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PERTURBATION METHOD FOR STRONGLY ELLIPTIC SECOND ORDER SYSTEMS WITH CONSTANT COEFFICIENTS

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Abstract. We consider a classical Dirichlet problem for a strongly elliptic second order system with constant coefficients in Jordan domains in the plane. We show that the solution of the problem can be represented as a functional series in the powers of the parameter governing the deviation of the operator of the system from the Laplacian. This series converges uniformly in the closure of the domain under the assumption that the boundary of the domain and the given boundary function satisfy sufficient regularity conditions: the composition of the boundary function with the trace of a conformal mapping of the unit circle on the domain belongs to the Hölder class with the exponent exceeding 1/2.

Keywords:strongly elliptic system, Dirichlet problem, perturbation method.

Mathematics Subject Classification: 30E25, 35J25

1. Introduction

In this work we consider a system of differential equations

$$\left(A\frac{\partial^2}{\partial x^2}+2B\frac{\partial^2}{\partial x\partial y}+C\frac{\partial^2}{\partial y^2}\right)\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \tag{1.1}$$

for real-valued functions $u(x,y)$ and $v(x,y)$ of real variables x and y with constant real matrices of the coefficients A, B, C of size 2×2. We study such systems of elliptic type. In accordance with the definition by Petrovskii [1], this means that

$$\det(A\xi^2+2B\xi\eta+C\eta^2)\neq 0 \quad \text{as} \quad (\xi,\eta)\in\mathbb{R}^2\setminus(0,0).$$

We introduce a complex-valued function $f=u+iv$ of a complex variable $z=x+iy$ and an operator of system (1.1)

$$Lf=\left(A\frac{\partial^2}{\partial x^2}+2B\frac{\partial^2}{\partial x\partial y}+C\frac{\partial^2}{\partial y^2}\right)\begin{pmatrix}u\\ v\end{pmatrix}.$$

A classical formulation of the Dirichlet problem for such operator in a Jordan domain is as follows.

Problem 1.1. *Let Ω be a Jordan domain with a boundary Γ. Given a boundary function $h\in C(\Gamma)$, find a function $f\in C(\overline{\Omega})\cap C^2(\Omega)$ such that $Lf=0$ in Ω and $f|_\Gamma=h$.*

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The study of the solvability of the Dirichlet problem led to selecting a subclass of strongly elliptic systems, which were defined in several ways; we shall make use of a definition from [2], in accordance with which we need

$$\det(A\alpha + 2\beta B + \gamma C) \neq 0 \quad \text{as} \quad \beta^2 - \alpha\gamma < 0.$$

It is equivalent to the well-known definition by Vishik [3].

The form of system (1.1) and its belonging to the class of elliptic or strongly elliptic systems are preserved under three classes of non-degenerate transformations: 1) linear change of the variables (x, y); 2) linear change of sought functions (u, v); 3) linear combinations of equations of system. At the same time, a specially chosen series of such transformations followed by the summation of the first of the obtained equations with the second one multiplied by the imaginary unit i allows us to reduce each elliptic system (1.1) to a complex equation

$$(\partial\overline{\partial} + \tau\partial^2)g(z) + \sigma(\tau\partial\overline{\partial} + \partial^2)\overline{g(z)} = 0 \tag{1.2}$$

for a complex-valued function g of a complex variable $z = x + iy$ with only two parameters $\tau \in [0, 1)$ and $\sigma \in [0, 1) \cup (1, \infty]$, see [4], [5]. Here

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

are the Cauchy-Riemann operators in the new coordinate system. As $\sigma = \infty$ we adopt that equation (1.2) becomes

$$(\tau\partial\overline{\partial} + \partial^2)\overline{g(z)} = 0.$$

In the case of the strong ellipticity we have $\sigma \in [0, 1)$.

Let us mention several well-known particular cases of equation (1.2). As $\tau = \sigma = 0$, we get a complex-valued Laplace equation $\Delta g(z) = 4\partial\overline{\partial}g(z) = 0$, while for $\tau = 0$, $\sigma = \infty$ this is the Bitsadze equation $\overline{\partial}^2 g(z) = 0$ [6]. For $\tau = 0$ there arises a planar isotropic Lamé equations of the elasticity theory written in the complex form $\partial\overline{\partial}g(z) + \sigma\partial^2\overline{f(z)} = 0$ and the parameter σ is related with the Poisson coefficient p by the relation $\sigma = 1/(3 - 4p)$, see [7], [8]. Since, as it is known [7], $p \in (0, 1/2)$, and hence, $\sigma \in (1/3, 1)$, corresponding system (1.2) is strongly elliptic. If $\sigma = 0$, we get a system called skew-symmetric, which can be written as the equation $ag_{xx} + 2bg_{xy} + cg_{yy} = 0$ with complex-valued coefficients a, b, c.

