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ON UNIFORM CONVERGENCE OF SEMI-ANALYTIC SOLUTION OF DIRICHLET PROBLEM FOR DISSIPATIVE HELMHOLTZ EQUATION IN VICINITY OF BOUNDARY OF TWO-DIMENSIONAL DOMAIN

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Abstract. In the framework of the collocation boundary element method, we propose a semi-analytic approximation of the double-layer potential, which ensures a uniform cubic convergence of the approximate solution to the Dirichlet problem for the Helmholtz equation in a two-dimensional bounded domain or its exterior with a boundary of class C^5. In order to calculate integrals on boundary elements, an exact integration over the variable $\rho := (r^2 - d^2)^{1/2}$ is used, where r and d are the distances from the observed point to integration point and to the boundary of the domain, respectively. Under some simplifications we prove that the use of a number of traditional quadrature formulas leads to a violation of the uniform convergence of potential approximations in the vicinity of the boundary of the domain. The theoretical conclusions are confirmed by a numerical solving of the problem in a circular domain.

Keywords: quadrature formula, double layer potential, Dirichlet problem, Helmholtz equation, boundary integral equation, almost singular integral, boundary layer phenomenon, uniform convergence.

Mathematics Subject Classification: 31-08, 31A10

1. Introduction

The boundary element method (BEM) [1, Sect. 2.5] is one of the main methods for approximate solution of problems of mathematical physics along with the finite element method (FEM) and the finite difference method (FDM). Unlike FEM and FDM, the implementation of FEM requires discretization of only the domain boundary. The solution to the problem in an open domain Ω is sought in the form of a potential $u(x)$ expressed using the integral operator via the density function $v(x')$ defined on the boundary $\partial\Omega$ and being a solution to boundary integral equations (BIE). Integrals in the BIE have the form of potentials and their derivatives calculated at the boundary $\partial\Omega$. In order to approximate potentials and their derivatives in the two-dimensional case, the boundary $\partial\Omega$ is divided into arcs Γ_i, the so-called boundary elements (BE), on each of which a polynomial interpolation of the function $v(x')$ is made. After this, the problem of calculating integrals on the BE arises.

Since the kernels of the integral operators has a singularity $x = x'$, the boundary elements Γ_i are divided into three types [2]:

1) singular boundary elements (SBE) if $x \in \Gamma_i$;

2) non-singular boundary elements (NSBE) if the point x is sufficiently far from Γ_i;

3) almost singular boundary elements (ASBE) if $x \in \Omega$, the distance from the point x to Γ_i is small in comparison with the size Γ_i and $x_0 \in \Gamma_i$, where x_0 is the projection of the point x onto the boundary $\partial\Omega$.

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The integrals on NSBE can be calculated with a good accuracy by using simple Gaussian quadrature formulas (SGQF) [1, Sect. 2.6]. The integrands on the SBE, as a rule, have an infinite discontinuity at $x' = x$; on ASBE at $x' = x_0$ they are continuous but grow unboundedly as $x \to x_0$. Therefore, the integrals on the SBE and ASBE cannot be satisfactorily calculated by using SGQF and special methods are used to calculate them, for example, semi-analytical [3]–[11], one of which is studied in this work, as well as methods of adaptive division of SBE and ASBE [12], [13] and methods of nonlinear transformation of the integration variable: exponential transformation [14], sinh-transformation [15], distance function transformation [2].

The inability to evaluate satisfactorily the integrals on ASBE using SGQF is called the boundary layer effect [15]. The need to calculate integrals on ASBE arises when solving problems in thin-walled and multilayer structures, thin coatings, films, and at the ends of cracks [5], [8], [12]. In these cases, FEM and FDM turn out to be ineffective, since their implementation requires oversampling of the boundary layer, and the advantage of FEM, which does not require sampling of the region, manifests itself. In addition, in such problems, high-precision approximation of the boundary $\partial\Omega$ is of a great importance. Therefore, the linear approximation of SBE and ASBE, used in works [2]–[4], [9]–[11], is considered unsatisfactory [6], [14], and special methods were implemented in the case of quadratic approximation of SBE and ASBE [4]–[7]. However, recently there was a need for even more accurate approximation of the boundary, in particular, using splines, and within the framework of such approximation, the ASBE adaptive division method [12] and methods of nonlinear transformation of the integration variable [14] were implemented. These methods are more easily adapted to more complex ASBE geometries compared to semi-analytical ones, since ultimately the integrals on the SBE and ASBE in them are calculated using SGQF, and therefore they can be implemented for any analytically specified boundary $\partial\Omega$. But these methods also have disadvantages: the method of adaptive division of ASBE does not provide accuracy or requires a lot of computer time at very small distances from point x to the boundary $\partial\Omega$ [2], [5], and the accuracy of methods for nonlinear transformation of the integration variable significantly depends on the position of the point x_0 on the SGQF [13].

The boundary is approximated because of two reasons. The first, irremovable, is that in practice the coordinates of only finitely many boundary points are known, with the help of which the boundary $\partial\Omega$ is interpolated. The second reason is the need to specify the boundary using simpler functions to be able to implement the computational algorithm. To implement methods for adaptive division of ASBE and methods for transforming the integration variable, as already noted, there is no need to specify the boundary using simpler functions. Semi-analytical methods are based on exact integration, which becomes possible, in particular, due to the approximation of the boundary, and refinement of the approximation is associated with a significant complication of the algorithm. We note that as the approximation of the boundary, we can also consider the replacement of the coordinate functions and the distance function by the first terms of their expansion in Taylor series formed by the powers of discretization steps of the curvilinear coordinates on the boundary.

Because of this, the semi-analytical methods proposed in works [4]–[6] are of interest for approximating the integrals on the SBE and ASBE that arise when calculating the two-dimensional simple layer potential (SLP) for the Laplace equation and its derivatives. In the works [5], [6], exact integration over the variable $\rho := (r^2 - d^2)^{1/2}$ is used to approximate integrals on ASBE, where d and r are the distances from the observed point x to the nearest boundary point x_0 and to the boundary integration point x', respectively. In order to an exact integration over ρ to become possible, the integrand function is represented as a product of two functions, one of which, for small values of d, is rapidly varying in the vicinity of $\rho = 0$ and is taken as a weight function, and the other is slowly varying and is approximated by using polynomial interpolation in the variable ρ. In an earlier work [4] a similar method was proposed for calculating integrals on the SBE when $\rho = r$. Although in works [4]–[6] this method was proposed for calculating integrals only on linear and quadratic SBE and ASBE, in fact it can be used for any sufficiently smooth analytically given curve $\partial\Omega$, since the integrals over the variable ρ depend on the curve $\partial\Omega$ only parametrically. Since the slowly varying function is included as a factor in the numerator of the integrand, the complexity of the integrals does not increase significantly with increasing powers of interpolants, and the order of approximation can be increased quite easily. The disadvantage of this method is that it is currently implemented only for a two-dimensional spatial domain.

Works [4]–[6] provide no rigorous justification of the method, and a number of theoretical issues related to its use for solving problems of mathematical physics is not resolved. For example, it is necessary to find out the conditions under which the change of variable and interpolation are possible. It is also necessary to prove that the convergence of the approximations of the potentials and their derivatives, solutions of the BIE and boundary value problems obtained on the base of this method is uniform in the vicinity of the boundary of domain. These issues were considered by the author in works [16]–[19]. Works [16]–[18] are devoted to applying the method to heat conduction problems: in work [16] the uniform convergence of approximate solutions of the BIE is proved, in work [17] the uniform convergence of the approximations of SLP and solutions to the Neumann and Robin problems was shown and in work [18] the uniform convergence of approximations of the double layer potential (DLP) and solutions to the Dirichlet problem was established. In work [19] the uniform convergence of DLP approximations for the Laplace equation was proved.

Here we study approximations of the DLP, approximate solutions of the BIE and the Dirichlet problem for the dissipative Helmholtz equation based on such approximations, as well as issues related to the continuity of approximations of the DLP, which were not reflected in works [18], [19]. In the fourth section we prove the uniform and stable cubic convergence of the DLP approximations in the vicinity of the boundary of the domain of class C^5. In order to construct approximations, we use the representation of the integrand obtained in the second section as a sum of functions with weaker singularities, see formula (2.6). We note that in works by [5], [6] such possibility was not used. Similar representations in the form of sums were used in works by the author [18], [19]. The exact integration over ρ is made not precisely on the SBE and ASBE, as in works [4]–[6], but in some domain of a fixed width including the point x_0. On the rest of the curve $\partial\Omega$, the integrals on the BE are calculated by using SGQF. The cubic rate of convergence is due to the use of piecewise quadratic interpolations (PQI). It is proved that such semi-analytic approximations of the DLP have a property that is similar to the property of the exact DLP: when passing the boundary $\partial\Omega$, they undergo a first kind discontinuity, the magnitude of which is proportional to the values of the PQI of the density function $v(x')$.

In the fifth section, the uniform and stable cubic convergence of semi-analytic approximations for the solution to the BIE and semi-analytic approximations for the solution to the Dirichlet problem is proved. For this purpose, in the third section we solve an auxiliary problem; sufficient conditions are obtained for the stable solvability of the BIE in spaces of differentiable functions. In the fifth section it is also proven that semi-analytic approximations of the solution to the Dirichlet problem, by analogy with the exact solution, have a finite discontinuity at the boundary $\partial\Omega$, the value of which is proportional to the values of the approximate solution of the BIE.

In the sixth section it is proven that if instead of exact integration over the variable ρ we use SGQF, then there is no uniform convergence of the DLP approximations in the vicinity of each boundary point, as a result of which there is a sharp decrease in accuracy in the vicinity of the boundary of the domain. Since the lack of uniform convergence in the vicinities of the nodes of the SGQF is quite often mentioned in the literature, see, for example, [10], [11], here we consider special approximations when the point x_0 never coincides with any SGQF node. In this case, the approximations are always continuous when passing through the boundary and do not have uniform convergence precisely for this reason, since the DLP itself has a finite discontinuity when passing through the boundary. We note that in works by the author [18],

[19] the lack of uniform convergence of DLP approximations based on SGQF was proven under some simplifying assumptions by estimating the remainder term of SGQF.

The final seventh section presents the results of a numerical solution of the Dirichlet problem in the unit disk for $k = \pi$, which confirm that the use of exact integration over ρ provides uniform convergence close to cubic, while using SGQF instead leads to a serious violation of accuracy in the vicinity of the boundary $\partial\Omega$.

2. Preliminary remarks

Let Ω_+ be a bounded open simply-connected two-dimensional domain with a boundary $\partial\Omega$, $\Omega_- := \mathbb{R}^2\backslash\overline{\Omega_+}$. In Cartesian coordinates (x_1, x_2) we define parametric equations of the curve $\partial\Omega$: $x_1 = \tilde{x}_1(s)$, $x_2 = \tilde{x}_2(s)$. The absolute value of the parameter s is equal to the length of the arc drawn from some fixed point and ending at the point $\tilde{x}(s) := (\tilde{x}_1(s), \tilde{x}_2(s))$, and it increases in passing the contour $\partial\Omega$ so that the domain Ω_+ is located to the left. The functions $\tilde{x}_1(s)$, $\tilde{x}_2(s)$ are $2S$-periodic, where S is the half of the length of $\partial\Omega$, and they make one-to-one mapping of the set $I_S := [-S, S)$ onto the set $\partial\Omega$. In what follows we write $\partial\Omega \in C^n$ if there exist derivatives $\tilde{x}_i^{(l)}(s)$ continuous on the closed set $\overline{I_S}$, $l = \overline{0, n}$, $i = 1, 2$, and $\tilde{x}_i^{(l)}(-S+0) = \tilde{x}_i^{(l)}(S-0)$. We assume that the boundary $\partial\Omega$ belongs to the smoothness class C^2, unless otherwise stated.

By $\vec{e}(s)$ we denote the tangential unit vector to the curve $\partial\Omega$ at a point $\tilde{x}(s)$ in the increasing direction of the parameter s, and $\vec{n}(s)$ stands for the unit normal to the curve $\partial\Omega$ passing through the point $\tilde{x}(s)$ and directed inside the domain Ω_+. The vectors $\vec{e}(s)$, $\vec{n}(s)$ form a right-handed system, and their coordinates (x_1, x_2) are calculated by the formulas [21]

$$\vec{e}(s) = (\tilde{x}_1'(s), \tilde{x}_2'(s)), \qquad \vec{n}(s) = (-\tilde{x}_2'(s), \tilde{x}_1'(s)).$$

By $C(\partial\Omega)$ we denote the Banach space of complex functions $f(s)$, which are $2S$-periodic and continuous on the entire real line $\mathbb{R}$ with norm $\|f\|_{C(\partial\Omega)} = \sup_{s\in\overline{I_S}} |f(s)|$. Let $C^n(\partial\Omega)$, $n \in \mathbb{Z}_+$, be the Banach spaces of functions $f \in C(\partial\Omega)$ having continuous derivatives $f^{(l)}(s)$, $s \in \mathbb{R}$, $l = \overline{1, n}$, with norm $\|f\|_{C^n(\partial\Omega)} = \sum_{l=0}^{n} \left\|f^{(l)}\right\|_{C(\partial\Omega)}$ and $C^0(\partial\Omega) = C(\partial\Omega)$ [20, Ch. IV, Part 2, Subp. 23]. We can suppose that a function $f \in C^n(\partial\Omega)$ is well-defined if it is defined on some set I_S and can be extended to the closed set $\overline{I_S}$ so that on the set $\overline{I_S}$ there are continuous derivatives $f^{(l)}(s)$, $l = \overline{0, n}$, and the identities $f^{(l)}(-S+0) = f^{(l)}(S-0)$ are satisfied.

We introduce interior and exterior Dirichlet problem:

$$\nabla^2 u_\pm = k^2 u_\pm \quad (x := (x_1, x_2) \in \Omega_\pm,\ \nabla := (\partial_{x_1}, \partial_{x_2})), \quad u_\pm(\tilde{x}(s)) = w(s), \tag{2.1}$$

where $u_\pm(x)$ and $w(s)$ are complex-valuied functions defined on the sets $\overline{\Omega_\pm}$ and I_S, respectively. We assume that $\operatorname{Re} k > 0$. Then it is known [22, Ch. 3] that under the condition $w \in C(\partial\Omega)$ problem (2.1) possesses a unique solution $u_\pm \in C(\overline{\Omega_\pm}) \cap C^2(\Omega_\pm)$, for which in the case of an external problem the Sommerfeld condition must be satisfied:

$$u_- = o(|x|^{-1/2}), \qquad |\nabla u_-| = o(|x|^{-1/2}) \qquad \text{for} \quad |x| := \sqrt{x_1^2 + x_2^2} \to \infty$$

uniformly in all directions $x/|x|$. The solution to problem (2.1) for $x \in \Omega_\pm$ can be obtained in the form of a DLP: $u_\pm(x) = G(x)v_\pm$, where

$$G(x)f := \int_{I_S} g(x, s')f(s')\,ds' \ (f \in C(\partial\Omega),\ x \in \Omega_\pm), \tag{2.2}$$

$$g(x, s') := -a_0(r^2)b, \quad a_0(r^2) := r^2 a(r^2), \quad a(r^2) := (2\pi r)^{-1}\partial_r K_0(kr), \quad b(x, s') := \partial_{\vec{n}(s')} \ln r^{-1},$$

$r(x, s') = |\vec{r}|$, $\vec{r}(x, s') := \overrightarrow{x\,\tilde{x}(s')}$. Here we adopt that sometimes for the sake of brevity we do not write the variables of a function if they are same as were used in the definition of the function;

the differentiation $\partial_{\vec{n}(s')}$ is made with respect to the variable $x' := \tilde{x}(s')$ in the direction $\vec{n}(s')$; $K_0(z)$ is the MacDonald function admitting the representation

$$K_0(z) = -\left[2^{-1}\ln(z^2/4) + C\right]\sum_{n=0}^{\infty}(n!)^{-2}\left(z^2/4\right)^n + \sum_{n=1}^{\infty}(n!)^{-2}\left(z^2/4\right)^n\sum_{m=1}^{n} m^{-1}, \qquad (2.3)$$

see [23, Sect. 3.71, Eq. (14)]), where $\arg z \in (-\pi, \pi]$, $z \neq 0$, and C is the Euler constant. For each boundary function $w \in C(\partial\Omega)$ the density function $v_\pm$ is the unique solution in the class $C(\partial\Omega)$ corresponding to BIE:

$$\mathbf{G}_\pm v_\pm = w, \qquad (2.4)$$

where

$$\mathbf{G}_\pm := \pm 2^{-1} + \mathbf{G}, \quad (\mathbf{G}v_\pm)(s) := G(\tilde{x}(s))\,v_\pm \quad (s \in I_S).$$

A solution to problem (2.1) can be written by the formula $u_\pm(x) = R_\pm(x)w$, $x \in \Omega_\pm$, where $R_\pm(x) := G(x)\mathbf{G}_\pm^{-1}$ are the resolvent functionals. By formula (2.3) the function $a(r^2)$ can be written as

$$a(r^2) = r^{-2}f_1(r^2) + \ln(r^2)f_2(r^2) + f_3(r^2), \qquad (2.5)$$

where $f_i(z)$, $i = \overline{1,3}$, are entire functions and $f_1(0) = -(2\pi)^{-1}$.

