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ON LINEAR-AUTONOMOUS SYMMETRIES OF GUÉANT–PU FRACTIONAL MODEL

Kh.V. YADRIKHINSKIY, V.E. FEDOROV

Abstract. We study the group properties of the Guéant-Pu model with a fractional order in time, which describes the dynamics of option pricing. We find the groups of linear-autonomous equivalence transformations of the corresponding equation. With their help, we obtain a group classification of the fractional Guéant-Pu model with a nonlinear free element. In the case of a non-zero risk-free interest rate r, the underlying Lie algebra of such a model is one-dimensional. For zero r, the main Lie algebra is three-dimensional in the case of a special right-hand side and it is two-dimensional otherwise.

Keywords: Riemann-Liouville fractional derivative, fractional Guéant-Pu model, symmetry analysis, linear-autonomous transformation, group of equivalence transformations, group classification.

Mathematics Subject Classification: 35R11, 26A33, 58J70

1. Introduction

More and more new nonlinear modifications of the Black-Scholes equation describing the dynamics of option pricing [1], [2], taking into consideration various properties of the real market that were idealized when deriving the linear equation, such as market illiquidity, hedging costs, and the impact of transactions on price formation, etc., are proposed by researchers over the last half century [1]–[10]. One of the nonlinear Black-Scholes models is the Guéant-Pu equation

$$\theta_t = r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F(t, \theta_q), \tag{1.1}$$

which models option pricing taking into consideration transaction costs and the impact of transactions on market under a number of assumptions [11], [12]. Here r is a constant risk-free rate; γ is an absolute risk aversion parameter; σ is a volatility; q is the number of shares in the hedged portfolio; S is a share price; μ is a trend forecast, an expected profit of the underlying asset; the function $\theta(t, S, q)$ models the price of indifference call option.

In [13]–[16], equation (1.1) was studied by methods of group analysis [17], [18] under various conditions for the function F of two variables. The present work is devoted to studying the symmetries for a fractional version of model (1.1)

$$D_t^\alpha\theta = r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F(t, \theta_q), \tag{1.2}$$

where D_t^α is the operator of Riemann-Liouville fractional derivative of order $\alpha \in (0, 1]$. As it is known, the fractional derivatives model processes with a memory [19], [20]. Fractional derivatives, as they are often called, were introduced into option pricing theory to take an

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advantage of their memory properties, allowing one to capture both large moves over short periods of time and long-term dependencies in markets [21]–[24].

In the present work, by the methods proposed in works by R.K. Gazizov, A.A. Kasatkin and S.Yu. Lukashchuk [25]–[28], we obtain the generators of the groups of linearly-autonomous transformations admitted by equations (1.2). As the same time, by Theorem 2.7 in work [29], equations resolved with respect to the fractional Riemann-Liouville derivative in a selected variable and involving the derivatives of only integer order in other variables, possess no other admitted groups. We obtain a group classification up to linearly-autonomous equivalence transformations of equation (1.2) with a nonlinear in θ_q free element F.

2. Preliminaries

A fractional Riemann-Liouville integral of order $\beta > 0$ and a fractional Riemann-Liouville derivative of order $\alpha \in (n-1, n]$ read as [20]

$$J_t^\beta \theta(t) := \frac{1}{\Gamma(\beta)} \int\limits_0^t (t-s)^{\beta-1}\theta(s)ds, \quad D_t^\alpha \theta := D_t^n J_t^{n-\alpha}\theta(t),$$

and we also assume that $J_t^0\theta(t) := \theta(t)$. Here D_t^n is the operator of differentiation of an integer order $n \in \mathbb{N}$. We also recall the definition of the Mittag-Leffler function [20]:

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha, \beta \in \mathbb{R}_+, \quad z \in \mathbb{C}.$$

The relations are known [20]

$$\begin{aligned} &D_t^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \qquad D_t^\alpha t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\alpha)} t^{\nu-\alpha}, \\ &D_t^\alpha t^{\alpha-k} = 0, \qquad k \in \mathbb{N}, \qquad k < \alpha+1, \end{aligned} \tag{2.1}$$

and a formula for the general solution

$$y = \sum_{j=1}^n b_j t^{\alpha-j} E_{\alpha,\alpha-j+1}(\lambda t^\alpha) + \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) f(s)\, ds, \tag{2.2}$$

of the fractional differential equations $D_t^\alpha y(t) - \lambda y(t) = f(t)$.

3. Equivalence transformations groups for Guéant-Pu equation

We consider equation (1.2), where $0 < \alpha < 1$, $\theta = \theta(t,S,q)$, $\gamma\sigma \neq 0$. In order to find the equivalence transformations groups, we consider the function F and all its derivatives as variables. We seek the generators of the equivalence transformations groups as

$$Y = \tau\partial_t + \xi\partial_S + \beta\partial_q + \eta\partial_\theta + \zeta\partial_F,$$

where τ, ξ, β, η depend on t, S, q, θ, while ζ does on t, S, q, θ, F, θ_t, θ_S, θ_q, $D_t^\alpha\theta$. Here $\partial_u := \frac{\partial}{\partial u}$ is the operator of the partial differentiation in the variable u.

Following [27], [28], we seek the operator Y in a linearly-autonomous form:

$$\xi_\theta = 0, \qquad \tau_\theta = 0, \qquad \beta_\theta = 0, \qquad \eta = p(t,S,q)\theta + g(t,S,q)$$

with the condition $\tau(0) = 0$. In order to take into consideration the dependence of F only on t and θ_q we add the equations

$$F_S = 0, \qquad F_q = 0, \qquad F_\theta = 0, \qquad F_{D_t^\alpha\theta} = 0, \qquad F_{\theta_t} = 0, \qquad F_{\theta_S} = 0. \tag{3.1}$$

We treat system (1.2), (3.1) as a manifold $\mathfrak{M}$ in an extended space of corresponding variables. The extended operator $\widetilde{Y}$ read as

$$\widetilde{Y} = Y + \eta^\alpha \partial_{D_t^\alpha \theta} + \eta^t \partial_{\theta_t} + \eta^S \partial_{\theta_S} + \eta^q \partial_{\theta_q} + \eta^{SS} \partial_{\theta_{SS}} + \zeta^t \partial_{F_t} + \zeta^S \partial_{F_S} + \zeta^q \partial_{F_q} + \zeta^\theta \partial_{F_\theta} + \zeta^{D_t^\alpha \theta} \partial_{F_{D_t^\alpha \theta}} + \zeta^{\theta_S} \partial_{F_{\theta_S}} + \zeta^{\theta_q} \partial_{F_{\theta_q}}.$$

Applying the operator $\widetilde{Y}$ to the both sides of identity (1.2), we obtain:

$$\eta^\alpha - r\eta - (\mu - rS)\beta + rq\xi + \mu\eta^S + \frac{\sigma^2}{2}\eta^{SS} - \frac{r}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2\tau + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q)(\eta^S - \beta) - \zeta|_{\mathfrak{M}} = 0. \tag{3.2}$$