Equation (1.2) can be regarded as perturbation of the Laplace equations in two parameters τ and σ and in the case of the strong ellipticity these parameters are relatively small: $\tau, \sigma \in [0, 1)$. In order to stress this fact, we make one more final transformation of equation (1.2) by separating the Laplacian from the rest of the equation:

$$\partial\overline{\partial}(\mathcal{T}_{1,\sigma\tau}g) + \partial^2(\mathcal{T}_{\tau,\sigma}g) = 0, \tag{1.3}$$

where we have employed the operator of an affine transformation

$$\mathcal{T}_{\alpha,\beta} := \alpha\mathcal{I} + \beta\mathcal{C}, \tag{1.4}$$

with, generally speaking, complex parameters α and β and which is expressed via the identity operator $\mathcal{I}\colon w \to w$ and the operator of the complex conjugation $\mathcal{C}\colon w \to \overline{w}$. If $|\alpha| \neq |\beta|$, then the inverse operator

$$\mathcal{T}^{-1}_{\alpha,\beta} = \frac{1}{|\alpha|^2 - |\beta|^2}\mathcal{T}_{\overline{\alpha},-\beta}$$

is well-defined. The norm of operator (1.4) as of a mapping from $\mathbb{C}$ into $\mathbb{C}$ is equal to $\|\mathcal{T}_{\alpha,\beta}\| = |\alpha| + |\beta|$.

Assuming that equation (1.3) is strongly elliptic, we replace the sought function g by $f = \mathcal{T}_{1,\sigma\tau}g$ (which is a non-degenerate transform in this case) and we rewrite equation (1.3) in the form

$$\mathcal{L}f := \partial\overline{\partial}f + \partial^2(Tf) = 0, \tag{1.5}$$

where

$$T = \mathcal{T}_{\tau,\sigma}\mathcal{T}_{1,\sigma\tau}^{-1} = \frac{\tau(1-\sigma^2)\mathcal{I} + \sigma(1-\tau^2)\mathcal{C}}{1-\sigma^2\tau^2}.$$

Equation (1.5) is the Laplace equation perturbed by the operator T with the norm

$$\|T\| = \frac{\tau+\sigma}{1+\sigma\tau},$$

which turns out to be strictly less than one in the considered strongly elliptic system as $\tau, \sigma \in [0,1)$. We introduce a normalized operator

$$T_0 = \|T\|^{-1}T = \mathcal{T}_{\alpha_0,\beta_0},$$

where

$$\alpha_0 = \frac{\tau(1-\sigma^2)}{(\tau+\sigma)(1-\sigma\tau)}, \qquad \beta_0 = \frac{\sigma(1-\tau^2)}{(\tau+\sigma)(1-\sigma\tau)},$$

and by means of this operator we rewrite (1.5) to a final form

$$\mathcal{L}f = \partial\overline{\partial}f + \|T\|\partial^2(T_0 f) = 0. \tag{1.6}$$

In this equation, a small parameter is $\|T\| < 1$.

2. Perturbation method

In order to solve the Dirichlet problem, we apply a perturbation method with respect to the quantity $\|T\|$, the matter of which is to seek the solution f as the series

$$f = \sum_{n=0}^{\infty} f_n \|T\|^n, \tag{2.1}$$

where the functions f_n are determined by substituting expansion (2.1) into the equation $\mathcal{L}f = 0$ and equating to zero the coefficients at the like powers of $\|T\|$; here we let $f_0|_\Gamma = h$ and $f_n|_\Gamma = 0$ for $n \geqslant 1$.

Thus, we obtain the following boundary value problems for successive determining of functions f_n:

$$\partial\overline{\partial}f_0 = 0 \quad \text{in} \quad \Omega, \qquad\qquad f_0|_\Gamma = h \tag{2.2}$$

and

$$\partial\overline{\partial}f_n = -\partial^2(T_0 f_{n-1}) \quad \text{in} \quad \Omega, \qquad\qquad f_n|_\Gamma = 0 \tag{2.3}$$

for $n \geqslant 1$.