We consider the functions via which the DLP is expressed on the boundary $\partial\Omega$ and in its vicinity. Let $\vec{r}_0(s,s') := \overrightarrow{\tilde{x}(s)\,\tilde{x}(s')}$, $r_0(s,s') := |\vec{r}_0|$. On the set

$$\Theta := \left\{(s,s') : s \in \overline{I_S}, s' - s \in \overline{I_S}\right\}$$

we define functions $\psi_i(s,s')$, $i = \overline{0,5}$, by the identities $\psi_i := \varphi_i/(s'-s)^2$, $i = \overline{0,2}$, and $\psi_i := \varphi_i/(s'-s)$, $i = \overline{3,5}$, as $s' \neq s$, where

$$\begin{aligned}
\varphi_0 :=& r_0^2 = \left[\tilde{x}_1(s') - \tilde{x}_1(s)\right]^2 + \left[\tilde{x}_2(s') - \tilde{x}_2(s)\right]^2,\\
\varphi_1 :=& 2^{-1}\partial_{\vec{n}(s')}r_0^2 = -\tilde{x}_2'(s')\left[\tilde{x}_1(s') - \tilde{x}_1(s)\right] + \tilde{x}_1'(s')\left[\tilde{x}_2(s') - \tilde{x}_2(s)\right] = (\vec{n}(s'), \vec{r}_0)_{\mathbb{R}^2},\\
\varphi_2 :=& 2^{-1}\partial_{\vec{n}(s)}r_0^2 = -\tilde{x}_2'(s)\left[\tilde{x}_1(s) - \tilde{x}_1(s')\right] + \tilde{x}_1'(s)\left[\tilde{x}_2(s) - \tilde{x}_2(s')\right] = -(\vec{n}(s), \vec{r}_0)_{\mathbb{R}^2},\\
\varphi_3 :=& 2^{-1}\partial_{s'}r_0^2 = \tilde{x}_1'(s')\left[\tilde{x}_1(s') - \tilde{x}_1(s)\right] + \tilde{x}_2'(s')\left[\tilde{x}_2(s') - \tilde{x}_2(s)\right] = (\vec{e}(s'), \vec{r}_0)_{\mathbb{R}^2},\\
\varphi_4 :=& 2^{-1}\partial_{s}r_0^2 = \tilde{x}_1'(s)\left[\tilde{x}_1(s) - \tilde{x}_1(s')\right] + \tilde{x}_2'(s)\left[\tilde{x}_2(s) - \tilde{x}_2(s')\right] = -(\vec{e}(s), \vec{r}_0)_{\mathbb{R}^2},\\
\varphi_5 :=& \partial_{s'}\varphi_2 = \tilde{x}_2'(s)\,\tilde{x}_1'(s') - \tilde{x}_1'(s)\,\tilde{x}_2'(s');
\end{aligned}$$

here $(\cdot,\cdot)_{\mathbb{R}^2}$ is the scalar product in the Euclidean space $\mathbb{R}^2$. As $s' = s$, we let

$$\psi_0 = \psi_3 = -\psi_4 := 1, \quad \psi_1 = \psi_2 = \psi_5 := 2^{-1}\left[\tilde{x}_2'(s)\tilde{x}_1''(s) - \tilde{x}_1'(s)\tilde{x}_2''(s)\right].$$

Also on the set Θ we define the functions $\hat{b}(s,s') := b(\tilde{x}(s), s')$ and $\rho_0(s,s')$: $\rho_0 := r_0$ if $s' \geqslant s$ and $\rho_0 := -r_0$ if $s' < s$.

Theorem 2.1 ([24, Lm.]). *Let I be a closed interval on the real line. Assume that some real function $f(z,\zeta)$ possesses continuous derivatives $\partial_z^i\partial_\zeta^j f$, $i = \overline{0,m}$, $j = \overline{0,m+q}$, on the set $I \times I$ and $m \in \mathbb{Z}_+$, $q \in \mathbb{N}$ and $\partial_\zeta^j f|_{\zeta=z} = 0$ as $z \in I$, $j = \overline{0,q-1}$. Then the function $h(z,\zeta)$ defined by the identity $h(z,\zeta) := f/(\zeta - z)^q$ as $\zeta \neq z$ and by $h(z,z) := \partial_\zeta^q f|_{\zeta=z}/q!$ as $\zeta = z$ possesses continuous derivatives $\partial_z^i\partial_\zeta^j h$ on the set $I \times I$ as $i = \overline{0,m-j}$, $j = \overline{0,m}$.*

Corollary 2.1. *Let $\partial\Omega \in C^{n+2}$, $n \in \mathbb{Z}_+$. Then on the set Θ there exist the following continuous derivatives:*

(i) $\partial_s^k\partial_{s'}^l\psi_i$, $k = \overline{0,n-l}$; $l = \overline{0,n}$ as $i = 0,1,2,5$; $l = \overline{0,n+1}$ as $i = 3,4$;

(ii) $\partial_s^k\partial_{s'}^l\hat{b}$, $\partial_s^{k+1}\partial_{s'}^l\rho_0$, $\partial_s^k\partial_{s'}^{l+1}\rho_0$ ($k = \overline{0,n-l}$, $l = \overline{0,n}$).

Proof. We observe the inclusion $\Theta \subset \overline{I_{2S}} \times \overline{I_{2S}}$. Let $I = \overline{I_{2S}}$, $z = s$, $\zeta = s'$. The assumptions of Theorem 2.1 hold in the following cases:

1) $f = \varphi_0$ or $f = \varphi_2$, $m = n$, $q = 2$;

2) $f = \varphi_4$, $m = n + 1$, $q = 1$;
3) $f = \varphi_5$, $m = n$, $q = 1$. Moreover, $\varphi_1(s, s') = \varphi_2(s', s)$, $\varphi_3(s, s') = \varphi_4(s', s)$.
This is why in accordance with Theorem 2.1 Statement (i) holds true. Since the contour $\partial\Omega \in C^2$ has no self-intersections, there exists a positive constant $c_r := \inf_{(s,s')\in\Theta} \psi_0$, $c_r \leqslant 1$. Since $\hat{b} = -\psi_1\psi_0^{-1}$, $\partial_{s'}\rho_0 = \psi_3\psi_0^{-1/2}$, $\partial_s\rho_0 = \psi_4\psi_0^{-1/2}$, then Statement (ii) is true. The proof is complete. □

The estimate

$$\vartheta \leqslant c_K\,|s' - s| \leqslant c'_K c_r^{-1/2} r_0,$$

where ϑ is the acute angle between the normals at the points $\tilde{x}(s')$ and

$$c_K := \sup_{s\in\overline{I_S}} K(s, s), \qquad c'_K := \sup_{(s,s')\in\Theta} K(s, s'), \qquad K(s, s') := \left|\partial^2_{s'}\varphi_2\right|,$$

and $K(s, s)$ is the curvature of the curve at the point $\tilde{x}(s)$. This is why the quantity $3D$, where $D := c_r^{1/2}\,(3c'_K)^{-1}$, can be used as the radius of the Lyapunov circle, see [25, Sect. 94, Cond. (3), (5)]. We introduce local systems of Cartesian coordinates (ξ_s, η_s) with origins at the points $\tilde{x}(s)$ and the ordinate axes directed along the normal inside the domain Ω_+. The points $\tilde{x}_d(s)$, $s \in I_S$, with local coordinates $(\xi_s, \eta_s) = (0, d)$ for a fixed $d \in I_D := [-D, 0) \cup (0, D]$ form a closed line $\partial\Omega_d \in C^1$, and the correspondence between the points $\tilde{x}_d(s)$ and $\tilde{x}(s)$ is one-to-one ($\tilde{x}_0(s) := \tilde{x}(s)$), and the normals $\tilde{x}(s)\tilde{x}_d(s)$ to the curve $\partial\Omega$ are also normals to the curve $\partial\Omega_d$, see [25, Sect. 102]. According to [25, Sect. 102], the curves $\partial\Omega_d$, $d \in I_D$ are called parallel to the curve $\partial\Omega$.

We suppose that the value of the parameter s corresponds to the observation point $x = \tilde{x}_d(s)$, while the value s' does to the integration point $x' = \tilde{x}(s')$ in expression (2.2) for the DLP. Local coordinates (ξ_s, η_s) of the points $\tilde{x}(s')$ are equal to

$$((\vec{e}(s), \vec{r}_0)_{\mathbb{R}^2}, (\vec{n}(s), \vec{r}_0)_{\mathbb{R}^2}) = (-\varphi_4, -\varphi_2)\,,$$

and this is why

$$r^2 = |\tilde{x}_d(s) - \tilde{x}(s')|^2 = \varphi'_0 + d^2,$$

where $\varphi'_0(d, s, s') := \varphi_0 + 2d\,\varphi_2$. Since the curve $\partial\Omega$ and the circle of radius $d \in I_D$ with center $\tilde{x}_d(s)$ have only one common point $\tilde{x}(s)$, then $2d\cos\alpha < r_0$, where α is the angle between the rays $\tilde{x}(s)\tilde{x}(s')$ and $\tilde{x}(s)\tilde{x}_d(s)$. Therefore, $\varphi'_0 = r_0^2 - 2dr_0\cos\alpha > 0$ for $(d, s, s') \in \Upsilon := \overline{I_D} \times \Theta$, $s \neq s'$. We define a function $\rho'(d, s, s')$: $\rho' = \sqrt{\varphi'_0}$ if $s' \geqslant s$; $\rho' = -\sqrt{\varphi'_0}$ if $s' < s$ ($\rho' = \rho_0$ for $d = 0$), and functions $\psi'_0(d, s, s') := \psi_0 + 2d\psi_2$, $\psi'_1(d, s, s') = \psi_3 + d\psi_5$. Under the condition $\partial\Omega \in C^{n+2}$ ($n \in \mathbb{Z}_+$) the derivatives of $\partial^j_{s'}\psi'_i$, $j = \overline{0, n}$, $i = 0, 1$, are continuous on the set Υ by Corollary 2.1. Since $\psi_0(s, s) = 1$, $|\psi_2(s, s)| = 2^{-1}K(s, s)$ and $D \leqslant (3c_K)^{-1}$, then for $(d, s) \in I_D \times I_S$ we have the estimate: $\psi'_0(d, s, s) \geqslant 2/3$. Therefore, $\psi'_0 > 0$ on the set Υ, and the derivatives $\partial^j_{s'}\rho'$, $j = \overline{0, n+1}$, are continuous on Υ once $\partial\Omega \in C^{n+2}$ since $\partial_{s'}\rho' = (\psi'_0)^{-1/2}\,\psi'_1$.

In view of the identity

$$2^{-1}\partial_{\vec{n}(s')} r^2 = \varphi_1 + d\,\varphi_6,$$

where

$$\varphi_6(s, s') := -\tilde{x}'_1(s)\tilde{x}'_1(s') - \tilde{x}'_2(s)\tilde{x}'_2(s'),$$

we can write the function $g(x, s')$ as $x = \tilde{x}_d(s)$, $(d, s, s') \in \Upsilon$ (except for the point x as $d = s' - s = 0$) in the following form:

$$g(\tilde{x}_d(s), s') = -a_1(d, \rho')\delta_1(d, s, s') - a_2(d, \rho')\delta_2(d, s, s'), \tag{2.6}$$

where

$$a_1(d, \rho) := \rho^2 a(\rho^2 + d^2), \quad a_2(d, \rho) := d\,a(\rho^2 + d^2); \quad \delta_1 := -\psi_1\,/\psi'_0\,, \quad \delta_2 := -\varphi_6.$$

Since $\psi'_0 > 0$, under the condition $\partial\Omega \in C^{n+2}$, $n \in \mathbb{Z}_+$, there exist continuous on the set Υ derivatives $\partial^j_{s'}\delta_i$, $j = \overline{0, n}$, $i = 1, 2$.

By E_s we denote the closed arc of the curve $\partial\Omega$ bounded by two parallel lines located at a distance D from the line $\tilde{x}(s)\tilde{x}_D(s)$ for a fixed $s \in I_S$ and $\tilde{x}(s) \in \mathrm{E}_s$. For a fixed s, we introduce a curvilinear coordinate σ_s of the point $\tilde{x}(s')$: $\sigma_s := s' - s$. The values of σ_s corresponding to the boundaries of the arc E_s are denoted by Σ'_s, Σ''_s, $\Sigma'_s < 0 < \Sigma''_s$, and then $\sigma_s \in \Xi_s := [\Sigma'_s, \Sigma''_s]$, $\xi_s \in \overline{I_D}$ if $\tilde{x}(s') \in \mathrm{E}_s$. We also define a function $\xi_s := \tilde{\xi}_s(\sigma_s)$, which describes the dependence of the local Cartesian coordinate ξ_s of a point $\tilde{x}(s')$ on its local curvilinear coordinate σ_s.

Lemma 2.1. *Under the condition $\partial\Omega \in C^2$ the values Σ'_s, Σ''_s depend continuously on $s \in \overline{I_S}$.*

Proof. Since the arc E_s is located inside the Lyapunov circle centered at the point $\tilde{x}(s)$ [25, Sect. 94], the angle between the normals $\vec{n}(s)$ and $\vec{n}(s')$ does not exceed $\pi/3$ if $\tilde{x}(s') \in \mathrm{E}_s$, see [25, Sect. 94, Estim. (7)]. This is why the derivative $d\tilde{\xi}_s(\sigma_s)/d\sigma_s$ is positive and continuous on the set Ξ_s for each $s \in I_S$ and the function $\tilde{\xi}_s(\sigma_s)$ is a C^1 diffeomorphism of the set Ξ_s onto the set $\overline{I_D}$. The derivative $\partial_{\sigma_s}\hat{\xi}$ of the function $\hat{\xi}(s,\sigma_s) := \tilde{\xi}_s(\sigma_s)$ is postivie and continuous on the set $\left\{(s,\sigma_s) : s \in \overline{I_S},\ \sigma_s \in \Xi_s\right\}$, and this is why the derivative $\partial_{\xi_s}\hat{\sigma} = (\partial_{\sigma_s}\hat{\xi})^{-1}$ of the function $\hat{\sigma}(s,\xi_s) := \tilde{\sigma}_s(\xi_s)$, where $\tilde{\sigma}_s(\xi_s)$ is a function inverse to the the function $\tilde{\xi}_s(\sigma_s)$, is continuous on the set $\overline{I_S} \times \overline{I_D}$. Therefore, the values $\Sigma'_s = \hat{\sigma}(s,-D)$, $\Sigma''_s = \hat{\sigma}(s,D)$ depend continuously on $s \in \overline{I_S}$. The proof is complete. □

Theorem 2.2 ([17, Thm. 5])**.** *Let $\partial\Omega \in C^{n+2}$, $n \in \mathbb{Z}_+$. Then the function*

$$\delta_0(d,s,s') := (\partial_{s'}\rho')^{-1} = \sqrt{\psi'_0/\psi'_1}$$

is positive and defined everywhere on the set

$$\Upsilon' := \left\{(d,s,s') : d \in \overline{I_D},\ s \in \overline{I_S},\ s'-s \in \Xi_s\right\},$$

and there exist continuous derivatives $\partial^j_{s'}\delta_0$, $j = \overline{0,n}$.