In order to calculate the coefficients of the extended operator $\widetilde{Y}$, we employ the operator of total differentiation

$$D_t = \frac{\partial}{\partial t} + \theta_t \frac{\partial}{\partial \theta} + \ldots, \qquad D_S = \frac{\partial}{\partial S} + \theta_S \frac{\partial}{\partial \theta} + \ldots, \qquad D_q = \frac{\partial}{\partial q} + \theta_q \frac{\partial}{\partial \theta} + \ldots,$$

$$\tilde{D}_t = \frac{\partial}{\partial t} + F_t \frac{\partial}{\partial F} + \ldots, \qquad \tilde{D}_S = \frac{\partial}{\partial S} + F_S \frac{\partial}{\partial F} + \ldots, \qquad \tilde{D}_q = \frac{\partial}{\partial q} + F_q \frac{\partial}{\partial F} + \ldots,$$

$$\tilde{D}_\theta = \frac{\partial}{\partial \theta} + F_\theta \frac{\partial}{\partial F} + \ldots, \qquad \tilde{D}_{D_t^\alpha \theta} = \frac{\partial}{\partial D_t^\alpha \theta} + F_{D_t^\alpha \theta} \frac{\partial}{\partial F} + \ldots,$$

$$\tilde{D}_{\theta_S} = \frac{\partial}{\partial \theta_S} + F_{\theta_S} \frac{\partial}{\partial F} + \ldots, \qquad \tilde{D}_{\theta_q} = \frac{\partial}{\partial \theta_q} + F_{\theta_q} \frac{\partial}{\partial F} + \ldots.$$

With help of them we define continuations in the variables S and q

$$\eta^S = D_S\eta - \theta_t D_S\tau - \theta_S D_S\xi - \theta_q D_S\beta, \qquad \eta^q = D_q\eta - \theta_t D_q\tau - \theta_S D_q\xi - \theta_q D_q\beta,$$

$$\eta^{SS} = D_S\eta^S - \theta_{St} D_S\tau - \theta_{SS} D_S\xi - \theta_{Sq} D_S\beta.$$

By Theorem 2.8 in [27] and in accordance with its generalizations for the case of many variables and by Theorem 3 in [28], we obtain a continuation in the fractional derivative

$$\eta^\alpha = D_t^\alpha(\eta - \tau\theta_t - \xi\theta_S - \beta\theta_q) + \tau D_t^{\alpha+1}\theta + \xi D_t^\alpha\theta_S + \beta D_t^\alpha\theta_q.$$

In order to exclude the derivatives of form θ_t under the fractional differentiations, we use the identity $D_t^\alpha(\tau\theta_t) = D_t^\alpha((\tau\theta)_t - \tau_t\theta)$. Using (1.8) from [27] or differentiating (2.43) in Theorem 2.2 in [20], we obtain the identity

$$D_t^\alpha(\tau\theta)_t = D_t^{\alpha+1}(\tau\theta) - (\tau\theta)(0)\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}.$$

In view of the condition $\tau(0) = 0$ this implies the identity

$$D_t^\alpha(\tau\theta_t) = D_t^{\alpha+1}(\tau\theta) - D_t^\alpha(\tau_t\theta).$$

Then the continuation in the fractional derivative becomes

$$\eta^\alpha = D_t^\alpha(\eta - \xi\theta_S - \beta\theta_q) + \xi D_t^\alpha\theta_S + \beta D_t^\alpha\theta_q + D_t^\alpha(\tau_t\theta) - D_t^{\alpha+1}(\tau\theta) + \tau D_t^{\alpha+1}\theta.$$

By the generalized Leibniz rule for the fractional derivatives we find:

$$\begin{aligned}\eta^\alpha = & D_t^\alpha\eta - \sum_{n=0}^\infty \binom{\alpha}{n} D_t^{\alpha-n}\theta_S D_t^n\xi - \sum_{n=0}^\infty \binom{\alpha}{n} D_t^{\alpha-n}\theta_q D_t^n\beta + \xi D_t^\alpha\theta_S \\ & + \beta D_t^\alpha\theta_q + \sum_{n=0}^\infty \binom{\alpha}{n} D_t^{\alpha-n}\theta D_t^{n+1}\tau - \sum_{n=0}^\infty \binom{\alpha+1}{n} D_t^{\alpha+1-n}\theta D_t^n\tau + \tau D_t^{\alpha+1}\theta \\ = & D_t^\alpha\eta - \sum_{n=0}^\infty \binom{\alpha}{n} D_t^{\alpha-n}\theta_S D_t^n\xi - \sum_{n=0}^\infty \binom{\alpha}{n} D_t^{\alpha-n}\theta_q D_t^n\beta + \xi D_t^\alpha\theta_S\end{aligned}$$

$$+\beta D_t^\alpha \theta_q + \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta D_t^{n+1}\tau - \sum_{n=0}^{\infty}\binom{\alpha+1}{n+1} D_t^{\alpha-n}\theta D_t^{n+1}\tau,$$

where

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)}.$$

Since

$$\binom{\alpha+1}{n+1} = \frac{\alpha+1}{n+1}\binom{\alpha}{n},$$

then

$$\begin{aligned}\eta^\alpha =& D_t^\alpha \eta - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta_S D_t^n \xi - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta_q D_t^n \beta \\ &+ \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{n-\alpha}{n+1} D_t^{\alpha-n}\theta D_t^{n+1}\tau.\end{aligned}$$

Further calculations for linearly-autonomous transformations give

$$\begin{aligned}\eta^\alpha =& D_t^\alpha g + \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta \left(D_t^n p + \frac{n-\alpha}{n+1} D_t^{n+1}\tau\right) \\ &- \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta_S D_t^n \xi - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta_q D_t^n \beta.\end{aligned}$$