Let $\omega\colon \mathbb{D} \to \Omega$ be some conformal mapping of the unit circle $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ onto the domain Ω. In the case of a Jordan domain Ω by the Caratheodory theorem the mapping ω is continued to a homomorphism of closed domains $\overline{\mathbb{D}}$ and $\overline{\Omega}$, so that $\omega \in C(\overline{\mathbb{D}})$. For further purposes it is convenient to transform problems (2.2) and (2.3) into the circle $\mathbb{D}$ by means of the introduced conformal mapping. Let

$$F = f \circ \omega, \qquad H = h \circ \omega, \qquad F_n = f_n \circ \omega.$$

Then

$$\mathcal{L}f = \frac{1}{|\omega'|^2}\left[\partial\overline{\partial}F + \|T\|\partial\left(\frac{\overline{\omega'}}{\omega'}\partial(T_0 F)\right)\right] =: \mathcal{M}F. \tag{2.4}$$

It follows from (2.1), (2.2) and (2.3) that

$$F = \sum_{n=0}^{\infty} F_n \|T\|^n, \tag{2.5}$$

where

$$\partial\overline{\partial} F_0 = 0 \quad \text{in} \quad \mathbb{D}, \qquad\qquad F_0|_{\mathbb{T}} = H \tag{2.6}$$

and

$$\partial\overline{\partial} F_n = -\partial\left(\frac{\overline{\omega'}}{\omega'}\partial(T_0 F_{n-1})\right) \quad \text{in} \quad \Omega, \qquad\qquad F_n|_{\mathbb{T}} = 0 \tag{2.7}$$

for $n \geqslant 1$. In the case of a sufficiently regular function F_{n-1} for a fixed n by means of the Green function

$$G(\zeta, z) = \frac{2}{\pi}\log\left|\frac{\zeta - z}{1 - \zeta\overline{z}}\right| \tag{2.8}$$

for the operator $\partial\overline{\partial}$ in the circle $\mathbb{D}$ we can write solutions to problems (2.6) and (2.7):

$$F_0(z) = \frac{1}{2i}\int_{\mathbb{T}} \partial_\zeta G(\zeta, z) h(\zeta) d\zeta, \qquad F_n(z) = -\int_{\mathbb{D}} G(\zeta, z)\partial\left(\frac{\overline{\omega'(\zeta)}}{\omega'(\zeta)}\partial(T_0 F_{n-1}(\zeta))\right) d\mu, \tag{2.9}$$

$n \geqslant 1$, where μ is the Lebesgue measure.

We define the operators

$$\mathcal{P}[\varphi(z)] := \frac{1}{2i}\int_{\mathbb{T}} \partial_\zeta G(\zeta, z)\varphi(\zeta) d\zeta, \qquad \mathcal{K}[\varphi(z)] := \int_{\mathbb{D}} \partial_\zeta G(\zeta, z)\varphi(\zeta) d\mu$$

and

$$\mathcal{K}_\partial[\varphi(z)] := \text{p.v.}\int_{\mathbb{D}} \partial_z\partial_\zeta G(\zeta, z)\varphi(\zeta) d\mu, \qquad \mathcal{K}_{\overline{\partial}}[\varphi(z)] := \text{p.v.}\int_{\mathbb{D}} \partial_{\overline{z}}\partial_\zeta G(\zeta, z)\varphi(\zeta) d\mu,$$

where $\mathcal{P}$ is defined on the class of functions $C(\mathbb{T})$, while the others are defined on $L_p(\mathbb{D})$ and two latter integrals are treated in the sense of the principal value. In what follows, for the sake of brevity, we omit the notation p.v. By formula (2.8) we obtain

$$\begin{gathered} \mathcal{P}[\varphi(z)] = \frac{1}{2\pi}\int_{\mathbb{T}}\left(\frac{1}{\zeta - z} + \frac{\overline{z}}{1 - \zeta\overline{z}}\right)\varphi(\zeta) d\zeta, \\ \mathcal{K}[\varphi(z)] = \frac{1}{\pi}\int_{\mathbb{D}}\left(\frac{1}{\zeta - z} + \frac{\overline{z}}{1 - \zeta\overline{z}}\right)\varphi(\zeta) d\mu \end{gathered} \tag{2.10}$$

and

$$\mathcal{K}_\partial[\varphi(z)] = \frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta) d\mu}{(\zeta - z)^2}, \qquad \mathcal{K}_{\overline{\partial}}[\varphi(z)] = \frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta) d\mu}{(1 - \zeta\overline{z})^2} - \varphi(z). \tag{2.11}$$

In terms of the introduced operators, formulas (2.9) for constructing the functions F_n can be written as

$$F_0 = \mathcal{P}[h], \qquad F_n = \mathcal{K}[(\overline{\omega'}/\omega')\partial(T_0 F_{n-1})], \qquad n \geqslant 1. \tag{2.12}$$

At the same time,

$$\partial F_n = \mathcal{K}_\partial[(\overline{\omega'}/\omega')\partial(T_0 F_{n-1})], \qquad \overline{\partial} F_n = \mathcal{K}_{\overline{\partial}}[(\overline{\omega'}/\omega')\partial(T_0 F_{n-1})], \quad n \geqslant 1.$$

We also introduce the notations for partial sums of series (2.1) and (2.5), respectively:

$$s_m = \sum_{n=0}^{m} f_n \|T\|^n, \qquad S_m = \sum_{n=0}^{m} F_n \|T\|^n. \tag{2.13}$$

In this work we prove the following convergence theorem.