Corollary 2.2. *Let $\partial\Omega \in C^{n+2}$, $n \in \mathbb{Z}_+$. The for all fixed $s \in I_S$, $d \in \overline{I_D}$ the function $\rho_{d,s}(\sigma) := \rho'(d,s,s+\sigma)$ diffeomorphically maps the set Ξ_s onto the $\rho_{d,s}(\Xi_s)$ with the smoothness C^{n+1}. The function $\sigma'(d,s,\rho) := \sigma_{d,s}(\rho)$, where $\sigma_{d,s}(\rho)$ is the inverse function for $\rho_{d,s}(\sigma)$, and*

$$\tilde{\delta}_0(d,s,\rho) := \delta_0(d,s,s+\sigma_{d,s}(\rho)), \qquad \tilde{\delta}_i(d,s,\rho) := \delta_i(d,s,s+\sigma_{d,s}(\rho))\,\tilde{\delta}_0, \quad i = 1,2,$$

possess continuous derivatives $\partial^j_\rho\sigma'$, $\partial^j_\rho\tilde{\delta}_i$, $j = \overline{0,n}$, $i = \overline{0,2}$, on the set

$$\tilde{\Upsilon}' := \left\{(d,s,\rho) : d \in \overline{I_D},\ s \in \overline{I_S},\ \rho \in \rho_{d,s}(\Xi_s)\right\}.$$

To conclude this section, we consider the functions that will be used for the interpolation. We denote by $\Lambda_m(t,\tau_1,\tau_2)$, $m = \overline{0,2}$, $t \in [\tau_1,\tau_2]$, Lagrange quadratic polynomials:

$$\Lambda_m(t,\tau_1,\tau_2) := \prod_{j=0 (j\neq m)}^{2} \frac{t-t_j}{t_m-t_j}.$$

Here

$$t_j := \bar{\tau} + q_j\triangle\tau, \quad j = \overline{0,2}, \qquad \triangle\tau := 2^{-1}(\tau_2-\tau_1),$$
$$\bar{\tau} := 2^{-1}(\tau_1+\tau_2), \qquad q_0 := -1, \qquad q_1 := 0, \qquad q_2 := 1,$$

[26, Ch. 2, Sect. 3, Item 2]. Let $f(t)$ be a complex-valued function on the segment $[\tau_1,\tau_2]$. Then for the function $\tilde{f}(t) := \sum_{m=0}^{2} f(t_m)\Lambda_m(t,\tau_1,\tau_2)$ and for its first two derivatives, as $t \in [\tau_1,\tau_2]$, the estimates hold:

$$\left|\tilde{f}(t) - f(t)\right| \leqslant c_\omega \sup_{t\in[\tau_1,\tau_2]} \left|f^{(3)}(t)\right| \triangle\tau^3 \quad (f \in C^3([\tau_1,\tau_2])), \tag{2.7}$$

$$\left|\tilde{f}^{(j)}(t)\right| \leqslant c_{\Lambda,j} \max_{m=\overline{0,2}} |f(t_m)|\,\triangle\tau^{-j} \quad (f \in C([\tau_1,\tau_2]), \quad j = \overline{0,2}), \tag{2.8}$$

$$\left|\tilde{f}^{(j)}(t)\right| \leqslant c'_{\Lambda,j} \sup_{t\in[\tau_1,\tau_2]} \left|f^{(j)}(t)\right| \quad (f \in C^j([\tau_1,\tau_2]), \quad j = 1, 2), \tag{2.9}$$

where

$$c_\omega := 2\sqrt{3}/9, \qquad c_{\Lambda,j} := 3, \quad j = \overline{0,2}, \qquad c'_{\Lambda,1} := 3, \quad c'_{\Lambda,2} := 2^{-1}.$$

Lemma 2.2. *Let a function $f(t,\alpha)$ be defined and continuous on a set*

$$\mathrm{T} := \left\{(t,\alpha) : t \in [\tau_1(\alpha), \tau_2(\alpha)]\,, \alpha \in \mathrm{A}\right\},$$

where A *is an n-dimensional rectangle and the functions $\tau_1(\alpha)$, $\tau_2(\alpha)$ are continuous and $\tau_1(\alpha) \leqslant \tau_2(\alpha)$ on the set* A. *Moreover, let on a subset $\mathrm{T}' \subseteq \mathrm{T}$ selected in* T *by the condition $\tau_1(\alpha) < \tau_2(\alpha)$ there exist continuous derivatives $\partial_t^j f$, $j = 1, 2$, which can be extended as $\tau_1(\alpha) = \tau_2(\alpha)$ to continuous on the set* T *functions. Then the function $\tilde{f}$ of the form*

$$\tilde{f}(t,\alpha) := \sum_{m=0}^{2} f\left(t_m(\alpha), \alpha\right) \Lambda_m\left(t, \tau_1(\alpha), \tau_2(\alpha)\right)$$

as $(t,\alpha) \in \mathrm{T}'$ and

$$\tilde{f}(t,\alpha) := f(t,\alpha)$$

as $(t,\alpha) \in \mathrm{T} \setminus \mathrm{T}'$ is continuous on the set T.

Proof. Under the condition $\tau_1 < \tau_2$ the function $\tilde{f}(t)$, $t \in [\tau_1, \tau_2]$ can be represented as

$$\tilde{f}(t) = f(\bar{\tau}) + f_1(\tau_1,\tau_2)(t - \bar{\tau}) + 2^{-1} f_2(\tau_1,\tau_2)(t - \bar{\tau})^2, \tag{2.10}$$

where

$$f_1 := 2^{-1} \triangle\tau^{-1} \left[f(\tau_2) - f(\tau_1)\right], \quad f_2 := \triangle\tau^{-2} \left[f(\tau_1) - 2f(\bar{\tau}) + f(\tau_2)\right].$$

Under the condition $f \in C^2([\tau_1,\tau_2])$, by the Taylor formula with the remainder in the integral form [27, Sect. 318] and the mean value theorem, the formulas

$$\begin{aligned} f_j(\tau_1,\tau_2) &= 2^{j-2} \triangle\tau^{-j} \left[(-1)^j \int_{\bar{\tau}}^{\tau_1} f^{(j)}(t)\,(\tau_1 - t)^{j-1}\, dt + \int_{\bar{\tau}}^{\tau_2} f^{(j)}(t)\,(\tau_2 - t)^{j-1}\, dt\right] \\ &= 2^{-1} \left[f^{(j)}(t_{1,j}) + f^{(j)}(t_{2,j})\right] \qquad t_{1,j} \in [\tau_1, \bar{\tau}]\,, \quad t_{2,j} \in [\bar{\tau}, \tau_2]\,, \quad j = 1, 2, \end{aligned} \tag{2.11}$$

hold true. Now by identities (2.10) and (2.11) we complete the proof. □

We note that if $\mathrm{T}' = \emptyset$, then on the set T as the functions $\partial_t^j f$, $j = 1, 2$, arbitrary continuous functions can serve.

3. Stable solvability of boundary integral equation in spaces of differentiable functions

In this section we solve an auxiliary problem: we study sufficient conditions under which BIE (2.4) are stably solvable in spaces of differentiable functions. These results will be used in what follows to justify the stable convergence of approximations of solutions of the BIE (2.4) and solutions of boundary value problems (2.1).

Let X and Y be some Banach spaces. As an operator $\mathbf{A}$ from the space X to the space Y, we call a linear operator mapping each element of some linear subspace $D(\mathbf{A}) \subseteq X$ into some element of the space Y, and if $X = Y$, then we call the operator $\mathbf{A}$ an operator in the space X. If $D(\mathbf{A}) = X$, then we say that the operator $\mathbf{A}$ from the space X into the space Y is defined everywhere and denote it as $\mathbf{A}\,[X \to Y]$, and if $X = Y$, then $\mathbf{A}\,[X]$.

Theorem 3.1. *Let $\partial\Omega \in C^{n+2}$, $n \in \mathbb{Z}_+$. Then the operator $\mathbf{G}$ in the space $C^n(\partial\Omega)$ is everywhere defined and is bounded.*

Proof. We introduce the functions

$$\tilde{g}(s,\sigma) := g(\tilde{x}(s), s+\sigma), \qquad \tilde{\psi}_0(s,\sigma) := \psi_0(s, s+\sigma),$$
$$\tilde{a}_0(s,\sigma) := a_0(\sigma^2\tilde{\psi}_0), \qquad \tilde{b}(s,\sigma) := \hat{b}(s, s+\sigma)$$

and operators $\mathbf{G}_\varepsilon$, $\varepsilon \in (0,S]$:

$$(\mathbf{G}_\varepsilon f)(s) := \int\limits_{-\varepsilon}^{\varepsilon} \tilde{g}(s,\sigma) f(s+\sigma)\, d\sigma \qquad f \in C(\partial\Omega), \quad s \in I_S.$$

We have the formula $\tilde{g} = -\tilde{a}_0\tilde{b}$. Since by Corollary 2.1 there exist continuous on the set Θ derivatives $\partial_s^k \partial_{s'}^l \psi_0$, $\partial_s^k \partial_{s'}^l \hat{b}(s,s')$, $k = \overline{0, n-l}$, $l = \overline{0,n}$, then there exist derivatives $\partial_s^l \tilde{\psi}_0$ continuous on the set $\overline{I_S} \times \overline{I_S}$, $\partial_s^l \tilde{b}$, $l = \overline{0,n}$. In addition, taking into consideration formula (2.5) and the strict positivity of the function $\tilde{\psi}_0$ on $\overline{I_S} \times \overline{I_S}$, we can represent the function $\tilde{a}_0$ as the sum $\sigma^2 \ln \sigma^2 f_1 + f_2$, where the functions $f_i(s,\sigma)$ $(i = 1,2)$ have continuous on the set $\overline{I_S} \times \overline{I_S}$ derivatives $\partial_s^l f_i$, $l = \overline{0,n}$. Therefore, the derivatives $\partial_s^l \tilde{g}$, $l = \overline{0,n}$, can be extended at $\sigma = 0$ to continuous on the set $\overline{I_S} \times \overline{I_S}$ functions. Let $f \in C^n(\partial\Omega)$. Then there exist continuous derivatives $j_\varepsilon^{(l)}(s)$, $s \in \overline{I_S}$, $l = \overline{0,n}$, of the functions $j_\varepsilon := \mathbf{G}_\varepsilon f$. Since

$$\tilde{x}_i^{(l)}(-S+0) = \tilde{x}_i^{(l)}(S-0), \quad l = \overline{0, n+2}, \quad i = 1,2,$$

and

$$f^{(l)}(-S+0) = f^{(l)}(S-0), \quad l = \overline{0,n},$$

then

$$j_\varepsilon^{(l)}(-S+0) = j_\varepsilon^{(l)}(S-0), \quad l = \overline{0,n}.$$

As a result we have $j_\varepsilon \in C^n(\partial\Omega)$. Taking into consideration the arbitrariness of the function $f \in C^n(\partial\Omega)$, the operator $\mathbf{G}$ in the space $C^n(\partial\Omega)$ is defined everywhere. The estimates hold:

$$\left|(\mathbf{G}_\varepsilon f)^{(l)}(s)\right| = \left| \sum_{k=0}^{l} C_l^k \int\limits_{-\varepsilon}^{\varepsilon} \partial_s^k \tilde{g}(s,\sigma)\, \partial_s^{l-k} f(s+\sigma)\, d\sigma \right| \leqslant 2^{n+1} \varepsilon\, c_n \|f\|_{C^n(\partial\Omega)}, \tag{3.1}$$

where

$$s \in \overline{I_S}, \qquad c_n := \sup_{0 \leqslant l \leqslant n,\, (s,\sigma) \in I_S \times I_S} |\partial_s^l \tilde{g}(s,\sigma)|, \qquad C_l^k := l!/(l-k)!/k!, \qquad l = \overline{0,n}.$$

In accordance with estimates (3.1), the operators $\mathbf{G}_\varepsilon$ $[C^n(\partial\Omega)]$ are bounded. Since $\mathbf{G} = \mathbf{G}_\varepsilon$ as $\varepsilon = S$, this completes the proof. □

Theorem 3.2. *Let $\partial\Omega \in C^{n+2}$, $w \in C^n(\partial\Omega)$, $n \in \mathbb{Z}_+$. Then there exists a unique solution $v_\pm \in C^n(\partial\Omega)$ to BIE (2.4).*

Proof. Let $\eta_\varepsilon(\sigma)$ be a n times continuously differentiable on the segment $\overline{I_S}$ real function:

$$\eta_\varepsilon(\sigma) = 1 \quad \text{as} \quad |\sigma| \leqslant \varepsilon/2, \qquad 0 < \eta_\varepsilon(\sigma) < 1 \quad \text{as} \quad \varepsilon/2 < |\sigma| < \varepsilon,$$
$$\eta_\varepsilon(\sigma) = 0 \quad \text{as} \quad |\sigma| \geqslant \varepsilon, \quad \varepsilon \in (0,S); \qquad g'_\varepsilon(s,\sigma) := \eta_\varepsilon \tilde{g}, \qquad g''_\varepsilon(s,s') := [1 - \eta_\varepsilon(s'-s)]\, g.$$

We represent the operator $\mathbf{G}$ as the sum $\mathbf{G} = \mathbf{G}'_\varepsilon + \mathbf{G}''_\varepsilon$, where the operators $\mathbf{G}'_\varepsilon$, $\mathbf{G}''_\varepsilon$ are defined by the identities

$$(\mathbf{G}'_\varepsilon f)(s) := \int\limits_{-S}^{S} g'_\varepsilon(s,\sigma) f(s+\sigma) d\sigma, \quad (\mathbf{G}''_\varepsilon f)(s) := \int\limits_{-S}^{S} g''_\varepsilon(s,s') f(s') ds'$$

($f \in C(\partial\Omega)$, $s \in I_S$). Similarly to Theorem 3.1, the operator $\mathbf{G}'_\varepsilon$ in the space $C^n(\partial\Omega)$ is defined everywhere, bounded and in view of the inequality $\eta_\varepsilon(\sigma) \leqslant 1$, it satisfies estimates (3.1). The operator $\mathbf{G}''_\varepsilon$ from the space $C(\partial\Omega)$ into the space $C^n(\partial\Omega)$ is also defined everywhere, since there exist continuous on set Θ derivatives $\partial^j_s g''_\varepsilon$, $j = \overline{0, n}$, and $\partial^j_s g''_\varepsilon(-S+0, s') = \partial^j_s g''_\varepsilon(S-0, s')$.

By the assumptions, condition, the right side of BIE (2.4) satisfies $w \in C(\partial\Omega)$, therefore, there exists a unique solution $v_\pm \in C(\partial\Omega)$ of BIE (2.4) [22, Ch. 3]. It remains to prove that $v_\pm \in C^n(\partial\Omega)$. Since $w \in C^n(\partial\Omega)$ and the operator $\mathbf{G}''_\varepsilon$ from the space $C(\partial\Omega)$ into the space $C^n(\partial\Omega)$ is defined everywhere, then $h^\pm_\varepsilon := \pm 2(w - \mathbf{G}''_\varepsilon v_\pm) \in C^n(\partial\Omega)$. BIE (2.4) can be written in the following form:

$$v_\pm \pm 2\mathbf{G}'_\varepsilon v_\pm = h^\pm_\varepsilon. \tag{3.2}$$

We note that the right side $h^\pm_\varepsilon$ of equation (3.2) depends on the solution $v_\pm$, but under the assumptions of this theorem we have $h^\pm_\varepsilon \in C^n(\partial\Omega)$.