The coefficients at the derivatives of F in the extended operator $\widetilde{Y}$ read as

$$\begin{aligned}
\zeta^S =& \tilde{D}_S\zeta - F_t\tilde{D}_S\tau - F_S\tilde{D}_S\xi - F_q\tilde{D}_S\beta - F_\theta\tilde{D}_S\eta \\
&- F_{D_t^\alpha\theta}\tilde{D}_S\eta^\alpha - F_{\theta_t}\tilde{D}_S\eta^t - F_{\theta_S}\tilde{D}_S\eta^S - F_{\theta_q}\tilde{D}_S\eta^q, \\
\zeta^q =& \tilde{D}_q\zeta - F_t\tilde{D}_q\tau - F_S\tilde{D}_q\xi - F_q\tilde{D}_q\beta - F_\theta\tilde{D}_q\eta \\
&- F_{D_t^\alpha\theta}\tilde{D}_q\eta^\alpha - F_{\theta_t}\tilde{D}_q\eta^t - F_{\theta_S}\tilde{D}_q\eta^S - F_{\theta_q}\tilde{D}_q\eta^q, \\
\zeta^\theta =& \tilde{D}_\theta\zeta - F_t\tilde{D}_\theta\tau - F_S\tilde{D}_\theta\xi - F_q\tilde{D}_\theta\beta - F_\theta\tilde{D}_\theta\eta \\
&- F_{D_t^\alpha\theta}\tilde{D}_\theta\eta^\alpha - F_{\theta_t}\tilde{D}_\theta\eta^t - F_{\theta_S}\tilde{D}_\theta\eta^S - F_{\theta_q}\tilde{D}_\theta\eta^q, \\
\zeta^{D_t^\alpha\theta} =& \tilde{D}_{D_t^\alpha\theta}\zeta - F_t\tilde{D}_{D_t^\alpha\theta}\tau - F_S\tilde{D}_{D_t^\alpha\theta}\xi - F_q\tilde{D}_{D_t^\alpha\theta}\beta - F_\theta\tilde{D}_{D_t^\alpha\theta}\eta \\
&- F_{D_t^\alpha\theta}\tilde{D}_{D_t^\alpha\theta}\eta^\alpha - F_{\theta_t}\tilde{D}_{D_t^\alpha\theta}\eta^t - F_{\theta_S}\tilde{D}_{D_t^\alpha\theta}\eta^S - F_{\theta_q}\tilde{D}_{D_t^\alpha\theta}\eta^q, \\
\zeta^{\theta_t} =& \tilde{D}_{\theta_t}\zeta - F_t\tilde{D}_{\theta_t}\tau - F_S\tilde{D}_{\theta_t}\xi - F_q\tilde{D}_{\theta_t}\beta - F_\theta\tilde{D}_{\theta_t}\eta \\
&- F_{D_t^\alpha\theta}\tilde{D}_{\theta_t}\eta^\alpha - F_{\theta_t}\tilde{D}_{\theta_t}\eta^t - F_{\theta_S}\tilde{D}_{\theta_t}\eta^S - F_{\theta_q}\tilde{D}_{\theta_t}\eta^q, \\
\zeta^{\theta_S} =& \tilde{D}_{\theta_S}\zeta - F_t\tilde{D}_{\theta_S}\tau - F_S\tilde{D}_{\theta_S}\xi - F_q\tilde{D}_{\theta_S}\beta - F_\theta\tilde{D}_{\theta_S}\eta \\
&- F_{D_t^\alpha\theta}\tilde{D}_{\theta_S}\eta^\alpha - F_{\theta_t}\tilde{D}_{\theta_S}\eta^t - F_{\theta_S}\tilde{D}_{\theta_S}\eta^S - F_{\theta_q}\tilde{D}_{\theta_S}\eta^q.
\end{aligned}$$

We apply the operator $\widetilde{Y}$ on equations (3.1) and we obtain

$$\zeta^S|_{\mathfrak{M}} = 0, \quad \zeta^q|_{\mathfrak{M}} = 0, \quad \zeta^\theta|_{\mathfrak{M}} = 0, \quad \zeta^{D_t^\alpha\theta}|_{\mathfrak{M}} = 0, \quad \zeta^{\theta_t}|_{\mathfrak{M}} = 0, \quad \zeta^{\theta_S}|_{\mathfrak{M}} = 0.$$

Expanding them and substituting (3.1), we get

$$\begin{aligned}
&\zeta^S|_{\mathfrak{M}} = \zeta_S - F_t\tau_S - F_{\theta_q}\eta^q_S|_{\mathfrak{M}} = 0, && \zeta^q|_{\mathfrak{M}} = \zeta_q - F_t\tau_q - F_{\theta_q}\eta^q_q|_{\mathfrak{M}} = 0, \\
&\zeta^\theta|_{\mathfrak{M}} = \zeta_\theta - F_t\tau_\theta - F_{\theta_q}\eta^q_\theta|_{\mathfrak{M}} = 0, && \zeta^{D_t^\alpha\theta}|_{\mathfrak{M}} = \zeta_{D_t^\alpha\theta}|_{\mathfrak{M}} = 0, \\
&\zeta^{\theta_t}|_{\mathfrak{M}} = \zeta_{\theta_t} - F_{\theta_q}\eta^q_{\theta_t}|_{\mathfrak{M}} = 0, && \zeta^{\theta_S}|_{\mathfrak{M}} = \zeta_{\theta_S} - F_{\theta_q}\eta^q_{\theta_S}|_{\mathfrak{M}} = 0.
\end{aligned}$$

We expand η^q and, passing to the manifold $\mathfrak{M}$, we obtain:

$$\zeta_S - F_t\tau_S - F_{\theta_q}(p_{Sq}\theta + p_S\theta_q + g_{Sq} - \theta_t\tau_{Sq} - \theta_S\xi_{Sq} - \theta_q\beta_{Sq}) = 0,$$
$$\zeta_q - F_t\tau_q - F_{\theta_q}(p_{qq}\theta + p_q\theta_q + g_{qq} - \theta_t\tau_{qq} - \theta_S\xi_{qq} - \theta_q\beta_{qq}) = 0,$$
$$\zeta_\theta - F_{\theta_q}p_q = 0, \quad \zeta_{D_t^\alpha\theta} = 0, \quad \zeta_{\theta_t} + F_{\theta_q}\tau_q = 0, \quad \zeta_{\theta_S} + F_{\theta_q}\xi_q = 0.$$

Separation of the variables gives

$$\begin{gathered}\tau_S = 0, \quad \tau_q = 0, \quad \xi_q = 0, \quad p_q = 0, \quad \zeta_S = 0, \quad \zeta_q = 0, \quad \zeta_\theta = 0, \quad \zeta_{D_t^\alpha\theta} = 0, \\ \zeta_{\theta_t} = 0, \quad \zeta_{\theta_S} = 0, \quad p_S - \beta_{Sq} = 0, \quad \beta_{qq} = 0, \quad g_{Sq} = 0, \quad g_{qq} = 0.\end{gathered} \tag{3.3}$$

Substituting into (3.2) for the formulas for the coefficients of the extended operator $\widetilde{Y}$ and the equation $\tau_S = 0$ from (3.3), we obtain:

$$\begin{aligned}&D_t^\alpha g + \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta\left(D_t^n p + \frac{n-\alpha}{n+1}D_t^{n+1}\tau\right)\\ &\quad - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta_S D_t^n\xi - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta_q D_t^n\beta - rp\theta - rg - (\mu - rS)\beta\\ &\quad + rq\xi + \mu(p_S\theta + p\theta_S + g_S - \theta_S\xi_S - \theta_q\beta_S) - \frac{r}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2\tau\\ &\quad + \frac{\sigma^2}{2}(p_{SS}\theta + g_{SS} + 2p_S\theta_S - \theta_S\xi_{SS} - \theta_q\beta_{SS} - 2\theta_{Sq}\beta_S + \theta_{SS}(p - 2\xi_S))\\ &\quad + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q)(p_S\theta + p\theta_S + g_S - \theta_S\xi_S - \theta_q\beta_S - \beta) - \zeta|_{\mathfrak{M}} = 0.\end{aligned} \tag{3.4}$$

