{(s)}(\lambda_l)\right)_{1\leqslant l\leqslant L, 0\leqslant s\leqslant k_l-1}\right)$$

is linear from $\mathbb{C}[z]$ into $\mathbb{C}^n$. Due to the uniqueness of the solution of the corresponding multiple interpolation Hermite problem in $\mathbb{C}[z]_{n-1}$, see [12, Ch. 4, Sect. 16.2], Φ bijectively maps $\mathbb{C}[z]_{n-1}$ onto $\mathbb{C}^n$. Since the system of polynomials $\mathcal{M}_n := \{m_j \,|\, 0 \leqslant j \leqslant n-1\}$ is linearly independent in $\mathbb{C}[z]_{n-1}$, then its image $\Phi(\mathcal{M}_n)$ is a linearly independent subset in $\mathbb{C}^n$. This implies that for each $(c_j)_{j=0}^{n-1} \in \mathbb{C}^n$ system (4.10) possesses a unique solution. The proof is complete. □

We introduce the functionals $\delta_{\lambda,j}(f) := f^{(j)}(\lambda)$, $\lambda \in \Delta$, $j \geqslant 0$, where $\delta_{\lambda,0}$ is the above considered delta function δ_λ. All of them are linear and continuous on $A_\omega(\Delta)$. For a subspace Q of the space $A_\omega(\Delta)$ by Q^0 we denote the annulator of Q in $A_\omega(\Delta)'$. We are going to prove the following statement on dividing by a polynomial in the space $A_\omega(\Delta)'$.

Theorem 4.3. *Let $q \in \mathbb{C}[z]$ be a polynomial of degree $n \in \mathbb{N}$.*
(i) For each $\psi \in A_\omega(\Delta)'$ the equation $q\varphi = \psi$ possesses a solution $\varphi \in A_\omega(\Delta)'$. This equation has a unique solution $\varphi_0 \in A_\omega(\Delta)'$ obeying the conditions $\varphi(m_j) = 0$, $0 \leqslant j \leqslant n-1$.
(ii) Let $\frac{\delta_0}{q} \in A_\omega(\Delta)'$ be a solution to the equation $q\varphi = \delta_0$ such that $\frac{\delta_0}{q}(m_j) = 0$, $0 \leqslant j \leqslant n-1$. Then for each $\psi \in A_\omega(\Delta)'$ the functional $\varphi_0 := \psi \circledast \frac{\delta_0}{q}$ is a solution to the equation $q\varphi = \psi$ obeying the conditions $\varphi_0(m_j) = 0$, $0 \leqslant j \leqslant n-1$.

Proof. (i): The adjoint operator for the operator $M_q : A_\omega(\Delta) \to A_\omega(\Delta)$ of multiplication by q is $M_q' : A_\omega(\Delta)' \to A_\omega(\Delta)'$, $\varphi \mapsto q\varphi$. Since M_q is injective and has a closed image, then M_q' is surjective [17, Ch. 8, Sect. 8.6; Thm 8.6.13]. At the same time,

$$\operatorname{Ker} M_q' = (\operatorname{Im} M_q)^0 = \operatorname{span}\{\delta_{\lambda_l,s} \,|\, 1 \leqslant l \leqslant L,\, 0 \leqslant s \leqslant k_l - 1\}. \tag{4.11}$$

Let $\lambda_l \in \mathbb{C}$, $1 \leqslant l \leqslant L$, $L \in \mathbb{N}$, be all pairwise different roots of q and k_l be the multiplicity of λ_l, while $\varphi \in A_\omega(\Delta)'$ be some solution of the equation $q\varphi = \psi$, $c_j := \varphi(m_j)$, $0 \leqslant j \leqslant n-1$. By Lemma 4.2 the system of equations

$$\sum_{l=1}^{L} \sum_{s=0}^{k_l-1} b_{l,s} m_j^{(s)}(\lambda_l) = c_j,\ \ 0 \leqslant j \leqslant n-1,$$

has a solution $b_{l,s} \in \mathbb{C}$, $1 \leqslant l \leqslant L$, $0 \leqslant s \leqslant k_l - 1$. The functional

$$\varphi_0 := \varphi - \sum_{1\leqslant l \leqslant L} \sum_{s=0}^{k_l-1} b_{l,s}\delta_{\lambda,s} \in A_\omega(\Delta)'$$

satisfy the identities $q\varphi_0 = \psi$ and $\varphi_0(m_j) = 0$, $0 \leqslant j \leqslant n-1$.

Let us show that such functional φ_0 is unique. Let $q\xi = 0$, that is, $M_q'(\xi) = 0$, where $\xi \in A_\omega(\Delta)'$. By (4.11) there exist numbers $d_{l,s}$, $1 \leqslant j \leqslant L$, $0 \leqslant s \leqslant k_l - 1$, for which

$$\xi = \sum_{j=1}^{L} \sum_{s=1}^{k_l-1} d_{l,s}\delta_{\lambda_l,s}.$$

If $\xi(m_j) = 0$, $0 \leqslant j \leqslant n-1$, then

$$\sum_{l=1}^{L} \sum_{s=0}^{k_l-1} d_{l,s} m_j^{(s)}(\lambda_l) = 0, \qquad 0 \leqslant j \leqslant n-1.$$

Due to Lemma 4.2 we have $\xi = 0$.