Let $\varepsilon < [(n+1)2^{n+2}c_n]^{-1}$. Since $(n+1)2^{n+2}c_n \geqslant 4c_0$, then, due to inequalities (3.1), for the operator $\mathbf{G}'_\varepsilon$ $[C(\partial\Omega)]$ we have the estimate $\|\mathbf{G}'_\varepsilon\| < 2^{-1}$, and therefore the operator $1 \pm 2\mathbf{G}'_\varepsilon$ is boundedly invertible in the space $C(\partial\Omega)$ and the inverse operator $(1 \pm 2\mathbf{G}'_\varepsilon)^{-1}$ can be represented as a Neumann series $\sum\limits_{i=0}^{\infty} (\mp 2\mathbf{G}'_\varepsilon)^i$ convergent in the operator norm [20, Ch. VII, Sect. 3, Subsect. 4, Lm.]. Hence, in view of identity (3.2), the function $v_\pm \in C(\partial\Omega)$ can be represented as a series $\sum\limits_{i=0}^{\infty} j^\pm_{\varepsilon,i}$ convergent in the $C(\partial\Omega)$-norm, where $j^\pm_{\varepsilon,i} := (\mp 2\mathbf{G}'_\varepsilon)^i\, h^\pm_\varepsilon$. Since $\varepsilon < [(n+1)2^{n+2}c_n]^{-1}$, due to inequalities (3.1) for the operator $\mathbf{G}'_\varepsilon$ $[C^n(\partial\Omega)]$ we have the estimate $q := 2\|\mathbf{G}'_\varepsilon\| < 1$. In addition, by Theorem 3.1, $j^\pm_{\varepsilon,i} \in C^n(\partial\Omega)$, $i \in \mathbb{Z}_+$, since $h^\pm_\varepsilon \in C^n(\partial\Omega)$. By induction we obtain the following inequalities:

$$\left|(j^\pm_{\varepsilon,i})^{(l)}(s)\right| \leqslant \left\|j^\pm_{\varepsilon,i}\right\|_{C_n(\partial\Omega)} \leqslant q^i \left\|h^\pm_\varepsilon\right\|_{C^n(\partial\Omega)} \quad s \in I_S, \quad l = \overline{0, n}, \quad i = 0, 1, \dots,$$

due to which the series $\sum\limits_{i=0}^{\infty} (j^\pm_{\varepsilon,i})^{(l)}(s)$ converges uniformly in $s \in \overline{I_S}$, and then

$$v^{(l)}_\pm = \sum_{i=0}^{\infty} (j^\pm_{\varepsilon,i})^{(l)} \in C(\partial\Omega), \qquad l = \overline{0, n},$$

that is, $v_\pm \in C^n(\partial\Omega)$. The proof is complete. □

We observe that we can not obtain the result of Theorem 3.2 on the base of the smoothness improving theorems for solutions to second order Fredholm equations proven in works [28]–[33]. Namely, let $\Phi := \{(s, s') \in (-S, S) \times (-S, S) :\ s \neq s'\}$. Under the condition $\partial\Omega \in C^{n+2}$, $n \in \mathbb{Z}_+$, the function $\hat{g}(s, s') := g(\tilde{x}(s), s')$ is n times continuously differentiable on the set Φ and the estimates hold:

$$\left|\partial^k_s(\partial_s + \partial_{s'})^l \hat{g}\right| \leqslant c_n \cdot \begin{cases} 1 & (k < 2), \\ \left|\ln\left|(s' - s)\big(s' - (s + 2S)\big)\big(s' - (s - 2S)\big)\right|\right| & (k = 2), \\ \left|(s' - s)\big(s' - (s + 2S)\big)\big(s' - (s - 2S)\big)\right|^{2-k} & (k > 2), \end{cases}$$

where c_n are positive constants, $k + l \leqslant n$, $(s, s') \in \Phi$. Thus, the derivatives of the kernels of the integral operator $\mathbf{G}$ possesses not only a diagonal singularity (as $s \to s'$), but also boundary singularities (as $s \to -S$, $s' \to S$ and as $s \to S$, $s' \to -S$). Theorem in [28], Theorem 1 in [29], Theorems 1, 2 in [30], Theorem 5 [31] were proved for the kernels of the integral operators, the derivatives of which possess only the diagonal singularity. Theorems 1.1–1.3 in [32], Theorem 4 in [33] were proved for the kernels, the derivatives of which possess also boundary singularities, but in contrast to the derivatives of the function $\hat{g}$, the boundary singularities of the derivatives of such kernels do not increase as the order of the partial derivative ∂^k_s grows.

On the base of Theorems 3.1, 3.2 and the Banach theorem on the inverse operator [20, Ch. II, Sect. 2, Subsect. 2, Thm.] we obtain the main result of the present section.

Corollary 3.1. *Let $\partial\Omega \in C^{n+2}$, $n \in \mathbb{Z}_+$. Then the operators $\mathbf{G}_\pm$ $[C^n(\partial\Omega)]$ are boundedly invertible.*

4. Semi-analytic approximation of double layer potential

We first consider some representations and properties of intergral operators of DLP, which will be employed for constructing semi-analytic approximations for DLPs and for studying their properties. By Ω_D we denote the set formed by the points $x = \tilde{x}_d(s)$ as $d \in I_D$, $s \in I_S$. Let

$$f \in C(\partial\Omega), \qquad B_i(d,s,\rho)f := \tilde{\delta}_i(d,s,\rho)f(s+\sigma_{d,s}(\rho)), \quad i=1,2,$$
$$B(d,s,s')f := g(\tilde{x}_d(s),s')f(s').$$

For $x \in \overline{\Omega_D}$ we define the functionals $G_i(x)$, $i = \overline{1,3}$:

$$\begin{aligned} G_i(x)f &:= -\int\limits_{\rho_{d,s}(\Xi_s)} a_i(d,\rho)B_i(d,s,\rho)f\,d\rho, \quad i=1,2,\\ G_3(x)f &:= \int\limits_{I_S\setminus\Xi_s} B(d,s,s+\sigma)f\,d\sigma. \end{aligned} \tag{4.1}$$

By formula (2.6) and Corollary 2.2 the functionals $G(x)$ can be represented as the sum

$$G(x) = G_1(x) + G_2(x) + G_3(x) \quad (x \in \overline{\Omega_D}), \tag{4.2}$$

and $G(\tilde{x}(s))f = G_1(\tilde{x}(s))f + G_3(\tilde{x}(s))f$ is the direct value of DLP on the boundary $\partial\Omega$, since $G_2(\tilde{x}(s))f = 0$ according to identities (4.1) ($s \in I_S$).

By Corollary 2.2 the functions $B_i(d,s,\rho)f$, $i = 1,2$, are continuous on the set $\tilde{\Upsilon}'$ since $\partial\Omega \in C^2$, $f \in C(\partial\Omega)$. Moreover, the values Σ'_s, Σ''_s depend continuously on $s \in \overline{I_S}$, see Lemma 2.1, and therefore, the values $\rho_{d,s}(\Sigma'_s)$, $\rho_{d,s}(\Sigma''_s)$ depend continuously on $(d,s) \in \overline{I_D}\times\overline{I_S}$ since the function ρ' is continuous on the set Υ. We also note that by formula (2.5) we can represent the functions $a_i(d,\rho)$, $i = 1,2$, as the sums

$$a_1 = -(2\pi)^{-1}\rho^2\left(\rho^2+d^2\right)^{-1} + a'_1, \qquad a_2 = -(2\pi)^{-1}d\left(\rho^2+d^2\right)^{-1} + a'_2, \tag{4.3}$$

where the functionals $a'_i(d,\rho)$ can be extended at $d = \rho = 0$ to continuous on the set $\overline{I_D}\times\overline{I_S}$ functions. This is why the function $G_1(x)f$ is continuous on the closed set $\overline{\Omega_D}$ by the theorem on the continuity of improper integrals with a parameter [34, Ch. XVII, Sect. 2, Stat.5], while the continuity of the function$G_2(x)f$ is guaranteed only on the set Ω_D.

The function $G_3(x)f$ is continuous on the set $\overline{\Omega_D}$. Indeed, as we have already mentioned, the values Σ'_s, Σ''_s depend continuously on $s \in \overline{I_S}$, and the function $B_3(d,s,s')f$ is continuous on the set $\overline{\Upsilon\setminus\Upsilon'}$ since here $r \geqslant D$.

We observe that by formula (2.5), $a'_2 \to 0$ as $d \to 0$ uniformly in $\rho \in \overline{I_S}$ and by the inequality $\psi'_0 > 0$, as $(d,s) \in \overline{I_D}\times I_S$, the inequalities hold:

$$|\rho_{d,s}(\Sigma'_s)| \geqslant c_0|\Sigma'_s| \geqslant c_0 D, \qquad \rho_{d,s}(\Sigma''_s) \geqslant c_0\Sigma''_s \geqslant c_0 D, \qquad c_0 := \inf_{(d,s,s')\in\Upsilon}\sqrt{\psi'_0} > 0.$$

This is why in view of second identity (4.3) and the boundedness of the function $B_2(d,s,\rho)f$ and the values Σ'_s, Σ''_s for each fixed $\varepsilon \in (0, c_0 D]$ we have limits

$$\lim_{d\to 0}\int\limits_{\rho_{d,s}(\Xi_s)\setminus[-\varepsilon,\varepsilon]} a_2 B_2 f\,d\rho = -(2\pi)^{-1}\lim_{d\to 0}\int\limits_{\rho_{d,s}(\Xi_s)\setminus[-\varepsilon,\varepsilon]} d\left(\rho^2+d^2\right)^{-1}B_2 f\,d\rho = 0$$

converging uniformly in $s \in I_S$. Taking into consideration identities $B_2(d,s,0)f = f(s)$ as $(d,s) \in \overline{I_D}\times I_S$ and the continuity of the function $B_2(d,s,\rho)f$ on the set $\tilde{\Upsilon}'$, we obtain uniformly convering on $s \in \overline{I_S}$ limits

$$
\begin{aligned}
\lim_{d\to\pm 0} G_2(\tilde{x}_d(s))f &= -\lim_{\varepsilon\to+0}\left(\lim_{d\to\pm 0}\int_{-\varepsilon}^{\varepsilon} a_2(d,\rho)B_2(d,s,\rho)f\,d\rho\right)\\
&= (2\pi)^{-1}\lim_{\varepsilon\to+0}\left(\lim_{d\to\pm 0} d\int_{-\varepsilon}^{\varepsilon}\left(\rho^2+d^2\right)^{-1}B_2(d,s,\rho)f\,d\rho\right)\\
&= \pi^{-1}\lim_{\varepsilon\to+0}\left(\lim_{d\to\pm 0} B_2(d,s,\tilde{\rho}_{d,s,\varepsilon})f\arctan\left(\varepsilon/d\right)\right)\\
&= \pm 2^{-1}f(s),
\end{aligned}
\tag{4.4}
$$

where $\tilde{\rho}_{d,s,\varepsilon} \in [-\varepsilon, \varepsilon]$, see generalized mean value theorem [27, Sect. 299]. On the base of identities (4.2), (4.4) we get well-known limiting relations for DLP, which hold as $\partial\Omega \in C^2$, $f \in C(\partial\Omega)$ uniformly in $s \in \overline{I_S}$:

$$
\lim_{d\to\pm 0} G(\tilde{x}_d(s))f = \pm 2^{-1}f(s) + G(\tilde{x}(s))f. \tag{4.5}
$$

Using identities (4.4) and (4.5), we can define the functions $G_2(x)f$ and $G(x)f$ ($f \in C(\partial\Omega)$) to functions continuous on closed sets $\overline{\Omega_D^\pm}$, $\Omega_D^\pm := \Omega_D \cap \Omega_\pm$. Under the condition $w \in C(\partial\Omega)$, the functions $u_\pm(x) := R_\pm(x)w$ are continuous on the corresponding sets $\Omega_D^\pm$ and they can be extended to continuous on the sets $\overline{\Omega_D^\pm}$ functions by means of uniformly converging in $s \in \overline{I_S}$ limits:

$$
\lim_{d\to\pm 0} u_\pm(\tilde{x}_d(s)) = \pm 2^{-1}v_\pm(s) + u_\pm(\tilde{x}(s)) = w(s), \tag{4.6}
$$

at the same time

$$
u_\pm(\tilde{x}(s)) := R_\pm(\tilde{x}(s))w, \qquad R_\pm(\tilde{x}(s)) := G(\tilde{x}(s))\mathbf{G}_\pm^{-1}, \quad s \in I_S,
$$

are direct values of the functions $u_\pm(x)$ and the functionals $R_\pm(x)$ on the boundary $\partial\Omega$.

In view of identities (4.3), the functions $a_i(d,\rho)$, $i = 1, 2$, are absolutely integrable in $\rho \in I_S$ and the integrals are uniformly bounded in $d \in \overline{I_D}$. The norms of the functionals $G_i(x)$ $[C(\partial\Omega) \to \mathbb{C}]$, $x \in \overline{\Omega_D}$, $i = \overline{1,3}$, satisfy the estimate

$$
\|G_i(x)\| \leqslant A_i\hat{c}_{i,0}, \quad i = 1, 2, \qquad \|G_3(x)\| \leqslant 2S\tilde{c}_{3,0}, \tag{4.7}
$$

where

$$
A_i := \sup_{d\in\overline{I_D}}\int_{I_S}|a_i(d,\rho)|\,d\rho, \qquad \hat{c}_{i,0} := \sup_{(d,s,\rho)\in\tilde{\Upsilon}'}|\tilde{\delta}_i|, \quad i = 1, 2; \qquad \tilde{c}_{3,0} := \sup_{(d,s,s')\in\overline{\Upsilon\setminus\Upsilon'}}|g(\tilde{x}_d(s), s')|.
$$

By identity (4.2) and estimates (4.7) we have the following estimates for the norms of the functionals $G(x)$ $[C(\partial\Omega) \to \mathbb{C}]$:

$$
\|G(x)\| \leqslant c_G := A_1\hat{c}_{1,0} + A_2\hat{c}_{2,0} + 2S\tilde{c}_{3,0} \quad (x \in \overline{\Omega_D}). \tag{4.8}
$$

By estimate (4.8) the functionals $G(x)$ $[C(\partial\Omega) \to \mathbb{C}]$ are bounded uniformly in $x \in \overline{\Omega_D}$.

Let us discretize BLP in the integration variable s' in accordance with BEM. In order to do this, we parition the boundary $\partial\Omega$ into boundary elements and on each of them we replace the density function by its quadratic interpolant. Namely, let $L/2 \in \mathbb{N}$, $h := S/(L+1)$, $s_l := lh$, $l \in \mathbb{Z}$ ($\tilde{x}(s_{l+2L+2}) = \tilde{x}(s_l)$). We introduce spaces H_L of complex-valued grid functions f with values f_l defined as the collocation points s_l on the set I_S ($f_{l+2L+2} = f_l$) with the norm $\|f\|_{H_L} = \max\limits_{-L-1\leqslant l\leqslant L}|f_l|$. We define projection operators $\mathbf{P}_L$ $[C(\partial\Omega) \to H_L]$: $(\mathbf{P}_Lf)_l := f(s_l)$

($\|\mathbf{P}_L\| \leqslant 1$). We introduce interpolation operators $\ddot{\mathbf{P}}_L\ [H_L \to C(\partial\Omega)]$:

$$(\ddot{\mathbf{P}}_L f)(s) := \sum_{m=0}^{2} f_{2l-1+m}\,\Lambda_m(s, s_{2l-1}, s_{2l+1}), \quad s \in [s_{2l-1}, s_{2l+1}]\,, \quad l = \overline{-L/2, L/2}.$$

By estimate (2.8) as $j = 0$ the operators $\ddot{\mathbf{P}}_L\ [H_L \to C(\partial\Omega)]$ are uniformly bounded: $\|\ddot{\mathbf{P}}_L\| \leqslant c_{\Lambda,0}$. On the base of inequality (2.7) we have the estimates

$$\left\|\ddot{\mathbf{P}}_L \mathbf{P}_L f - f\right\|_{C(\partial\Omega)} \leqslant c_\omega \left\|f^{(3)}\right\|_{C(\partial\Omega)} h^3 \quad , f \in C^3(\partial\Omega). \tag{4.9}$$

By means of identities $\ddot{G}(x)f := G(x)\ddot{\mathbf{P}}_L f$ ($f \in H_L$) we define the functionals $\ddot{G}(x)\ [H_L \to \mathbb{C}]$, $x \in \overline{\Omega_D}$. Using inequalities (4.8), (4.9), we obtain the estimates of approximation of the functionals $G(x)$ by the functionals $\ddot{G}(x)\mathbf{P}_L$:

$$\left|\ddot{G}(x)\mathbf{P}_L f - G(x)f\right| \leqslant c_G\, c_\omega \left\|f^{(3)}\right\|_{C(\partial\Omega)} h^3, x \in \overline{\Omega_D}, \quad f \in C^3(\partial\Omega). \tag{4.10}$$