Passing to the manifold $\mathfrak{M}$ in equation (3.4) by means of the expression for $D_t^\alpha\theta$ in (1.2), we obtain

$$\begin{aligned}&(p - \alpha\tau_t)\left(r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F\right)\\ &\quad + D_t^\alpha g + \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta\left(D_t^n p + \frac{n-\alpha}{n+1}D_t^{n+1}\tau\right)\\ &\quad - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta_S D_t^n\xi - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\theta_q D_t^n\beta\\ &\quad - rp\theta - rg - (\mu - rS)\beta + rq\xi\\ &\quad + \mu(p_S\theta + g_S + p\theta_S - \theta_S\xi_S - \theta_q\beta_S) - \frac{r\tau}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2\\ &\quad + \frac{\sigma^2}{2}\left(p_{SS}\theta + g_{SS} + 2p_S\theta_S - \theta_S\xi_{SS} - \theta_q\beta_{SS} - 2\theta_{Sq}\beta_S + \theta_{SS}(p - 2\xi_S)\right)\\ &\quad + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q)(p_S\theta + g_S + p\theta_S - \theta_S\xi_S - \theta_q\beta_S - \beta) - \zeta = 0.\end{aligned}$$

Separation of variables gives:

$$D_t^{\alpha-n}\theta: \qquad D_t^n p + \frac{n-\alpha}{n+1}D_t^{n+1}\tau = 0, \quad n = 1, 2, \ldots, \tag{3.5}$$

$$D_t^{\alpha-n}\theta_S: \qquad D_t^n\xi = 0, \qquad D_t^{\alpha-n}\theta_q : D_t^n\beta = 0, \quad n = 1, 2, \ldots, \tag{3.6}$$

$$\theta_{SS}: \qquad \frac{\sigma^2}{2}(-(p - \alpha\tau_t) + p - 2\xi_S) = 0, \tag{3.7}$$

$$\theta_{Sq}: \qquad \beta_S = 0, \quad p_S = 0, \tag{3.8}$$

$$\theta_S^2: \qquad -\frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(p - \alpha\tau_t) - \frac{r\tau}{2}\gamma\sigma^2 e^{r(T-t)} + \gamma\sigma^2 e^{r(T-t)}(p - \xi_S) = 0, \tag{3.9}$$

$$\theta_S: \quad (p-\alpha\tau_t)(-\mu+q\gamma\sigma^2e^{r(T-t)})+\mu(p-\xi_S)-\frac{\sigma^2}{2}\xi_{SS} \\ +r\tau q\gamma\sigma^2e^{r(T-t)}+\gamma\sigma^2e^{r(T-t)}(g_S-\beta-q(p-\xi_S))=0, \tag{3.10}$$

$$1: \quad (p-\alpha\tau_t)\left(r\theta+(\mu-rS)q-\frac{q^2}{2}\gamma\sigma^2e^{r(T-t)}+F\right) \\ +D_t^\alpha g-rp\theta-rg-(\mu-rS)\beta+rq\xi+\mu g_S+\frac{\sigma^2}{2}g_{SS} \\ -\frac{rq^2}{2}\gamma\sigma^2e^{r(T-t)}\tau-q\gamma\sigma^2e^{r(T-t)}(g_S-\beta)-\zeta=0. \tag{3.11}$$

By (3.6) we find that $\xi_t=0$, $\beta_t=0$. Identity (3.7) implies one more identity $\alpha\tau_t-2\xi_S=0$ and hence, $\tau_{tt}=0$, $\xi_{SS}=0$. Then by (3.5) with $n=1$ we obtain $p_t=0$. Taking into consideration that $p_q=0$ by (3.3) and $p_S=0$ by (3.8), we have a constant $p=p_0$.

For the functions τ, ξ, β we have $\tau_{tt}=0$, $\tau_q=0$ and $\tau_S=0$ by (3.3); $\xi_t=0$ and $\xi_{SS}=0$, $\xi_q=0$ is implied by (3.3); $\beta_t=0$, $\beta_S=0$ is yielded by (3.8) and $\beta_{qq}=0$ follows from (3.3). In view of the identity $\alpha\tau_t-2\xi_S=0$, by (3.7) and the condition $\tau(0)=0$, an integration gives:

$$p=p_0,\quad \tau=Mt,\quad \xi=\frac{\alpha M}{2}S+A,\quad \beta=Bq+K, \tag{3.12}$$

where M, A, B, K are various arbitrary constants. The substitution of (3.12) into (3.9), (3.10) followed by a cancellation gives the identities

$$\theta_S^2: \quad -rMt+p_0=0, \tag{3.13}$$

$$\theta_S: \quad \mu\frac{\alpha M}{2}+r\gamma\sigma^2e^{r(T-t)}Mtq+\gamma\sigma^2e^{r(T-t)}\left(g_S-Bq-K-q\frac{\alpha M}{2}\right)=0. \tag{3.14}$$

By (3.13) we obtain $rM=0$, $p_0=0$. Since $g_{Sq}=0$ in (3.3), by differentiating (3.14) with respect to q we obtain $B=rMt-\alpha M/2=-\alpha M/2$. Therefore, it follows from (3.13), (3.14) that

$$rM=0,\quad p_0=0,\quad B=-\frac{\alpha M}{2},\quad g_S=K-\mu\frac{\alpha Me^{r(t-T)}}{2\gamma\sigma^2}. \tag{3.15}$$

Since $g_{qq}=0$ by (3.3), the integration of g_S gives

$$g=S\left(K-\mu\frac{\alpha Me^{r(t-T)}}{2\gamma\sigma^2}\right)+N(t)q+V(t), \tag{3.16}$$

where $N(t)$, $V(t)$ are arbitrary functions. We substitute identities (3.12), (3.15), (3.16) into (3.11), then

$$-\alpha M\left((\mu-rS)q-\frac{q^2}{2}\gamma\sigma^2e^{r(T-t)}+F\right)+SD_t^\alpha\left(K-\mu\frac{\alpha Me^{r(t-T)}}{2\gamma\sigma^2}\right) \\ +qD_t^\alpha N(t)+D_t^\alpha V(t)-rS\left(K-\mu\frac{\alpha Me^{r(t-T)}}{2\gamma\sigma^2}\right) \\ -rN(t)q-rV(t)-(\mu-rS)\left(-\frac{\alpha M}{2}q+K\right)+rqA \\ +\mu K-\mu^2\frac{\alpha Me^{r(t-T)}}{2\gamma\sigma^2}+q\mu\frac{\alpha M}{2}-\gamma\sigma^2e^{r(T-t)}\frac{\alpha M}{2}q^2-\zeta=0. \tag{3.17}$$