(ii): First we are going to show that $q\varphi_0 = \psi$. For $f \in A_\omega(\Delta)$ we have

$$\begin{aligned}(q\varphi_0)(f) &= q\left(\psi \circledast \frac{\delta_0}{q}\right)(f) = \left(\psi \circledast \frac{\delta_0}{q}\right)(qf) \\ &= \psi_z\left(\frac{\delta_0}{q}(T_z(qf))\right) = \psi_z\left(\left(\frac{\delta_0}{q}\right)_t\left(\frac{tq(t)f(t) - zq(z)f(z)}{t-z}\right)\right) \\ &= \psi_z\left(\left(\frac{\delta_0}{q}\right)_t\left(tq(t)\frac{f(t)-f(z)}{t-z} + f(z)\frac{tq(t)-zq(z)}{t-z}\right)\right).\end{aligned}$$

Since

$$\left(\frac{\delta_0}{q}\right)_t\left(tq(t)\frac{f(t)-f(z)}{t-z}\right) = (\delta_0)_t\left(t\frac{f(t)-f(z)}{t-z}\right) = 0,$$

then

$$(q\varphi_0)(f) = \psi_z\left(f(z)\left(\frac{\delta_0}{q}\right)_t\left(\frac{tq(t)-zq(z)}{t-z}\right)\right).$$

Let

$$q(z) = \sum_{j=0}^{n} a_j z^j, \qquad a_n \neq 0.$$

Then

$$\frac{tq(t)-zq(z)}{t-z}=\frac{1}{t-z}\sum_{j=0}^{n}a_j(t^{j+1}-z^{j+1})=\sum_{j=0}^{n}a_j\sum_{k=0}^{j}t^kz^{j-k}.$$

This is why

$$(q\varphi_0)(f)=\psi_z\left(f(z)\left(\frac{\delta_0}{q}\right)_t(a_nt^n)\right)=\psi(f)\left(\frac{\delta_0}{q}\right)_t(a_nt^n).$$

Since

$$\left(\frac{\delta_0}{q}\right)_t(a_nt^n)=\left(\frac{\delta_0}{q}\right)_t\left(q(t)-\sum_{j=0}^{n-1}a_jt^j\right)=\left(\frac{\delta_0}{q}\right)(q)=1,$$

then $(q\varphi_0)(f)=\psi(f)$ for each function $f\in A_\omega(\Delta)$. Thus, $q\varphi_0=\psi$.

Let us confirm the initial conditions $\varphi_0(m_j)=0$, $0\leqslant j\leqslant n-1$. For j such that $0\leqslant j\leqslant n-1$ we obtain:

$$\left(\psi\circledast\frac{\delta_0}{q}\right)(m_j)=\psi_z\left(\left(\frac{\delta_0}{q}\right)_t\left(\frac{t^{j+1}-z^{j+1}}{t-z}\right)\right)=\psi_z\left(\left(\frac{\delta_0}{q}\right)_t\left(\sum_{k=0}^{j}t^kz^{j-k}\right)\right)=0.$$

The proof is complete. □

Since $\mathcal{F}'$ is an isomorphism of $A_\omega(\Delta)'$ onto $\mathcal{E}_\omega(\Delta)$, then (4.2) and the identity, see Section 4.2,

$$\begin{aligned}\widehat{v\varphi}(x)&=(v\varphi)(e_x)=\varphi(ve_x)=\varphi(\mathcal{F}(\mathcal{F}^{-1}(ve_x)))=\mathcal{F}^{-1}(ve_x)(\mathcal{F}'(\varphi))\\&=\langle ve_x,\widehat{\varphi}\rangle=v(d)(\widehat{\varphi})(x),\quad v\in\mathbb{C}[z],\quad \varphi\in A_\omega(\Delta)',\quad x\in\Delta,\end{aligned}$$

yield such statement.

Corollary 4.2. *For each function $g\in\mathcal{E}_\omega(\Delta)$ equation (4.9) has a unique solution $f\in\mathcal{E}_\omega(\Delta)$ obeying the conditions $f^{(j)}(0)=0$, $0\leqslant j\leqslant n-1$.*

If $f_1\in\mathcal{E}_\omega(\Delta)$ is a solution of equation (4.9) with the right hand side $g\equiv 1$ saitsfying the conditions $f_1^{(j)}(0)=0$, $0\leqslant j\leqslant n-1$, then for each function $g\in\mathcal{E}_\omega(\Delta)$ the function $f=g*f_1\in\mathcal{E}_\omega(\Delta)$ is a solution of (4.9) such that $f^{(j)}(0)=0$, $0\leqslant j\leqslant n-1$.

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