The functionals $\ddot{G}(x)$ are expressed via the integrals

$$\int_{s_{2l-1}}^{s_{2l+1}} g(x, s')\ddot{f}(s')\,ds', \qquad \ddot{f} := \ddot{\mathbf{P}}_L f, \quad f \in H_L,$$

which in the general case can not be calculated exactly. This is why we need an additional approximation. We note that in accordance with identities (4.2) the functionals $\ddot{G}(x)$ can be represented as the sums $\ddot{G}(x) = \ddot{G}_1(x) + \ddot{G}_2(x) + \ddot{G}_3(x)$ as $x \in \overline{\Omega_D}$, where $\ddot{G}_i(x) := G_i(x)\ddot{\mathbf{P}}_L$, $i = \overline{1,3}$. We introduce the functionals $\hat{G}_i(x)$ approximating $\ddot{G}_i(x)$, $x \in \overline{\Omega_D}$, $i = 1, 2$. In order to do this, in expressions (4.1) for $\ddot{G}_i(\tilde{x}_d(s))f$ we replace the functions $\ddot{B}_i(d, s, \rho)f := \tilde{\delta}_i\,\ddot{f}(s + \sigma_{d,s}(\rho))$, $f \in H_L$, $\ddot{f} := \ddot{\mathbf{P}}_L f$, $(d, s, \rho) \in \tilde{\Upsilon}'$, by their piece-wise-quadratic interpolants $\hat{B}_i(d, s, \rho)f$ in the variable ρ:

$$\hat{G}_i(\tilde{x}_d(s))\,f := -\int_{\rho_{d,s}(\Xi_s)} a_i(d, \rho)\hat{B}_i(d, s, \rho)f\,d\rho. \tag{4.11}$$

Here

$$\hat{B}_i(d, s, \rho)f := \hat{B}_{i,l}(d, s, \rho)f, \qquad \rho \in [\rho_{d,s,l}, \rho_{d,s,l+1}]\,, \quad l \in \mathbb{Z},$$

$$\hat{B}_{i,l}(d, s, \rho)f := \begin{cases} \displaystyle\sum_{m=0}^{2} \ddot{B}_i(d, s, \rho_{d,s,l,m})f\,\Lambda_m(\rho, \rho_{d,s,l}, \rho_{d,s,l+1}) & (\rho_{d,s,l} < \rho_{d,s,l+1}), \\ \ddot{B}_i(d, s, \rho_{d,s,l}) & (\rho_{d,s,l} = \rho_{d,s,l+1}); \end{cases}$$

$$\rho_{d,s,l,m} := 2^{-1}\left(\rho_{d,s,l} + \rho_{d,s,l+1}\right) + q_m h'_{d,s,l}, \qquad h'_{d,s,l} := 2^{-1}\left(\rho_{d,s,l+1} - \rho_{d,s,l}\right), \qquad \rho_{d,s,l} := \rho_{d,s}(\alpha_{s,l}).$$

We define the values $\alpha_{s,l}$, $l \in \mathbb{Z}$, $s \in \overline{I_S}$, so that to satisfy the conditions:

(1) $\bigcup_{l\in\mathbb{Z}} [\alpha_{s,l}, \alpha_{s,l+1}] = \Xi_s$; (2) $\left(\{s_k - s\}_{k\in\mathbb{Z}} \cap \Xi_s\right) \subseteq \{\alpha_{s,l}\}_{l\in\mathbb{Z}}$; (3) $\alpha_{s,l} \leqslant \alpha_{s,l+1}$ ($l \in \mathbb{Z}$);

(4) $\alpha_{s,l}$, $l \in \mathbb{Z}$, continuously depend on $s \in \overline{I_S}$ if Σ'_s, Σ''_s continuously depend on $s \in \overline{I_S}$. Let $s \in [s_k, s_{k+1})$ for k from $-L-1$ to L. Then we let

$$\alpha_{s,2l+k} := \min\left\{s_{l+k} - s, \Sigma''_s\right\}, \qquad \alpha_{s,2l+1+k} := \min\left\{lh, \Sigma''_s\right\}, \qquad l \geqslant 0;$$
$$\alpha_{s,2l+k} := \max\left\{s_{l+k} - s, \Sigma'_s\right\}, \qquad \alpha_{s,2l+1+k} := \max\left\{lh, \Sigma'_s\right\}, \qquad l < 0.$$

We note that in formula (4.11), it is sufficient to prescribe the functions $\hat{B}_i(d, s, \rho)f$ on the segments $[\rho_{d,s,l}, \rho_{d,s,l+1}]$ only for $l = \overline{-3L-2, 3L+2}$ since for other values $l \in \mathbb{Z}$ the lengths of these segments are zero ($\rho_{d,s,l} = \rho_{d,s,l+1}$) for all $s \in \overline{I_S}$ even if $\Sigma'_s = -S$, $\Sigma''_s = S$.

We introduce the functions $\ddot{B}(d,s,s')f := g(\tilde{x}_d(s),s')\ddot{f}(s')$ ($f \in H_L$, $\ddot{f} := \ddot{\mathbf{P}}_L f$, $(d,s,s') \in \Upsilon$) and approximate the functionals $\ddot{G}_3(\tilde{x}_d(s))$ for $(d,s) \in \overline{I_D} \times I_S$ by the functionals $\tilde{G}_3(\tilde{x}_d(s))$ constructed with help of SGQF with γ nodes:

$$\tilde{G}_3(\tilde{x}_d(s))f := \sum_{l=-3L-2}^{3L+2} h''_{s,l}\tilde{B}_{3,l}(d,s)f, \tag{4.12}$$

where

$$\tilde{B}_{3,l}(d,s)f := \sum_{j=1}^{\gamma} \omega_j \ddot{B}(d,s,s+\beta_{s,l,j})f,$$

$$\beta_{s,l,j} := 2^{-1}(\beta_{s,l}+\beta_{s,l+1}) + h''_{s,l}z_j, \quad h''_{s,l} := 2^{-1}(\beta_{s,l+1}-\beta_{s,l}).$$

Here z_j are the roots of the polynomial $P_\gamma(z) := (d^\gamma/dz^\gamma)\,(z^2-1)^\gamma$ on the interval $(-1,1)$ [26, Ch. 3, Sect. 5, Item 2] $(z_1 < z_2 < \ldots < z_\gamma)$. For the weight coefficients ω_j the relations $\sum\limits_{j=1}^{\gamma}\omega_j = 2$ and $\omega_j > 0$ hold [26, Ch. 3, Sect. 5, Item 1].

We prescribe the values $\beta_{s,l}$, $l \in \mathbb{Z}$, $s \in \overline{I_S}$, so that to satisfy the conditions:

(1) $\bigcup\limits_{l\in\mathbb{Z}} [\beta_{s,l},\beta_{s,l+1}] = \overline{I_S \setminus \Xi_s}$; (2) $\left(\{s_k - s\}_{k\in\mathbb{Z}} \cap \overline{I_S \setminus \Xi_s}\right) \subseteq \{\beta_{s,l}\}_{l\in\mathbb{Z}}$;

(3) $\beta_{s,l} \leqslant \beta_{s,l+1}$;

(4) $\beta_{s,l}$ $(l \in \mathbb{Z})$ continuously depend on $s \in \overline{I_S}$ if Σ'_s, Σ''_s continuously depend on $s \in \overline{I_S}$. Let $s \in [s_k, s_{k+1})$ for some k from $-L-1$ to L. Then we let

$$\beta_{s,2l+k} := \min\bigl\{\max\{s_{l+k}-s, \Sigma''_s\}, S\bigr\}, \qquad \beta_{s,2l+1+k} := \min\bigl\{\max\{lh, \Sigma''_s\}, S\bigr\}, \quad l \geqslant 0;$$

$$\beta_{s,2l+k} := \max\bigl\{\min\{s_{l+k}-s, \Sigma'_s\}, -S\bigr\}, \quad \beta_{s,2l+1+k} := \max\bigl\{\min\{lh, \Sigma'_s\}, -S\bigr\}, \quad l < 0.$$

We note that in formula (4.12) the sum over $l = \overline{-3L-2, 3L+2}$ can be replaced by that over $l \in \mathbb{Z}$ since all additional terms are zero $(h''_{s,l} = 0)$ for all $s \in \overline{I_S}$.

Let $\hat{G}(x) := \hat{G}_1(x) + \hat{G}_2(x) + \tilde{G}_3(x)$. By estimate (2.8) as $j = 0$ the functionals $\hat{B}_i(d,s,\rho)$ $[H_L \to \mathbb{C}]$, $i = 1,2$, is uniformly bounded in $(d,s,\rho) \in \tilde{\Upsilon}'$:

$$\left\|\hat{B}_i(d,s,\rho)\right\| \leqslant c^2_{\Lambda,0}\hat{c}_{i,0},$$

while the functionals $\tilde{B}_{3,l}(d,s)$ $[H_L \to \mathbb{C}]$ are bounded uniformly in $(d,s) \in \overline{I_D} \times \overline{I_S}$, $l \in \mathbb{Z}$:

$$\left\|\tilde{B}_{3,l}(d,s)\right\| \leqslant 2c_{\Lambda,0}\hat{c}_{3,0}.$$

This is why as $x \in \overline{\Omega_D}$ we have inequalities

$$\left\|\hat{G}_i(x)\right\| \leqslant A_i c^2_{\Lambda,0}\hat{c}_{i,0} \qquad i = 1,2, \qquad \left\|\tilde{G}_3(x)\right\| \leqslant 2S c_{\Lambda,0}\tilde{c}_{3,0},$$

on the base of which as well as of the inequality $\|\mathbf{P}_L\| \leqslant 1$ we obtain the following statement.

Theorem 4.1. *Let $\partial\Omega \in C^2$, $\gamma, L/2 \in \mathbb{N}$. Then the functionals $\hat{G}(x)$ $[H_L \to \mathbb{C}]$, $\hat{G}(x)\mathbf{P}_L$ $[C(\partial\Omega) \to \mathbb{C}]$ are equibounded uniformly in $x \in \overline{\Omega_D}$.*

By Corollary 2.2 and the inequality $r \geqslant D$, which holds if $(d,s,s') \in \overline{\Upsilon \setminus \Upsilon'}$, under the mentioned smoothness of the curve $\partial\Omega$ and for $j = \overline{0,n}$, $n \in \mathbb{Z}_+$ we can define the constants

$$\hat{c}_{i,j} := \sup_{(d,s,\rho)\in\tilde{\Upsilon}'} \left|\partial^j_\rho \tilde{\delta}_i\right|, \quad i = 1,2, \quad \partial\Omega \in C^{n+2},$$

$$\tilde{c}_{3,j} := \sup_{(d,s,s')\in\overline{\Upsilon\setminus\Upsilon'}} \left|\partial^j_{s'} g(\tilde{x}_d(s),s')\right|, \quad \partial\Omega \in C^{n+1}.$$

Employing inequalities (2.7)–(2.9) and $h'_{d,s,l} \leqslant 2^{-1} c_h h$ ($c_h := \sup\limits_{(d,s,s')\in\Upsilon'} \partial_{s'}\rho'$) and letting $f \in C^2(\partial\Omega)$, $x = \tilde{x}_d(s)$, $(d,s) \in \overline{I_D} \times I_S$, under the mentioned smoothness of the curve $\partial\Omega$ we obtain the estimates

$$\begin{aligned} \left|\hat{G}_i(x)\mathbf{P}_L f - \ddot{G}_i(x)\mathbf{P}_L f\right| &\leqslant 8^{-1} A_i c_h^3 c_\omega \operatorname*{ess\,sup}_{(d,s,\rho)\in\tilde{\Upsilon}'} \left|\partial_\rho^3 \ddot{B}_i(d,s,\rho)\mathbf{P}_L f\right| h^3 \\ &\leqslant c_i \|f\|_{C^2(\partial\Omega)} h^3, \quad \partial\Omega \in C^5, \quad i = 1,2, \end{aligned} \tag{4.13}$$

$$\begin{aligned} \left|\tilde{G}_3(x)\mathbf{P}_L f - \ddot{G}_3(x)\mathbf{P}_L f\right| &\leqslant 2S \operatorname*{ess\,sup}_{(d,s,s')\in\overline{\Upsilon\backslash\Upsilon'}} \left|\partial_{s'}^{2\gamma} \ddot{B}(d,s,s')\mathbf{P}_L f\right| h^{2\gamma} \\ &\leqslant c_3 \|f\|_{C^2(\partial\Omega)} h^{2\gamma} \quad (\partial\Omega \in C^{2\gamma+1}), \end{aligned} \tag{4.14}$$

where the constants c_i, $i = 1,2$, c_3 are defined by the identities

$$c_i := 8^{-1} A_i c_h^3 c_\omega \left[\hat{c}_{i,3} c_{\Lambda,0} + (3\hat{c}_{i,2}\hat{c}_{0,0} + 3\hat{c}_{i,1}\hat{c}_{0,1} + \hat{c}_{i,0}\hat{c}_{0,2})\, c'_{\Lambda,1} + 3\left(\hat{c}_{i,1}\hat{c}_{0,0}^2 + \hat{c}_{i,0}\hat{c}_{0,1}\hat{c}_{0,0}\right) c'_{\Lambda,2}\right],$$
$$c_3 := 2S(\gamma!)^4 \left[(2\gamma)!\right]^{-3} (2\gamma+1)^{-1} \left[\tilde{c}_{3,2\gamma} c_{\Lambda,0} + 2\gamma\tilde{c}_{3,2\gamma-1} c'_{\Lambda,1} + \gamma(2\gamma-1)\tilde{c}_{3,2\gamma-2} c'_{\Lambda,2}\right].$$

Here ess sup is the essential supremum [20, Ch. III, Sect. 1, Subsect. 11]. While obtaining estimate (4.14), we have employed an estimate for the remainder in SGQF [26, Ch. 3, Sect. 5, Item 2]. By estimates (4.10), (4.13), (4.14) we arrive at the statement.

Theorem 4.2. *Let $\partial\Omega \in C^{2\gamma+1}$, $\gamma \geqslant 2$, $\gamma, L/2 \in \mathbb{N}$. Then the functionals $\hat{G}(x)\mathbf{P}_L$ $[C^3(\partial\Omega) \to \mathbb{C}]$ converge as $L \to \infty$ in the uniform operator topology to corresponding functionals $G(x)$ $[C^3(\partial\Omega) \to \mathbb{C}]$ uniformly in $x \in \overline{\Omega_D}$ with the approximation order $O(L^{-3})$.*

For definitions and basic information concerning the convergence of the operators in various topologies we refer to [20, Ch. VI, Sect. 1, Subsect. 1–3].

By estimates (4.8), (4.10), (4.13), (4.14), Theorem 4.1 and the density of the set $C^3(\partial\Omega)$ in the space $C(\partial\Omega)$ we also obtain the following statement.

Corollary 4.1. *Let $\partial\Omega \in C^{2\gamma+1}$, $\gamma \geqslant 2$, $\gamma, L/2 \in \mathbb{N}$. Then, as $L \to \infty$, the functionals $\hat{G}(x)\mathbf{P}_L$ $[C(\partial\Omega \to \mathbb{C}]$ converge to corresponding functionals $G(x)$ $[C(\partial\Omega \to \mathbb{C}]$ in the strong operator topology uniformly in $x \in \overline{\Omega_D}$.*

We observe that in view of formula (2.3) the integrals in ρ in expressions for $\hat{G}_i(x)$, $i = 1,2$, can be calculated analytically.