In view of the identities $\zeta_S = 0$, $\zeta_q = 0$, $\zeta_\theta = 0$ from (3.3), we consider (3.17) as a polynomial of S, q, θ and by means of the identity $rM = 0$ we obtain the following equations:

$$D_t^\alpha N(t) - rN(t) + rA = 0, \qquad D_t^\alpha \left(K - \mu \frac{\alpha M e^{r(t-T)}}{2\gamma\sigma^2} \right) = 0, \tag{3.18}$$
$$- \alpha MF + D_t^\alpha V(t) - rV(t) - \mu^2 \frac{\alpha M e^{r(t-T)}}{2\gamma\sigma^2} - \zeta = 0.$$

Under the assumption $r \neq 0$, by the identity $rM = 0$ we get $M = 0$. Then by (2.1) the second equation in (3.18) gives

$$D_t^\alpha K = \frac{Kt^{-\alpha}}{\Gamma(1-\alpha)} = 0.$$

Therefore, $K = 0$. In accordance with (2.2), as $0 < \alpha < 1$, the first equation in (3.18) has a solution of the form

$$N(t) = Ht^{\alpha-1} E_{\alpha,\alpha}(rt^\alpha) - rA \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(r(t-s)^\alpha) ds,$$

where H is an arbitrary constant. Then the operators become

$$\tau = 0, \qquad \xi = A, \qquad \beta = 0, \qquad \zeta = D_t^\alpha V(t) - rV(t),$$
$$\eta = \left(Ht^{\alpha-1} E_{\alpha,\alpha}(rt^\alpha) - rA(t-s)^\alpha E_{\alpha,\alpha+1}(rt^\alpha) \right) q + V(t).$$

If $r = 0$, then by (2.1) the solution of (3.18) is

$$N(t) = Ht^{\alpha-1}, \qquad K = \mu \frac{\alpha M}{2\gamma\sigma^2}.$$

Taking into consideration also results (3.12), (3.15), (3.16), we obtain the following statement.

Theorem 3.1. *1. A basis of the Lie algebra of the groups of linearly-autonomous transformations of equation* (1.2) *as $r \neq 0$ is formed by the operators*

$$Y_1 = t^{\alpha-1} E_{\alpha,\alpha}(rt^\alpha) q \partial_\theta,$$
$$Y_2 = \partial_S - rt^\alpha E_{\alpha,\alpha+1}(rt^\alpha) q \partial_\theta,$$
$$Y_V = V(t)\partial_\theta + (D_t^\alpha V(t) - rV(t))\partial_F,$$

where $V(t)$ is an arbitrary function.

2. A basis of the Lie algebra of the groups of linearly-autonomous transformations of equation (1.2) *as $r = 0$ is formed by the operators*

$$Y_1 = t^{\alpha-1} q \partial_\theta, \qquad Y_2 = \partial_S, \qquad Y_V = V(t)\partial_\theta + D_t^\alpha V(t)\partial_F,$$
$$Y_3 = 2\gamma\sigma^2 t \partial_t + \alpha\gamma\sigma^2 S \partial_S + \alpha \left(\mu - \gamma\sigma^2 q \right) \partial_q - \alpha \left(2\gamma\sigma^2 F + \mu^2 \right) \partial_F,$$

where $V(t)$ is an arbitrary function.

As $\alpha = 1$ and $r \neq 0$, equation (1.1) possesses, among other, the equivalence transformations groups generated by the operators

$$Y_1|_{\alpha=1} = e^{rt} q \partial_\theta \quad \text{and} \quad Y_V|_{\alpha=1} = V(t)\partial_\theta + (D_t^1 V(t) - rV(t))\partial_F,$$

see [14, Thm. 1].

It follows from the first part of Theorem 3.1 the groups of linearly-autonomous transformations admitted by equation (1.2) as $r \neq 0$ and for all F are generated only by the operators Y_V for V such that $D_t^\alpha V(t) - rV(t) \equiv 0$, that is, the function V reads as

$$V(t) = t^{\alpha-1} E_{\alpha,\alpha}(rt^\alpha), \qquad Y_{t^{\alpha-1} E_{\alpha,\alpha}(rt^\alpha)} = t^{\alpha-1} E_{\alpha,\alpha}(rt^\alpha) \partial_\theta.$$

By the second part of Theorem 3.1, the group of linearly-autonomous transformations admitted by equation (1.2) as $r = 0$ and for all F generated by the operators Y_2 and Y_V with $V(t) = t^{\alpha-1}$.

Some of the equivalence transformations groups obtained in Theorem 3.1 will be employed in the group classification of equation (1.2), while other can be formally used for simplifying the form of the equation.

4. Group classification as $F_{\theta_q\theta_q} \neq 0$

Now we are going to seek admissible groups of linearly-autonomous transformations of equation (1.2) as $0 < \alpha < 1$, since due to Theorem 2.7 in [29] the systems of equations resolved with respect to the fractional Riemann-Liouville derivative in time and involving only derivatives of integer order in other variables possess no other symmetries except for linearly-autonomous ones. We consider a free element $F = F(t, \theta_q)$ nonlinear in θ_q, that is, under the condition $F_{\theta_q\theta_q} \neq 0$. The linear case has its own features, see, for instance, [15], and we shall study it later.

We seek the symmetry operator in the form $X = \tau\partial_t + \xi\partial_S + \beta\partial_q + \eta\partial_\theta$. The action of the extended operator

$$\widetilde{X} = X + \eta^\alpha \partial_{D_t^\alpha \theta} + \eta^S \partial_{\theta_S} + \eta^q \partial_{\theta_q} + \eta^{SS} \partial_{\theta_{SS}}$$

on (1.2) after the restriction to the manifold $\mathfrak{N}$ defined in the extended space of variables by equation (1.2) gives

$$\begin{aligned}\eta^\alpha - r\eta - (\mu - rS)\beta + rq\xi + \mu\eta^S + \frac{\sigma^2}{2}\eta^{SS} - \frac{r}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2\tau \\ + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q)(\eta_S - \beta) - F_t\tau - F_{\theta_q}\eta^q|_{\mathfrak{N}} = 0.\end{aligned} \tag{4.1}$$