Theorem 4.3. *Let $\partial\Omega \in C^4$, $L/2 \in \mathbb{N}$, $f \in H_L$, $\ddot{f} := \ddot{\mathbf{P}}_L f$. Then the functions $\hat{G}(x)f$ is continuous on the set Ω_D and it can be extended to a continuous function on the closure $\overline{\Omega_D^\pm}$ by means of the corresponding limits converging uniformly in $s \in I_S$:*

$$\lim_{d\to\pm 0} \hat{G}(\tilde{x}_d(s))f = \pm 2^{-1}\ddot{f}(s) + \hat{G}(\tilde{x}(s))f. \tag{4.15}$$

Proof. Since Σ'_s, Σ''_s are continuous in $s \in \overline{I_S}$ (see Lemma 2.1), then by definition, all $\alpha_{s,l}$, $l \in \mathbb{Z}$, depend continuously on $s \in \overline{I_S}$, while $\alpha_{s,l} \leqslant \alpha_{s,l+1}$. Therefore, by Theorem 2.2, the values $\rho_{d,s,l}$, $l \in \mathbb{Z}$, depend continuously on $(d,s) \in \overline{I_D} \times \overline{I_S}$ and $\rho_{d,s,l} \leqslant \rho_{d,s,l+1}$. We also note that $\ddot{f} \in C(\partial\Omega)$, and since $\left(\{s_k - s\}_{k\in\mathbb{Z}} \cap \Xi_s\right) \subseteq \{\alpha_{s,l}\}_{l\in\mathbb{Z}}$, then on each subset selected from the set

$$\Upsilon_l := \left\{(s,\sigma):\ s \in \overline{I_S},\ \sigma \in [\alpha_{s,l}, \alpha_{s,l+1}]\right\}, \qquad l \in \mathbb{Z}$$

by the inequality $\alpha_{s,l} < \alpha_{s,l+1}$, there exist continuous derivatives $\partial_\sigma \ddot{f}^{(j)}(s+\sigma)$, $j = 1,2$, which for $\alpha_{s,l} = \alpha_{s,l+1}$ can be extended by continuity to the entire set Υ_l. Therefore, by Corollary 2.2 the functions $\ddot{B}_i(d,s,\rho)f$ $(i = 1,2)$ are continuous on the set $\tilde{\Upsilon}'$, and on each subset selected from the set

$$\tilde{\Upsilon}'_l := \left\{(d,s,\rho):\ d \in \overline{I_D},\ s \in \overline{I_S},\ \rho \in [\rho_{d,s,l}, \rho_{d,s,l+1}]\right\}, \qquad l \in \mathbb{Z},$$

by the inequality $\rho_{d,s,l} < \rho_{d,s,l+1}$, there exist continuous derivatives $\partial_\rho^j \ddot{B}_i(d,s,\rho)f$, $j = 1,2$, which for $\rho_{d,s,l} = \rho_{d,s,l+1}$ can be extended by continuity to the entire set $\tilde{\Upsilon}'_l$. Therefore, by Lemma 2.2 and the identities $\bigcup_{l\in\mathbb{Z}} \tilde{\Upsilon}'_l = \tilde{\Upsilon}'$, $\hat{B}_i(d,s,\rho_{d,s,l})f = \ddot{B}_i(d,s,\rho_{d,s,l})f$ (the values $\rho_{d,s,l}$ are collocation points) the functions $\hat{B}_i(d,s,\rho)f$, $i = 1,2$, are continuous on the set $\tilde{\Upsilon}'$. In view of the fact that the value $\rho = 0$ is a collocation point ($\rho_{d,s,k+1} = 0$ if $s \in [s_k, s_{k+1})$), for all $(d,s) \in \overline{I_D} \times \overline{I_S}$ we have the following identities:

$$\hat{B}_2(d,s,0)f = \ddot{B}_2(d,s,0)f = \tilde{\delta}_2(d,s,0)\ddot{f}(s+\sigma_{d,s}(0)) = \ddot{f}(s). \tag{4.16}$$

Similarly to the functions $G_i(x)f$, $f \in C(\partial\Omega)$, $i = \overline{1,2}$, we conclude that the functions $\hat{G}_1(x)f$, $\hat{G}_2(x)f$, $f \in H_L$, are continuous on the sets $\overline{\Omega_D}$, Ω_D, respectively, and by identities (4.16) the function $\hat{G}_2(x)f$ can be extended to a continuous one on the closures $\overline{\Omega_D^\pm}$ by means of limits convergent uniformly in $s \in \overline{I_S}$:

$$\lim_{d\to\pm 0} \hat{G}_2(\tilde{x}_d(s))f = \pm 2^{-1}\hat{B}_2(d,s,0)f = \pm 2^{-1}\ddot{f}(s), \tag{4.17}$$

cf. limits (4.4).

Since Σ'_s, Σ''_s depend continuously on $s \in \overline{I_S}$, then by definition all $\beta_{s,l}$ ($l \in \mathbb{Z}$) depend continuously on $s \in \overline{I_S}$, while $\beta_{s,l} \leqslant \beta_{s,l+1}$. Therefore, the nodes of SGQF $\tilde{x}(s+\beta_{s,l,j})$, $l \in \mathbb{Z}$, $j = \overline{1,\gamma}$, continuously depend on $s \in \overline{I_S}$ and, moreover, they are separated from the observation point $\tilde{x}_d(s)$ by a distance at least D. Hence, the functions $\tilde{B}_{3,l}(d,s)f$, $l \in \mathbb{Z}$, are continuous on the set $\overline{I_D} \times \overline{I_S}$, and the function $\tilde{G}_3(x)f$, in its turn, is continuous on the set $\overline{\Omega_D}$.

Since, in accordance with definitions (4.11) and (4.12) the identities $\hat{G}_2(\tilde{x}(s)) = 0$ and $\hat{G}(\tilde{x}(s)) = \hat{G}_1(\tilde{x}(s)) + \tilde{G}_3(\tilde{x}(s))$, $s \in I_S$, hold, identities (4.17) imply identities (4.15) are valid. The proof is complete □

5. Approximate solutions to boundary integral equations and boundary value problems

We employ semi-analytic approximations of DLP obtained in the previous section for approximate solving BIE (2.4). In the space H_L we define the operators $\hat{\mathbf{G}}$: $(\hat{\mathbf{G}}f)_l := \hat{G}(\tilde{x}(s_l))f$, $f \in H_L$, $l = \overline{-L-1,L}$, as well as the operators $\hat{\mathbf{G}}_\pm := \pm 2^{-1} + \hat{\mathbf{G}}$. By Theorem 4.2, the operators $\hat{\mathbf{G}}_\pm$ are grid approximations of the operators of BIE (2.4) on the grid with nodes $\tilde{x}(s_l)$. Similarly to the operators $\hat{\mathbf{G}}$, in the space H_L we define the operators $\ddot{\mathbf{G}}$, $\ddot{\mathbf{G}}_i$, $i = 1,3$, $\hat{\mathbf{G}}$, $\hat{\mathbf{G}}_1$, $\tilde{\mathbf{G}}_3$. Then $\ddot{\mathbf{G}} = \ddot{\mathbf{G}}_1 + \ddot{\mathbf{G}}_3$, $\hat{\mathbf{G}} = \hat{\mathbf{G}}_1 + \tilde{\mathbf{G}}_3$.

Theorem 5.1. *Let $\partial\Omega \in C^{2\gamma+1}$, $\gamma \geqslant 2$, $\gamma, L/2 \in \mathbb{N}$. Then the operators $\hat{\mathbf{G}}_\pm$ $[H_L]$ are invertible for sufficiently large L and the inverse operators $\hat{\mathbf{G}}_\pm^{-1}$ $[H_L]$ are equibounded.*

Proof. Let $f \in H_L$. By a bounded invertibility of the operators $\mathbf{G}_\pm$ $[C(\partial\Omega)]$ (see Corollary 3.1) we have the estimates

$$\left\|\mathbf{G}_\pm \ddot{\mathbf{P}}_L f\right\|_{C(\partial\Omega)} \geqslant c_{-1}\left\|\ddot{\mathbf{P}}_L f\right\|_{C(\partial\Omega)} \geqslant c_{-1}\|f\|_{H_L}, \quad c_{-1} := \left\|\mathbf{G}_\pm^{-1}\right\|^{-1}. \tag{5.1}$$

We note that $\hat{g} = -a_0(\rho_0^2)\hat{b}$. By Corollary 2.1 the derivatives $\partial_s\hat{b}$ and $\partial_s\rho_0$ are continuous on the set Θ. This is in view of formula (2.5) the derivative $\partial_s a_0(\rho_0^2)$ is continuous on the set Θ and in view of estimate (2.8) (for $j = 0$) we have the inequality

$$\max_{-L-1\leqslant l\leqslant L}\ \sup_{s\in[s_l,s_{l+1}]}\left|\ddot{\mathbf{G}}(\tilde{x}(s_l))f - \mathbf{G}(\tilde{x}(s))\ddot{\mathbf{P}}_L f\right| \leqslant c' c_{\Lambda,0}\|f\|_{H_L} h, \tag{5.2}$$

where $c' := 2S \sup_{(s,s')\in\Theta}|\partial_s\hat{g}(s,s')|$.

Let $h \leqslant 1$. Employing estimates (2.8) for $j = 1, 2$ instead of (2.9), similarly to inequalities (4.13), (4.14), under the condition $h \leqslant 1$ we obtain the following inequalities:

$$\left\| \hat{\mathbf{G}}_1 f - \ddot{\mathbf{G}}_1 f \right\|_{H_L} \leqslant c_1' \|f\|_{H_L} h, \quad \left\| \tilde{\mathbf{G}}_3 f - \ddot{\mathbf{G}}_3 f \right\|_{H_L} \leqslant c_3' \|f\|_{H_L} h, \tag{5.3}$$

where

$$c_1' := A_1 c_h^3 c_\omega \left[\hat{c}_{1,3} c_{\Lambda,0} + (3\hat{c}_{1,2}\hat{c}_{0,0} + 3\hat{c}_{1,1}\hat{c}_{0,1} + \hat{c}_{1,0}\hat{c}_{0,2}) c_{\Lambda,1} + 3(\hat{c}_{1,1}\hat{c}_{0,0}^2 + \hat{c}_{1,0}\hat{c}_{0,1}\hat{c}_{0,0}) c_{\Lambda,2} \right],$$

$$c_3' := 2S(\gamma!)^4 \left[(2\gamma)!\right]^{-3} (2\gamma+1)^{-1} \left[\tilde{c}_{3,2\gamma} c_{\Lambda,0} + 2\gamma \tilde{c}_{3,2\gamma-1} c_{\Lambda,1} + \gamma(2\gamma-1)\tilde{c}_{3,2\gamma-2} c_{\Lambda,2} \right].$$

Let $h \leqslant 3^{-1} c_{-1} / \max\{c', c_1' + c_3'\}$, and $h \leqslant 1$. Then by inequalities (5.1)–(5.3) for all $f \in H_L$ we have the estimates $\left\| \hat{\mathbf{G}}_\pm f \right\|_{H_L} \geqslant 3^{-1} c_{-1} \|f\|_{H_L}$, which mean that the operators $\hat{\mathbf{G}}_\pm\ [H_L]$ are invertible and the inverse operators $\hat{\mathbf{G}}_\pm^{-1}$ are equibounded: $\left\| \hat{\mathbf{G}}_\pm^{-1} \right\| \leqslant 3/c_{-1}$. The proof is complete. □

Definition 5.1. *We say that sequences of bounded operators $\{\mathbf{A}_n [X \to Y]\}_{n=1}^\infty$ and $\{\mathbf{B}_n [X \to Y]\}_{n=1}^\infty$ are approximately equivalent in the uniform operator topology with the approximation order $O(n^{-k})$, $k \in \mathbb{N}$, if there exists a constant $c > 0$ independent of n such that for all $n \in \mathbb{N}$, $f \in X$, the inequality holds: $\|\mathbf{A}_n f - \mathbf{B}_n f\|_Y \leqslant c\, n^{-k} \|f\|_X$.*

By Theorems 4.2, 5.1, Corollary 3.1, estimates (4.9) and $\|\mathbf{P}_L\| \leqslant 1$, $\|\ddot{\mathbf{P}}_L\| \leqslant c_{\Lambda,0}$ we obtain the following statements.

Corollary 5.1. *Let $\partial\Omega \in C^{2\gamma+1}$, $\gamma \geqslant 2$, $\gamma, L/2 \in \mathbb{N}$. Then the sequence of the operators $\{\hat{\mathbf{G}}_\pm^{-1} \mathbf{P}_L [C^3(\partial\Omega) \to H_L]\}_{L/2=1}^\infty$ and $\{\mathbf{P}_L \mathbf{G}_\pm^{-1} [C^3(\partial\Omega) \to H_L]\}_{L/2=1}^\infty$ are approximately equivalent in the uniform operator topology with the approximation order $O(L^{-3})$. As $L \to \infty$, the operators $\ddot{\mathbf{P}}_L \hat{\mathbf{G}}_\pm^{-1} \mathbf{P}_L\ [C^3(\partial\Omega) \to C(\partial\Omega)]$ converge in the uniform operator topology to corresponding operators $\mathbf{G}_\pm^{-1}\ [C^3(\partial\Omega) \to C(\partial\Omega)]$ with the approximation order $O(L^{-3})$. As $L \to \infty$, the operators $\ddot{\mathbf{P}}_L \hat{\mathbf{G}}_\pm^{-1} \mathbf{P}_L\ [C(\partial\Omega)]$ converge to corresponding operators $\mathbf{G}_\pm^{-1}\ [C(\partial\Omega)]$ in the strong operator topology.*

Corollary 5.1 allows us to obtain grid and approximate solutions to BIE (2.4): $\hat{v}_\pm := \hat{\mathbf{G}}_\pm^{-1} \mathbf{P}_L w \in H_L$, $\ddot{v}_\pm := \ddot{\mathbf{P}}_L \hat{\mathbf{G}}_\pm^{-1} \mathbf{P}_L w \in C(\partial\Omega)$, respectively.

We define functionals $\hat{R}_\pm(x) := \hat{G}(x) \hat{\mathbf{G}}_\pm^{-1}$, $x \in \overline{\Omega_D^\pm}$, as $x \in \Omega_D^\pm$ being approximations for the resolvent functionals $R_\pm(x)$ of problems (2.1), while as $x = \tilde{x}(s)$, $s \in I_S$, they are approximations of direct values of the functionals $R_\pm(x)$ on the boundary $\partial\Omega$. In view of Theorems 4.1, 4.2, 5.1 and Corollaries 3.1, 5.1 we get the following corollary.

Corollary 5.2. *Let $\partial\Omega \in C^{2\gamma+1}$, $\gamma \geqslant 2$, $\gamma, L/2 \in \mathbb{N}$. Then the functionals $\hat{R}_\pm(x)\ [H_L \to \mathbb{C}]$, $\hat{R}_\pm(x) \mathbf{P}_L\ [C(\partial\Omega) \to \mathbb{C}]$ are equibounded uniformly in $x \in \overline{\Omega_D^\pm}$. As $L \to \infty$, the functionals $\hat{R}_\pm(x) \mathbf{P}_L\ [C^3(\partial\Omega) \to \mathbb{C}]$ converge in the uniform operator topology to corresponding functionals $R_\pm(x)\ [C^3(\partial\Omega) \to \mathbb{C}]$ uniformly in $x \in \overline{\Omega_D^\pm}$ with the approximation order $O(L^{-3})$. As $L \to \infty$, the functionals $\hat{R}_\pm(x) \mathbf{P}_L\ [C(\partial\Omega) \to \mathbb{C}]$ converge in the strong operator topology to corresponding functionals $R_\pm(x)\ [C(\partial\Omega) \to \mathbb{C}]$ uniformly in $x \in \overline{\Omega_D^\pm}$.*

We define functions $\hat{u}_\pm(x) := \hat{R}_\pm(x) \mathbf{P}_L w$ $(x \in \overline{\Omega_D^\pm})$ as $x \in \Omega_D^\pm$ being approximations of solutions $u_\pm(x)$ of problem (2.1), while as $x = \tilde{x}(s)$, $s \in I_S$, they are approximations of direct values of the functions $u_\pm(x)$ on the boundary $\partial\Omega$. On the base of Corollary 5.2 we formulate the main result of the present paper.

Corollary 5.3. *Let $\partial\Omega \in C^{2\gamma+1}$, $\gamma \geqslant 2$, $\gamma, L/2 \in \mathbb{N}$, $c > 0$. Then as $L \to \infty$ the functions $\hat{u}_\pm(x)$ converge with a cubic rate to corresponding functions $u_\pm(x)$ (2.1) uniformly in $x \in \overline{\Omega_D^\pm}$ and boundary functions $w \in C^3(\partial\Omega)$ obeying the condition $\|w\|_{C^3(\partial\Omega)} \leqslant c$. Moreover, as $L \to$*

∞, $\varepsilon \to +0$, the functions $\hat{R}_\pm(x)\mathbf{P}_L w_\varepsilon$ converge uniformly in $x \in \overline{\Omega_D^\pm}$ to the function $u_\pm(x)$ once $\|w_\varepsilon - w\|_{C(\partial\Omega)} \leqslant \varepsilon$, $w, w_\varepsilon \in C(\partial\Omega)$.

In view of identities $\hat{u}_\pm(x) = \hat{G}(x)\hat{v}_\pm$, $\ddot{\mathbf{P}}_L\hat{v}_\pm = \ddot{v}_\pm$ and Theorem 4.3, Corollaries 5.1, 5.3 and formula (4.6) we obtain the following statement.

Corollary 5.4. *Let $\partial\Omega \in C^{2\gamma+1}$, $\gamma \geqslant 2$, $\gamma, L/2 \in \mathbb{N}$, $c > 0$. Then under the condition $w \in C(\partial\Omega)$ the functions $\hat{u}_\pm(x)$ are continuous on the corresponding sets $\Omega_D^\pm$ and they can be extended to continuous functions on the closures $\overline{\Omega_D^\pm}$ by means of limits convergent uniformly in $s \in I_S$:*

$$\hat{u}_\pm(\tilde{x}_{\pm 0}(s)) := \lim_{d\to\pm 0} \hat{u}_\pm(\tilde{x}_d(s)) = \pm 2^{-1}\ddot{v}_\pm(s) + \hat{u}_\pm(\tilde{x}(s)).$$

As $L \to \infty$, the functions $\hat{u}_\pm(\tilde{x}_{\pm 0}(s))$ converge to the corresponding boundary functions $w(s)$ with a cubic rate uniformly in $s \in I_S$ and functions $w \in C^3(\partial\Omega)$ obeying the condition $\|w\|_{C^3(\partial\Omega)} \leqslant c$. Moreover, as $L \to \infty$, $\varepsilon \to +0$, the functions $\hat{R}_\pm(\tilde{x}_{\pm 0}(s))\mathbf{P}_L w_\varepsilon$ converge to the functions $w(s)$ uniformly in $s \in I_S$ if $\|w_\varepsilon - w\|_{C(\partial\Omega)} \leqslant \varepsilon$, $w, w_\varepsilon \in C(\partial\Omega)$.

6. Absence of uniform convergence of approximations for double layer potential while using standard quadrature formulas

In this section we examine the reasons for the occurrence of the boundary layer effect in the case of DLP. As it has been noted in the Introduction, SGQFs are traditionally used to calculate the DLP in the domain Ω. Computational experiments show that the rate of convergence of approximate solutions to problems (2.1) decreases significantly in the vicinity of the boundary $\partial\Omega$. We relate this phenomenon with the lack of uniform convergence of such approximations for DLP in the vicinity of the boundary of the domain.

The absence of uniform convergence of approximations of the functionals $\ddot{G}(x)\mathbf{P}_L$ in the vicinity of the boundary is quite obvious in cases when the point $\tilde{x}(s)$, the projection of the observation point $x = \tilde{x}_d(s)$ onto the curve $\partial\Omega$, can coincide with one of the SGQF nodes for arbitrarily large values of L. For example, consider the approximations $\breve{G}(x)$ of the functionals $\ddot{G}(x)$, $x = \tilde{x}_d(s)$, $s \in I_S$, $d \in I_D$, made exclusively on the basis of SGQF:

$$\breve{G}(x)f := 2^{-1}h \sum_{l=-L-1}^{L} \sum_{j=1}^{\gamma} \omega_j \ddot{B}(d, s, s_{l,j})f, \quad s_{l,j} := 2^{-1}(s_l + s_{l+1}) + 2^{-1}h z_j, \quad f \in H_L.$$

By identities (2.5), (2.6) and $\varphi_6(s,s) = -1$ we have $g(\tilde{x}_d(s), s) \sim d^{-1} \to \infty$ as $d \to 0$ for fixed $s \in I_S$. This is why the absolute values of $\ddot{B}(d, s_{l,j}, s_{l,j})f$ grow unboundedly as $d \to 0$, $\ddot{f}(s_{l,j}) \neq 0$ ($\ddot{f} := \ddot{\mathbf{P}}_L f$) for fixed L. Since $\ddot{B}(d, s_{l,j}, s')f$ are continuous in $d \in \overline{I_D}$ for fixed $s' \neq s_{l,j}$, the absolute values of $\breve{G}(\tilde{x}_d(s_{l,j}))f$ also grow unboundedly as $d \to 0$, $\ddot{f}(s_{l,j}) \neq 0$ for fixed L. Let $f \in C^3(\partial\Omega)$ and $f(s) \neq 0$ for some $s \in I_S$. Then according to estimate (4.9) for sufficiently large L there exists $s_{l,j}$ depending on L such that $\ddot{f}(s_{l,j}) \neq 0$ ($\ddot{f} := \ddot{\mathbf{P}}_L \mathbf{P}_L f$). Therefore, a uniform in $(d,s) \in I_D \times I_S$ convergence of approximating functions $\breve{G}(\tilde{x}_d(s))\mathbf{P}_L f$ to exact function $G(\tilde{x}_d(s))f$ as $L \to \infty$ is impossible since exact functions $G(\tilde{x}_d(s))f$ can be extended to continuous ones on the sets $\overline{\Omega_D^\pm}$ by means of identities (4.5).

We consider approximations $\tilde{G}(\tilde{x}_d(s))$ of the functionals $\ddot{G}(\tilde{x}_d(s))$ made exclusively on the base of SGQF in such a way that the point $\tilde{x}(s)$ cannot coincide with any of the nodes of SGQF. In order to do this, we introduce approximations of the functionals $\ddot{G}_{1-2}(x) := G_{1-2}(x)\ddot{\mathbf{P}}_L$ $[H_L \to \mathbb{C}]$, where $G_{1-2}(x) := G_1(x) + G_2(x)$, by using SGQF γ' nodes:

$$\tilde{G}_{1-2}(x)f := \sum_{l=-3L-2}^{3L+2} h_{s,l} \sum_{j=1}^{\gamma'} \omega_j \ddot{B}'(d, s, s + \alpha_{s,l,j})f.$$

Here

$$x=\tilde{x}_d(s),\quad d\in\overline{I_D},\quad s\in I_S,\quad f\in H_L,\qquad \alpha_{s,l,j}:=\bar{\alpha}_{s,l}+h_{s,l}z_j,$$
$$\bar{\alpha}_{s,l}:=2^{-1}\left(\alpha_{s,l}+\alpha_{s,l+1}\right),\qquad h_{s,l}:=2^{-1}\left(\alpha_{s,l+1}-\alpha_{s,l}\right);$$
$$\ddot{B}'(d,s,s'):=\ddot{B}(d,s,s'),\quad \text{if}\quad s\neq s',\qquad \ddot{B}'(d,s,s):=0.$$

We observe that the sum over $l=\overline{-3L-2,3L+2}$ can be replaced by sum over $l\in\mathbb{Z}$ since all additional terms vanish ($h_{s,l}=0$) for all $s\in\overline{I_S}$ even if $\Sigma'_s=-S$, $\Sigma''_s=S$.

We let $\tilde{G}(x):=\tilde{G}_{1-2}(x)+\tilde{G}_3(x)$. The approximations $\tilde{G}(x)$ differ from the approximations $\hat{G}(x)$ only by the fact that for calculating the functionals $\ddot{G}_{1-2}(x)$, instead of exact integration in the variable ρ, SGQF are used.

Theorem 6.1. *Let $\partial\Omega\in C^2$. Then the function $\tilde{G}(\tilde{x}_d(s))f$ is continuous on the set $d\in\overline{I_D}$ for all fixed $s\in I_S$, $L/2\in\mathbb{N}$, $f\in H_L$.*

Proof. We fix $s\in[s_k,s_{k+1})$ $(-L-1\leqslant k\leqslant L)$, $L/2\in\mathbb{N}$, $f\in H_L$. Since SGQFs are fomrulas of open kind, see [26, Ch. 3, Sect. 5, Item 1], then $z_j\neq\pm1$, $j=\overline{1,\gamma}$, and the point $\tilde{x}(s)$ coincides with none of the SGQF nodes $\tilde{x}(s+\alpha_{s,l,j})$ except for $\tilde{x}(s+\alpha_{s,k,j})$ if $h_{s,k}=0$, but then $\ddot{B}'(d,s,s+\alpha_{s,k,j})f:=0$ as $d\in\overline{I_D}$. The derivative $d\xi_s(\sigma)/d\sigma$ is positive and continuous on the set Ξ_s, see Lemma 2.1, and $\alpha_{s,l,j}\in\Xi_s$. Therefore, we can determine the constant $c_s>0$, which the smallest of the distances from the points $\tilde{x}_d(s)$ $(d\in\overline{I_D})$ to the nodes of SGQF $\tilde{x}(s+\alpha_{s,l,j})$ $(l\neq k$ if $h_{s,k}=0)$. Namely, taking into consideration that $\alpha_{s,l,1}<\alpha_{s,l,2}<\ldots<\alpha_{s,l,\gamma}$, the following formulas hold:

$$c_s:=\begin{cases}\min\left\{-\xi_s(\alpha_{s,k,\gamma}),\xi_s(\alpha_{s,k+1,1})\right\} & \big(s\in(s_k,s_{k+1})\big),\\ \min\left\{-\xi_s(\alpha_{s,k-1,\gamma}),\xi_s(\alpha_{s,k+1,1})\right\} & (s=s_k).\end{cases}$$

This is why all functions $\ddot{B}(d,s,s+\alpha_{s,l,j})f$, $l\in\mathbb{Z}$, $j=\overline{1,\gamma}$, and as a consequence, the function, $\tilde{G}_{1-2}(\tilde{x}_d(s))f$ are continuous in $d\in\overline{I_D}$. It was shown in Theorem 4.3 that the function $\tilde{G}_3(\tilde{x}_d(s))f$ is also continuous on the set $\overline{I_D}$. The proof is complete. □

Under the assumption $\partial\Omega\in C^{n+1}$, for each closed domain $\Omega'\subset\Omega_D$ we can define the constants

$$\tilde{c}_{1-2,j}:=\sup_{x\in\Omega',\,s'\in I_S}\left|\partial^j_{s'}g(x,s')\right|,\qquad j=\overline{0,n}$$

and to make sure that as $x\in\Omega'$, there hold the estimate

$$\left\|\tilde{G}(x)\right\|\leqslant 2Sc_{\Lambda,0}\left(\tilde{c}_{1-2,0}+\tilde{c}_{3,0}\right)$$

and

$$\left|\tilde{G}(x)\mathbf{P}_Lf-\ddot{G}(x)\mathbf{P}_Lf\right|\leqslant\left(c_{1-2}+c_3\right)\,\|f\|_{C^2(\partial\Omega)}\,h^{2\gamma'}\quad(\partial\Omega\in C^{2\gamma'+1}),$$

where

$$c_{1-2}:=2S\left(\gamma'!\right)^4/\left[(2\gamma')!\right]^3/\left(2\gamma'+1\right)\left[\tilde{c}_{1-2,2\gamma'}c_{\Lambda,0}+2\gamma'\tilde{c}_{1-2,2\gamma'-1}c'_{\Lambda,1}+\gamma'(2\gamma'-1)\tilde{c}_{1-2,2\gamma'-2}c'_{\Lambda,2}\right].$$

Similarly to Theorems 4.1, 4.2 and Corollary 4.1, we obtain the following statement.

Theorem 6.2. *Let $\partial\Omega\in C^{2\gamma+1}\cap C^{2\gamma'+1}$, $\gamma,\gamma'\geqslant 2$, γ, γ', $L/2\in\mathbb{N}$ and Ω' be a closed subset of the domain Ω_D. Then the functionals $\tilde{G}(x)$ $[H_L\to\mathbb{C}]$, $\tilde{G}(x)\mathbf{P}_L$ $[C(\partial\Omega)\to\mathbb{C}]$ are equibounded uniformly in $x\in\Omega'$. As $L\to\infty$, the functionals $\tilde{G}(x)\mathbf{P}_L$ $[C^3(\partial\Omega)\to\mathbb{C}]$ converge in the uniform operator topology to the corresponding functionals $G(x)$ $[C^3(\partial\Omega)\to\mathbb{C}]$ uniformly in $x\in\Omega'$ with the approximation order $O(L^{-3})$. As $L\to\infty$, the functionals $\tilde{G}(x)\mathbf{P}_L$ $[C(\partial\Omega)\to\mathbb{C}]$ converge in the strong operator topology to the corresponding functionals $G(x)$ $[C(\partial\Omega)\to\mathbb{C}]$ uniformly in $x\in\Omega'$.*

The properties of approximations $\tilde{G}(x)$ on the boundary $\partial\Omega$ were studied in the following theorem.

Theorem 6.3. *Let $\partial\Omega \in C^{2\gamma+1}$, $\gamma \geqslant 2$, γ, γ', $L/2 \in \mathbb{N}$. Then the functionals $\tilde{G}(x)$ $[H_L \to \mathbb{C}]$, $\tilde{G}(x)\mathbf{P}_L$ $[C(\partial\Omega) \to \mathbb{C}]$ are equibounded uniformly in $x \in \partial\Omega$. As $L \to \infty$, the functionals $\tilde{G}(x)\mathbf{P}_L$ $[C^3(\partial\Omega) \to \mathbb{C}]$ converge in the uniform operator topology to corresponding functionals $G(x)$ $[C^3(\partial\Omega) \to \mathbb{C}]$ uniformly in $x \in \partial\Omega$ with the approximation order $O(L^{-2}\ln L^{-1})$. As $L \to \infty$, the functionals $\tilde{G}(x)\mathbf{P}_L$ $[C(\partial\Omega) \to \mathbb{C}]$ converge to the corresponding functionals $G(x)$ $[C(\partial\Omega) \to \mathbb{C}]$ in the strong operator topology uniformly in $x \in \partial\Omega$.*

Proof. We have the identity $\tilde{g} = -\tilde{a}_0\tilde{b}$, where

$$\tilde{g}(s,\sigma) := \hat{g}(s, s+\sigma), \qquad \tilde{a}_0(s,\sigma) := a_0(\tilde{\rho}_0^2),$$
$$\tilde{\rho}_0(s,\sigma) := \rho_0(s, s+\sigma), \qquad \tilde{b}(s,\sigma) := \hat{b}(s, s+\sigma).$$

By Corollary 2.1, the derivatives $\partial_\sigma^j\tilde{b}$, $\partial_\sigma^j\tilde{\rho}_0$, $j = \overline{0,2}$, are continuous on the set $\overline{I_S} \times \overline{I_S}$. In view of formula (2.5), the functions $\tilde{a}_0$, $\partial_\sigma\tilde{a}_0$ are continuous on the set $\overline{I_S} \times \overline{I_S}$, while the function $\partial_\sigma^2\tilde{a}_0$ can be represented as the sum $\tilde{a}_1 \ln\sigma^2 + \tilde{a}_2$, where the functions $\tilde{a}_i(s,\sigma)$, $i = 1, 2$, are continuous on $\overline{I_S} \times \overline{I_S}$. Therefore, the functions $\tilde{g}_0 := \tilde{g}$, $\tilde{g}_1 := \partial_\sigma\tilde{g}_0$ are continuous on the set $\overline{I_S} \times \overline{I_S}$, while the function $\partial_\sigma^2\tilde{g}$ can be represented as the sum $\tilde{g}_2 \ln\sigma^2 + \tilde{g}_3$, where the functions $\tilde{g}_j(s,\sigma)$, $j = 2, 3$, are continuous on $\overline{I_S} \times \overline{I_S}$. We define the constants: $\tilde{c}_{1,j} := \sup\limits_{(s,\sigma)\in\overline{I_S}\times\overline{I_S}} |\tilde{g}_j|$, $j = \overline{0,3}$. By the estimate $\left\|\tilde{G}(\tilde{x}(s))\right\| \leqslant 2Sc_{\Lambda,0}\left(\tilde{c}_{1,0} + \tilde{c}_{3,0}\right)$, which is true for each $s \in I_S$, and by the inequality $\|\mathbf{P}_L\| \leqslant 1$ the functionals $\tilde{G}(x)$ $[H_L \to \mathbb{C}]$, $\tilde{G}(x)\mathbf{P}_L$ $[C(\partial\Omega) \to \mathbb{C}]$ are bounded uniformly in $x \in \partial\Omega$.