We substitute the extension formulas, which have been obtained above while calculating the equivalence transformations under the assumption $\eta(t, S, q, \theta) = p(t, S, q)\theta + g(t, S, q)$, into (4.1) and we obtain

$$\begin{aligned}
&D_t^\alpha g + \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}\theta \left(D_t^n p + \frac{n-\alpha}{n+1} D_t^{n+1}\tau\right) - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}\theta_S D_t^n \xi \\
&- \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}\theta_q D_t^n \beta - r(p\theta + g) - (\mu - rS)\beta + rq\xi \\
&+ \mu\left(p_S\theta + g_S + p\theta_S - \theta_t\tau_S - \theta_S\xi_S - \theta_q\beta_S\right) - \frac{r}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2\tau \\
&+ \frac{\sigma^2}{2}(p_{SS}\theta + g_{SS} + 2p_S\theta_S - \theta_t\tau_{SS} - \theta_S\xi_{SS} - \theta_q\beta_{SS} - 2\theta_{St}\tau_S - 2\theta_{Sq}\beta_S \\
&+ \theta_{SS}(p - 2\xi_S)) + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q)(p_S\theta + g_S + p\theta_S - \theta_t\tau_S - \theta_S\xi_S - \theta_q\beta_S - \beta) \\
&- F_t\tau - F_{\theta_q}(p_q\theta + g_q + p\theta_q - \theta_t\tau_q - \theta_S\xi_q - \theta_q\beta_q)|_{\mathfrak{N}} = 0.
\end{aligned}$$

The passage to the manifold $\mathfrak{N}$ gives the equation

$$\begin{aligned}
&(p - \alpha\tau_t)\left(r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F\right) \\
&\quad + D_t^\alpha g + \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}\theta \left(D_t^n p + \frac{n-\alpha}{n+1} D_t^{n+1}\tau\right) - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}\theta_S D_t^n \xi \\
&\quad - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}\theta_q D_t^n \beta - r(p\theta + g) - (\mu - rS)\beta + rq\xi
\end{aligned}$$

$$
\begin{aligned}
&+\mu\left(p_S\theta+g_S+p\theta_S-\theta_t\tau_S-\theta_S\xi_S-\theta_q\beta_S\right)-\frac{r}{2}\gamma\sigma^2e^{r(T-t)}(\theta_S-q)^2\tau \\
&+\frac{\sigma^2}{2}(p_{SS}\theta+g_{SS}+2p_S\theta_S-\theta_t\tau_{SS}-\theta_S\xi_{SS} \\
&-\theta_q\beta_{SS}-2\theta_{St}\tau_S-2\theta_{Sq}\beta_S+\theta_{SS}(p-2\xi_S)) \\
&+\gamma\sigma^2e^{r(T-t)}(\theta_S-q)(p_S\theta+g_S+p\theta_S-\theta_t\tau_S-\theta_S\xi_S-\theta_q\beta_S-\beta) \\
&-F_t\tau-F_{\theta_q}(p_q\theta+g_q+p\theta_q-\theta_t\tau_q-\theta_S\xi_q-\theta_q\beta_q)=0.
\end{aligned} \tag{4.2}
$$

By Theorem 2.7 in [29],

$$
\tau(t,S,q)=At+Bt^2,\qquad \xi_t=\xi_\theta=\beta_t=\beta_\theta=0,\qquad p(t,S,q)=h(S,q)+\kappa(A+2Bt),
$$

where A, B are some constants, $\kappa=0$ as $B=0$ and $\kappa=(\alpha-1)/2$ as $B\neq 0$. This is equation (4.2) becomes

$$
\begin{aligned}
&(p-\alpha\tau_t)\left(r\theta+(\mu-rS)q-\mu\theta_S-\frac{\sigma^2}{2}\theta_{SS}-\frac{1}{2}\gamma\sigma^2e^{r(T-t)}(\theta_S-q)^2+F\right) \\
&\quad+D_t^\alpha g-r(p\theta+g)-(\mu-rS)\beta+rq\xi \\
&\quad+\mu\left(h_S\theta+g_S+p\theta_S-\theta_S\xi_S-\theta_q\beta_S\right)-\frac{r}{2}\gamma\sigma^2e^{r(T-t)}(\theta_S-q)^2\tau \\
&\quad+\frac{\sigma^2}{2}(h_{SS}\theta+g_{SS}+2h_S\theta_S-\theta_S\xi_{SS}-\theta_q\beta_{SS}-2\theta_{Sq}\beta_S+\theta_{SS}(p-2\xi_S)) \\
&\quad+\gamma\sigma^2e^{r(T-t)}(\theta_S-q)(h_S\theta+g_S+p\theta_S-\theta_S\xi_S-\theta_q\beta_S-\beta) \\
&\quad-F_t\tau-F_{\theta_q}(h_q\theta+g_q+p\theta_q-\theta_S\xi_q-\theta_q\beta_q)=0.
\end{aligned} \tag{4.3}
$$

We separate the variables θ_S, θ_{SS}, θ_{Sq} in (4.3) and we obtain

$$
\theta_{Sq}:\qquad \beta_S=0, \tag{4.4}
$$

$$
\theta_{SS}:\qquad \alpha\tau_t-2\xi_S=0, \tag{4.5}
$$

$$
\theta_S^2:\qquad p-r\tau=0, \tag{4.6}
$$

$$
\begin{aligned}
\theta_S:\qquad &(p-\alpha\tau_t)(-\mu+\gamma\sigma^2e^{r(T-t)}q)+\mu(p-\xi_S)+\frac{\sigma^2}{2}(2h_S-\xi_{SS}) \\
&+r\gamma\sigma^2e^{r(T-t)}q\tau+\gamma\sigma^2e^{r(T-t)}(h_S\theta+g_S-\beta-q(p-\xi_S))+F_{\theta_q}\xi_q=0,
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
1:\qquad &(p-\alpha\tau_t)\left(r\theta+(\mu-rS)q-\frac{1}{2}\gamma\sigma^2e^{r(T-t)}q^2+F\right) \\
&+D_t^\alpha g-r(p\theta+g)-(\mu-rS)\beta+rq\xi+\mu\left(h_S\theta+g_S\right)-\frac{r}{2}\gamma\sigma^2e^{r(T-t)}q^2\tau \\
&+\frac{\sigma^2}{2}(h_{SS}\theta+g_{SS})-\gamma\sigma^2e^{r(T-t)}q(h_S\theta+g_S-\beta) \\
&-F_t\tau-F_{\theta_q}(h_q\theta+g_q+p\theta_q-\theta_q\beta_q)=0.
\end{aligned} \tag{4.8}
$$

It follows from (4.5) that $\tau_{tt}=0$, $\xi_{SS}=0$, this is why $B=\kappa=0$, $\tau=At$, $p=h(S,q)$. In view of the assumption $F_{\theta_q\theta_q}\neq 0$, the differentiation of (4.7) in θ_q gives $\xi_q=0$. Thus, in view of (4.4), (4.6),

$$
\alpha A-2\xi_S=0,\quad r\tau_t=p_t=0,\quad h_S=h_q=0,\quad \xi_{SS}=0,\quad \xi=\xi(S),\quad \beta=\beta(q).
$$