Let $x = \tilde{x}(s)$ $(s \in I_S)$, $f \in C^3(\partial\Omega)$, $\ddot{f} := \ddot{\mathbf{P}}_L\mathbf{P}_L f$, $h < \min\{S^{-1}, e^{-1}\}$. Employing the Taylor formula with a remainder in the integral form [27, Sect. 318] and the formulas [26, Ch. 3, Sect. 5, Subsect. 1]

$$\sum_{j=1}^{\gamma'} \omega_j = 2, \qquad \sum_{j=1}^{\gamma'} \omega_j(\alpha_{s,l,j} - \bar{\alpha}_{s,l}) = 0, \qquad \omega_j > 0$$

and

$$\bigcup_{l\in\mathbb{Z}} [\alpha_{s,l}, \alpha_{s,l+1}] = \Xi_s, \quad |\Xi_s| \leqslant 2S,$$

we obtain the estimates

$$\begin{aligned}
&\left|\tilde{G}_{1-2}(x)\mathbf{P}_L f - \ddot{G}_{1-2}(x)\mathbf{P}_L f\right| \\
&= \left|\sum_{l=-3L-2}^{3L+2}\left(\int\limits_{\alpha_{s,l}}^{\alpha_{s,l+1}} \tilde{g}(s,\sigma)\ddot{f}(s+\sigma)\,d\sigma \right.\right. \\
&\qquad \left.\left. -h_{s,l}\sum_{j=1}^{\gamma'}\omega_j\,\tilde{g}(s,\alpha_{s,l,j})\,\ddot{f}\,(s+\alpha_{s,l,j})\right)\right| \\
&= \left|\sum_{l=-3L-2}^{3L+2}\left[\int\limits_{\alpha_{s,l}}^{\alpha_{s,l+1}}\int\limits_{\bar{\alpha}_{s,l}}^{\sigma}\partial_\zeta^2\left(\tilde{g}(s,\zeta)\ddot{f}(s+\zeta)\right)(\sigma-\zeta)\,d\zeta\,d\sigma\right.\right. \\
&\qquad \left.\left. -h_{s,l}\sum_{j=1}^{\gamma'}\omega_j\int\limits_{\bar{\alpha}_{s,l}}^{\alpha_{s,l,j}}\partial_\zeta^2\left(\tilde{g}(s,\zeta)\ddot{f}(s+\zeta)\right)(\alpha_{s,l,j}-\zeta)\,d\zeta\right]\right| \\
&\leqslant 2S\left(c_1h^2\ln h^{-2}+c_2h^2\right)\|f\|_{C^2(\partial\Omega)}\,.
\end{aligned} \tag{6.1}$$

Here

$$c_1 := 2\tilde{c}_{1,2}c_{\Lambda,0}, \qquad c_2 := \big((-4\ln 2+4)\tilde{c}_{1,2}+\tilde{c}_{1,3}\big)c_{\Lambda,0}+2\tilde{c}_{1,1}c'_{\Lambda,1}+\tilde{c}_{1,0}c'_{\Lambda,2}.$$

While obtaining inequalities (6.1), we have employed the following estimates:

$$\left|\int\limits_{\bar{\alpha}_{s,l}}^{\sigma}\ln\zeta^{-2}\,d\zeta\right| \leqslant \max\left\{\int\limits_0^{2h}\ln\zeta^{-2}\,d\zeta,\ 2^{-1}h\sup_{\zeta\in[h\,,\,S]}\left|\ln\zeta^{-2}\right|\right\} = 2h\ln(2h)^{-2}+4h,$$

where $\sigma\in[\alpha_{s,l},\alpha_{s,l+1}]$. Together with estimates (4.14), (4.10), estimates (6.1) prove the uniform in $x\in\partial\Omega$ convergence of the functionals $\tilde{G}(x)\mathbf{P}_L$ $[C^3(\partial\Omega)\to\mathbb{C}]$ in the uniform operator topology with the approximation order $O(h^2\ln h)$. Taking into consideration the equiboundedness of the set of functionals $\tilde{G}(x)\mathbf{P}_L$ $[C(\partial\Omega)\to\mathbb{C}]$, $x\in\partial\Omega$, estimates (4.8), (6.1) and the density of the set $C^3(\partial\Omega)$ in the space $C(\partial\Omega)$ we also obtain uniform in $x\in\partial\Omega$ strong operator convergence of functionals $\tilde{G}(x)\mathbf{P}_L$ $[C(\partial\Omega)\to\mathbb{C}]$. The proof is complete. □

Corollary 6.1. *Let $\partial\Omega\in C^{2\gamma'+1}\cap C^{2\gamma+1}$, $\gamma,\gamma'\geqslant 2$, $\gamma,\gamma',L/2\in\mathbb{N}$. Then as $L\to\infty$, for fixed $s\in I_S$, $d_0\in(0,D]$ there is no uniform in $|d|\in(0,d_0]$ convergence of the functionals $\tilde{G}(\tilde{x}_d(s))\mathbf{P}_L$ $[C(\partial\Omega)\to\mathbb{C}]$ to the corresponding functionals $G(\tilde{x}_d(s))$ $[C(\partial\Omega)\to\mathbb{C}]$ in the strong operator topology.*

Proof. Let $f\in C(\partial\Omega)$ and $f(s)\neq 0$ for fixed $s\in I_S$. Then the function $j(d) := G\left(\tilde{x}_d(s)\right)f$ is continuous as $d\in I_D$ and by limiting identities (4.3), it has a discontinuity at the point $d=0$: $\lim\limits_{d\to\pm 0} j(d)-j(0) = \pm 2^{-1}f(s)\neq 0$. In accordance with Theorem 6.1, the function $\tilde{j}(d) := \tilde{G}\left(\tilde{x}_d(s)\right)\mathbf{P}_L f$ is continuous on the set $\overline{I_D}$ and in accordance with Theorems 6.2, 6.3 we have a pointwise convergence of the approximations $\tilde{j}(d)$ to the exact function: $\tilde{j}(d)\to j(d)$ as $L\to\infty$ for each fixed $d\in\overline{I_D}$. This is why for each $d_0\in(0,D]$ a uniform in $|d|\in(0,d_0]$ convergence of the approximations $\tilde{j}(d)$ to the exact function $j(d)$ is impossible. The proof is complete. □

We define functionals $\tilde{R}_\pm(x) := \left(\tilde{G}_{1-2}(x)+\tilde{G}_3(x)\right)\hat{\mathbf{G}}_\pm^{-1}$ and functions $\tilde{u}_\pm(x) := \tilde{R}_\pm(x)w$ $(x\in\overline{\Omega_\pm})$, which for $x\in\Omega_\pm$ are approximations for the resolvent functionals $R_\pm(x)$ and solutions $u_\pm(x)$ of boundary value problem (2.1), while as $x = \tilde{x}(s)$, $s \in I_S$, these are approximations of direct values of the functionals $R_\pm(x)$ and the functions $u_\pm(x)$ on the boundary $\partial\Omega$. By Theorems 6.2, 6.3 similarly to Corollary 5.3 we obtain the following statement.

Corollary 6.2. *Let $\partial\Omega \in C^{2\gamma'+1} \cap C^{2\gamma+1}$, $\gamma, \gamma' \geqslant 2$, $\gamma, \gamma', L/2 \in \mathbb{N}$, $c > 0$, and Ω' be a closed subset of the domain Ω_D, $d_0 \in (0, D]$. Then, as $L \to \infty$, the functions $\tilde{u}_\pm(x)$ converge to the corresponding solutions $u_\pm(x)$ of boundary value problem (2.1) with a cubic rate uniformly in $x \in \Omega'$ and boundary functions $w \in C^3(\partial\Omega)$ satisfying the condition $\|w\|_{C^3(\partial\Omega)} \leqslant c$. As $L \to \infty$, the functions $\tilde{u}_\pm(\tilde{x}(s))$ converge to the corresponding functions $u_\pm(\tilde{x}(s))$ with the approximation order $O(L^{-2} \ln L^{-1})$ uniformly in $s \in I_S$ and boundary functions $w \in C^3(\partial\Omega)$ obeying the condition $\|w\|_{C^3(\partial\Omega)} \leqslant c$. Moreover, as $L \to \infty$, $\varepsilon \to +0$, the functions $\tilde{R}_\pm(x)\mathbf{P}_L w_\varepsilon$ converge to the function $u_\pm(x)$ uniformly in $x \in \Omega'$ and $x \in \partial\Omega$ if $\|w_\varepsilon - w\|_{C(\partial\Omega)} \leqslant \varepsilon$, $w, w_\varepsilon \in C(\partial\Omega)$.*

On the base of Corollary 6.2, the limiting identities $\lim\limits_{d\to\pm 0} u_\pm(\tilde{x}_d(s)) - u_\pm(\tilde{x}(s)) = \pm 2^{-1} v_\pm(s)$ and the continuity in $d \in \overline{I_D}$ of the functions $\tilde{u}_\pm(\tilde{x}_d(s))$ for fixed $s \in I_S$, similarly to Corollary 6.1 we obtain a statement on the absence of the uniform convergence of approximate solutions $\tilde{u}_\pm(x)$ in the vicinity of the boundary $\partial\Omega$.

Corollary 6.3. *Let $\partial\Omega \in C^{2\gamma'+1} \cap C^{2\gamma+1}$, $\gamma', \gamma \geqslant 2$, $\gamma, \gamma', L/2 \in \mathbb{N}$, $d_0 \in (0, D]$, $w \in C(\partial\Omega)$, and also $v_\pm(s) \neq 0$ for some $s \in I_S$. Then as $L \to \infty$ there is no uniform in $\pm d \in (0, d_0]$ convergence of the functions $\tilde{u}_\pm(\tilde{x}_d(s))$ to the solution $u_\pm(\tilde{x}_d(s))$ of boundary value problem (2.1).*

7. Numerical experiments

Here we consider a numerical solution of problem (2.1) in the exterior of a unit circle with the boundary function $w = \cos\varphi$ and the dissipation coefficient $k = \pi$. The exact solution $\bar{u}$ of such a problem is calculated by the formula: $\bar{u} = \cos\varphi K_1(\pi r') \,/\, K_1(\pi)$. Here $\varphi = s \in [-\pi, \pi)$, $r' = 1 - d > 1$, are the polar angle and radius with the pole at the center of the circle, $K_1(z)$ is the MacDonald function. For this geometry, a third of the radius of the Lyapunov circle is $D = 2/(3\pi)$; half-widths of arc lengths over which exact integration over the ρ variable is made are $\Sigma''_s = -\Sigma'_s = \arcsin(2\,/\,(3\pi))$. When calculating semi-analytic solutions $\hat{u}$ (see Corollary 5.3), the integrals in ρ in expressions for $\hat{G}_i(x)$, $i = 1, 2$, are calculated using the Newton-Leibniz formula. In order to do this, the series in formula (2.3) are replaced by finite sums formed by powers of z^{2k}, $k = \overline{0, 10}$. In the expressions for $\tilde{G}_3(x)$ we use SGQFs with $\gamma = 2$ nodes. Approximate solutions $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$ based on SGQF (see Corollary 6.2) are calculated with the values $\gamma' = 12$, 24, 48, respectively, and $\gamma = 2$. All calculations are made with a double precision. The solutions $\hat{u}$, $\tilde{u}_i$, $i = \overline{1, 3}$, $\bar{u}$ are found for fixed $d \in [-D, 0)$ at the points $\tilde{x}_d(s_{l/4})$, $s_{l/4} = lh/4$, $l = \overline{-4L - 4, 4L + 3}$, so these solutions can be considered as functions in the space H_{4L+3}. For a fixed d, we calculate the maximums of the absolute values of the errors of the approximate solutions $\hat{u}$, $\tilde{u}_i$: $\triangle\hat{u} := \|\hat{u} - \bar{u}\|_{H_{4L+3}}$, $\triangle\tilde{u}_i := \|\tilde{u}_i - \bar{u}\,|_{H_{4L+3}}$. In Table 1, in each main cell, we show the values of $\triangle\hat{u}$, $\triangle\tilde{u}_1$, $\triangle\tilde{u}_2$, $\triangle\tilde{u}_3$ in the appropriate order from the top to bottom.

We observe that the solution $\hat{u}$ has a cubic convergence rate, which is preserved even at very small distances $|d|$ to the $\partial\Omega$ boundary, which is in good agreement with Corollary 5.3. The convergence rates of solutions $\tilde{u}_i$ decrease from cubic to zero as points x approach the boundary $\partial\Omega$ at fixed sampling steps h and are restored to cubic as h decreases for fixed d, which is consistent with Corollaries 6.2 and 6.3.

We made calculations, in which in order to approximate the functionals $\ddot{G}_{1-2}(x)$, instead of SGQF, closed-type Newton-Cotes quadrature formulas were used [26, Ch. 3, Sect. 4, Item 1]. Then, as the points x approach the nodes of quadrature approximations, namely: as the points $\tilde{x}_d(s_l)$ approach the points $\tilde{x}(s_l)$ ($l = \overline{-L - 1, L}$) as $d \to 0$ for a fixed L, the errors $\triangle\tilde{u}_i$ grow catastrophically.

In conclusion we note that in a similar way, using exact integration in the variable ρ, approximate solutions of the internal Dirichlet problem for the Laplace equation can be obtained, since this problem, like the Dirichlet problem for the dissipative Helmholtz equation, has a unique solution. The corresponding approximations of the DLP were studied in work by the author [19].

d	$h_1=\pi/3$	$h_2=\pi/7$	$h_3=\pi/15$	$h_4=\pi/31$	$h_5=\pi/63$
-10^{-2}	$6.60\cdot10^{-2}$	$5.87\cdot10^{-3}$	$5.72\cdot10^{-4}$	$5.60\cdot10^{-5}$	$4.90\cdot10^{-6}$
	$6.59\cdot10^{-2}$	$6.61\cdot10^{-3}$	$5.72\cdot10^{-4}$	$5.60\cdot10^{-5}$	$4.90\cdot10^{-6}$
	$6.60\cdot10^{-2}$	$5.87\cdot10^{-3}$	$5.72\cdot10^{-4}$	$5.60\cdot10^{-5}$	$4.90\cdot10^{-6}$
	$6.60\cdot10^{-2}$	$5.87\cdot10^{-3}$	$5.72\cdot10^{-4}$	$5.60\cdot10^{-5}$	$4.90\cdot10^{-6}$
-10^{-3}	$6.88\cdot10^{-2}$	$6.27\cdot10^{-3}$	$6.52\cdot10^{-4}$	$7.31\cdot10^{-5}$	$8.46\cdot10^{-6}$
	$3.03\cdot10^{-1}$	$2.71\cdot10^{-1}$	$3.95\cdot10^{-2}$	$2.99\cdot10^{-2}$	$3.65\cdot10^{-3}$
	$7.12\cdot10^{-2}$	$3.82\cdot10^{-2}$	$6.35\cdot10^{-3}$	$7.88\cdot10^{-4}$	$2.59\cdot10^{-5}$
	$6.88\cdot10^{-2}$	$6.27\cdot10^{-3}$	$6.52\cdot10^{-4}$	$7.31\cdot10^{-5}$	$8.46\cdot10^{-6}$
-10^{-4}	$6.90\cdot10^{-2}$	$6.31\cdot10^{-3}$	$6.60\cdot10^{-4}$	$7.51\cdot10^{-5}$	$8.93\cdot10^{-6}$
	$1.05\cdot10^{0}$	$1.03\cdot10^{0}$	$9.36\cdot10^{-1}$	$7.20\cdot10^{-1}$	$3.64\cdot10^{-1}$
	$7.58\cdot10^{-1}$	$7.40\cdot10^{-1}$	$4.13\cdot10^{-1}$	$9.02\cdot10^{-2}$	$5.09\cdot10^{-2}$
	$1.13\cdot10^{-1}$	$6.72\cdot10^{-2}$	$5.45\cdot10^{-2}$	$1.47\cdot10^{-2}$	$1.29\cdot10^{-3}$
-10^{-5}	$6.91\cdot10^{-2}$	$6.32\cdot10^{-3}$	$6.61\cdot10^{-4}$	$7.53\cdot10^{-5}$	$8.98\cdot10^{-6}$
	$1.11\cdot10^{0}$	$1.14\cdot10^{0}$	$1.13\cdot10^{0}$	$1.11\cdot10^{0}$	$1.07\cdot10^{0}$
	$1.12\cdot10^{0}$	$1.09\cdot10^{0}$	$1.07\cdot10^{0}$	$9.84\cdot10^{-1}$	$8.12\cdot10^{-1}$
	$9.98\cdot10^{-1}$	$9.73\cdot10^{-1}$	$8.30\cdot10^{-1}$	$5.30\cdot10^{-1}$	$1.53\cdot10^{-1}$
-10^{-15}	$6.91\cdot10^{-2}$	$6.32\cdot10^{-3}$	$6.61\cdot10^{-4}$	$7.53\cdot10^{-5}$	$8.98\cdot10^{-6}$
	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$
	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$
	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$	$1.16\cdot10^{0}$

Table 1

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