The integration gives the identities

$$
\tau=At,\quad rA=0,\quad \xi=\frac{\alpha A}{2}S+E,\quad \beta=\beta(q),\quad p=0,\quad \eta=g=g(t,S,q), \tag{4.9}
$$

where A, E are constants. Equations (4.7), (4.8) now become

$$\mu\frac{\alpha A}{2}+\gamma\sigma^2e^{r(T-t)}\left(g_S-\beta-q\frac{\alpha A}{2}\right)=0, \tag{4.10}$$

$$\begin{aligned}-\alpha A\left(\mu q-\frac{1}{2}\gamma\sigma^2e^{r(T-t)}q^2+F\right)+D_t^\alpha g-rg-(\mu-rS)\beta+rqE+\mu g_S\\+\frac{\sigma^2}{2}g_{SS}-q\gamma\sigma^2e^{r(T-t)}(g_S-\beta)-F_tAt-F_{\theta_q}(g_q-\theta_q\beta_q)=0.\end{aligned} \tag{4.11}$$

We express $\gamma\sigma^2e^{r(T-t)}(g_S-\beta)$ from equation (4.10) and substitute into (4.11) and we obtain

$$\gamma\sigma^2e^{r(T-t)}(g_S-\beta)=\gamma\sigma^2e^{r(T-t)}\frac{\alpha A}{2}q-\mu\frac{\alpha A}{2}, \tag{4.12}$$

$$\begin{aligned}-\alpha A\left(\frac{\mu q}{2}+F\right)+D_t^\alpha g-rg-(\mu-rS)\beta+rqE+\mu g_S\\+\frac{\sigma^2}{2}g_{SS}-F_tAt-F_{\theta_q}(g_q-\theta_q\beta_q)=0.\end{aligned} \tag{4.13}$$

By equation (4.12) we obtain

$$g=\beta(q)S+\frac{\alpha A}{2}Sq-\mu\frac{\alpha A}{2\gamma\sigma^2}e^{r(t-T)}S+G(t,q), \tag{4.14}$$

$G(t,q)$ is some function.

The differentiation of (4.13) by θ_q gives

$$(\beta_q-\alpha A)F_{\theta_q}-F_{t\theta_q}At-F_{\theta_q\theta_q}(g_q-\theta_q\beta_q)=0. \tag{4.15}$$

Differentiating equation (4.15) in S, we obtain $g_{Sq}=0$. By (4.14) this implies the identity $\beta_q=-\alpha A/2$, $\beta=-\alpha Aq/2+L$, where L is a constant. Hence, $\beta_{qq}=0$ and by differentiating (4.15) in q we get $g_{qq}=G_{qq}=0$. Thus,

$$\beta=-\frac{\alpha A}{2}q+L,\qquad g=LS-\mu\frac{\alpha A}{2\gamma\sigma^2}e^{r(t-T)}S+N(t)q+M(t), \tag{4.16}$$

where $N(t)$, $M(t)$ are some functions.

Substituting (4.16) into (4.13) gives

$$\begin{aligned}&-\alpha A\left(\frac{\mu q}{2}+F\right)+D_t^\alpha\left(L-\mu\frac{\alpha Ae^{r(t-T)}}{2\gamma\sigma^2}\right)S\\&\qquad+D_t^\alpha N(t)q+D_t^\alpha M(t)-r\left(L-\mu\frac{\alpha Ae^{r(t-T)}}{2\gamma\sigma^2}\right)S-rN(t)q-rM(t)\\&\qquad-(\mu-rS)\left(-\frac{\alpha A}{2}q+L\right)+rqE+\mu\left(L-\mu\frac{\alpha Ae^{r(t-T)}}{2\gamma\sigma^2}\right)\\&\qquad-F_tAt-F_{\theta_q}\left(N(t)+\theta_q\frac{\alpha A}{2}\right)=0.\end{aligned} \tag{4.17}$$

We equate to zero the coefficients of polynomial (4.17) on S and q:

$$S:\quad D_t^\alpha\left(L-\mu\frac{\alpha Ae^{r(t-T)}}{2\gamma\sigma^2}\right)=0,\qquad q:\quad D_t^\alpha N(t)-rN(t)+rE=0, \tag{4.18}$$

$$1:\quad D_t^\alpha M(t)-rM(t)-\mu^2\frac{\alpha Ae^{r(t-T)}}{2\gamma\sigma^2}-\alpha AF-F_tAt-F_{\theta_q}\left(N(t)+\theta_q\frac{\alpha A}{2}\right)=0. \tag{4.19}$$

Since a solution of equations (4.18), (4.19) depends on r, we should consider the cases $r=0$ and $r\neq 0$.

4.1. Case $\mathbf{r \neq 0}$. In this case $A = 0$ and this is why identities (4.9), (4.16), (4.18), (4.19) cast into the form

$$\tau = 0, \qquad \xi = E, \qquad \beta = L, \qquad \eta = g = LS + N(t)q + M(t),$$

$$D_t^\alpha L = 0, \qquad D_t^\alpha N(t) - rN(t) + rE = 0, \qquad D_t^\alpha M(t) - rM(t) - F_{\theta_q} N(t) = 0. \tag{4.20}$$

Differentiating the third equation in (4.20) with respect to θ_q, we obtain $N = 0$. We substitute the obtained identity $N = 0$ into the second equation in (4.20) and we obtain $E = 0$. In view (2.1) the first equation in (4.20) gives the expression $Lt^{-\alpha}/\Gamma(1-\alpha)$ and hence, $L = 0$. Thus, the function F in this case is arbitrary up to the nonlinearity condition $F_{\theta_q\theta_q} \neq 0$. Solving the equation $D_t^\alpha M - rM = 0$ by means of (2.2), we obtain the following theorem.

Theorem 4.1. *The main Lie algebra of equation* (1.2)*, where $F_{\theta_q\theta_q} \neq 0$ and $r \neq 0$ is generated by the operator $X_1 = t^{\alpha-1}E_{\alpha,\alpha}(rt^\alpha)\partial_\theta$.*

Hereinafter the notion "main Lie algebra" is used in the sense of monograph by L.V. Ovsyannikov, see [17].

Comparing with the symmetries of equation (1.1) [14], we can observe that the condition $\tau(0) = 0$ leads to the loss of symmetry with the shift in time and several symmetries of the equations of the first order in time are absent in the case of a fractional time derivative because of the equation $\xi_t = 0$, $\beta_t = 0$ in the determining system of equations. The symmetry $\partial_q + S\partial_\theta$ of equation (1.1) in this case is absent since the fractional Riemann-Liouville derivative of a constant is not the zero. The operator X_1 in the case of equation (1.1) becomes $X_1|_{\alpha=1} = e^{rt}\partial_\theta$.

4.2. Case $\mathbf{r = 0}$. We substitute $r = 0$ into (4.18), (4.19) and we obtain

$$D_t^\alpha \left(L - \mu\frac{\alpha A}{2\gamma\sigma^2}\right) = \left(L - \mu\frac{\alpha A}{2\gamma\sigma^2}\right) D_t^\alpha 1 = 0, \qquad D_t^\alpha N(t) = 0, \tag{4.21}$$

$$D_t^\alpha M(t) - \mu^2\frac{\alpha A}{2\gamma\sigma^2} - \alpha AF - F_t At - F_{\theta_q}\left(N(t) + \theta_q\frac{\alpha A}{2}\right) = 0. \tag{4.22}$$

Solving (4.21) by means of (2.1), we get

$$N(t) = Pt^{\alpha-1}, \qquad L = \mu\frac{\alpha A}{2\gamma\sigma^2}.$$

Substitution of the obtained relations into (4.9), (4.16) gives

$$\tau = At, \qquad \xi = \frac{\alpha A}{2}S + E, \qquad \beta = -\frac{\alpha A}{2}q + \mu\frac{\alpha A}{2\gamma\sigma^2}, \qquad \eta = Pt^{\alpha-1}q + M(t),$$

where A, E, P are some constants. We represent (4.22) as

$$\alpha AF + F_t At + F_{\theta_q}\left(Pt^{\alpha-1} + \theta_q\frac{\alpha A}{2}\right) - R(t) = 0, \tag{4.23}$$

where $R(t) = D_t^\alpha M(t) - \mu^2\frac{\alpha A}{2\gamma\sigma^2}$.

4.2.1. Case $A \neq 0$. Integrating equation (4.23), we obtain

$$F = t^{-\alpha}\Phi\left(t^{-\alpha/2}\theta_q - \frac{Pt^{\alpha/2-1}}{A(\frac{\alpha}{2}-1)}\right) + \frac{t^{-\alpha}}{A}\int_0^t R(t)t^{\alpha-1}dt,$$

where Φ is an arbitrary function. On the obtained expression we apply the equivalence transformation $\bar\theta = \theta + a_1 t^{\alpha-1}q$ from the group generated by the operator Y_1 in the second part of Theorem 3.1 with the group parameter $a_1 = \frac{P}{A(\frac{\alpha}{2}-1)}$ and then we apply the equivalence transformation

$$\bar{\theta} = \theta + a_V V(t), \qquad \bar{F} = F + a_V D_t^\alpha V(t)$$

generated by the operator Y_V of the group (see Theorem 2) with $a_V = 1$ and a function $V(t)$ such that

$$D_t^\alpha V(t) = -\frac{t^{-\alpha}}{A} \int\limits_0^t R(t) t^{\alpha-1}\, dt.$$

The function $V(t)$ is defined by formula (2.2) with $\lambda = 0$. As a result we obtain the function $F = t^{-\alpha}\Phi\left(t^{-\alpha/2}\theta_q\right)$. Substituting it into determining equation (4.22) gives

$$D_t^\alpha M(t) - \mu^2 \frac{\alpha A}{2\gamma\sigma^2} - N(t) t^{-3\alpha/2}\Phi' = 0.$$

Since by our assumption $F_{\theta_q\theta_q} \neq 0$, we obtain $\Phi'' \neq 0$ and hence

$$N(t) = 0, \qquad D_t^\alpha M(t) - \mu^2 \frac{\alpha A}{2\gamma\sigma^2} = 0.$$

Thus, integrating $M(t)$ by means of (2.1), we get

$$M(t) = M_0 t^{\alpha-1} + \mu^2 \frac{\alpha A}{2\gamma\sigma^2} \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

As a result we have

$$\tau = At, \qquad \xi = \frac{\alpha A}{2} S + E, \qquad \beta = -\frac{\alpha A}{2} q + \mu \frac{\alpha A}{2\gamma\sigma^2},$$
$$\eta = \frac{M_0 t^{\alpha-1}}{\Gamma(\alpha)} + \mu^2 \frac{\alpha A}{2\gamma\sigma^2} \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

4.2.2. Case $A = 0$. We obtain identities (4.20) with $r = 0$, that is,

$$\tau = 0, \qquad \xi = E, \qquad \beta = L, \qquad \eta = g = LS + N(t)q + M(t),$$
$$D_t^\alpha L = 0, \qquad D_t^\alpha N(t) = 0, \qquad D_t^\alpha M(t) - F_{\theta_q} N(t) = 0.$$

Then $L = 0$ and differentiating the third equation in θ_q, we obtain $N(t) \equiv 0$, $M(t) = M_0 t^{\alpha-1}$, F is an arbitrary function. We have proved the following statement for equation (1.2) as $r = 0$.

Theorem 4.2. *The main Lie algebra of the equation*

$$D_t^\alpha \theta = \mu q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{1}{2}\gamma\sigma^2(\theta_S - q)^2 + t^{-\alpha}\Phi\left(t^{-\alpha/2}\theta_q\right),$$

where $\Phi'' \neq 0$, is generated by the operators

$$X_1 = \partial_S, \qquad X_2 = t^{\alpha-1}\partial_\theta,$$
$$X_3 = 2t\partial_t + \alpha S\partial_S + \left(-\alpha q + \frac{\mu\alpha}{\gamma\sigma^2}\right)\partial_q + \frac{\mu^2\alpha t^\alpha}{\gamma\sigma^2\Gamma(\alpha+1)}\partial_\theta.$$

For the equation

$$D_t^\alpha \theta = \mu q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{1}{2}\gamma\sigma^2(\theta_S - q)^2 + F(t, \theta_q),$$

where $F_{\theta_q\theta_q} \neq 0$ and $F(t,\theta_q)$ is not equivalent to the function $t^{-\alpha}\Phi\left(t^{-\alpha/2}\theta_q\right)$ in the sense of equivalence transformations generated by the operators Y_1, Y_2, Y_3, Y_V in the second part of Theorem 3.1, the main Lie algebra is generated by the operators $X_1 = \partial_S$, $X_2 = t^{\alpha-1}\partial_\theta$.

We observe that the form of the obtained symmetries agrees with the results of Theorem 2.7 [29], in which there was found a general form of symmetries for such systems.

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Khristofor Vasilievich Yadrikhinskiy,
M.K. Ammosov North-Eastern Federal University
Yakutsk Branch of
Far Eastern center of mathematical studies,
Belinsky str. 58,
677000, Yakutsk, Russia
E-mail: ghdsfdf@yandex.ru

Vladimir Evgenievich Fedorov,
M.K. Ammosov North-Eastern Federal University
Yakutsk Branch of
Far Eastern center of mathematical studies,
Belinsky str. 58,
677000, Yakutsk, Russia
Chelyabinsk State University,
Br. Kashiriny str. 129,
450001, Chelyabinsk, Russia
E-mail: kar@csu.ru