Theorem 2.1. *Let the Jordan domain Ω and a function h defined on its boundary Γ be such that $h \circ \omega \in C^\alpha(\mathbb{T})$ as $1/2 < \alpha < 1$, where ω is some conformal mapping of the unit circle $\mathbb{D}$ onto Ω. Then for each value $\|T\| \in [0, 1)$ series (2.1) with the functions $f_n = F_n \circ \omega^{-1}$, where F_n are defined in (2.12) converges in the norm of the space $C(\overline{\Omega})$ to a function $f \in C(\overline{\Omega})$ solving the equation $\mathcal{L}f = 0$ in Ω and coinciding with h on Γ.*

The condition $h \circ \omega \in C^\alpha(\mathbb{T})$, $\alpha \in (1/2, 1)$, holds, for instance, if $h \in C^\beta(\Gamma)$ and $\omega \in C^\gamma(\mathbb{T})$, where $\beta\gamma = \alpha \in (1/2, 1)$. Indeed, in this case,

$$|h \circ \omega(z_1) - h \circ \omega(z_2)| \leqslant [h]_\alpha |\omega(z_1) - \omega(z_2)|^\beta \leqslant [h]_\beta [\omega|_{\mathbb{T}}]_\gamma^\beta |z_1 - z_2|^{\beta\gamma},$$

where

$$[\varphi]_\alpha := \sup_{\zeta_1 \neq \zeta_2} \frac{|\varphi(\zeta_1) - \varphi(\zeta_2)|}{|\zeta_1 - \zeta_2|^\alpha}.$$

Theorem 2.1 is an extension of a similar result obtained in [9] for a skew-symmetric strongly elliptic system being a particular case of a considered here system corresponding to $\sigma = 0$.

We note that not all considered here elliptic systems (1.2) possess an energy functional, by means of which a variational reformulation of the Dirichlet problem is possible; this reformulation is a base for the proof of the Lebesgue theorem on the general solvability of the Dirichlet problem in a simply-connected domain, see [10]. System of canonical form (1.2) possesses an energy functional, which is an integral over the domain of the quadratic form of the first derivatives only as $\sigma > \tau$; such systems are called symmetrizable, see [8]. This fact is the reason why the issue on the solvability of the Dirichlet problem for general strongly elliptic systems (1.1) in simply-connected or at least in Jordan domains with arbitrary continuous boundary data is open.

At present the most advance in the issue on solvability of Problem 1.1 is a result by Verchota and Vogel [11], which establishes the general solvability of the Dirichlet problem in domains with piece-wise smooth Lipschitz boundaries for arbitrary continuous boundary data. In our Theorem 2.1 the boundary function are taken from a narrower Hölder class but the domain can belong to a wider class in comparison with [11].

3. Proof of convergence of perturbation method

Lemma 3.1. *If $\varphi \in C^\alpha(\mathbb{T})$, where $1/2 < \alpha < 1$, then $\mathcal{P}[\varphi] \in W^1_p(\mathbb{D})$ with an arbitrary exponent $0 < p < (2(1-\alpha))^{-1}$.*

Proof. Let $\psi = \mathcal{P}\varphi$. Since $\varphi \in C(\mathbb{T})$, by the property of the Poisson integral we have $\psi \in C(\overline{\mathbb{D}})$ and hence $\psi \in L_p(\mathbb{D})$. Let us prove the L_p-integrability of the first derivatives. We represent the function ψ as a sum $\psi(z) = \psi_1(z) + \psi_2(z)$ of holomorphic and antiholomorphic components

$$\psi_1(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\varphi(\zeta) d\zeta}{\zeta - z}, \qquad \psi_2(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\varphi(\zeta) d\overline{\zeta}}{\overline{\zeta} - \overline{z}} - \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(\zeta) |d\zeta|.$$

By Privalov theorem [12] for the Cauchy type integrals, the belonging $\varphi \in C^\alpha(\mathbb{T})$ for $1/2 < \alpha < 1$ implies that $\psi_1 \in C^{2\alpha-1}(\overline{\mathbb{D}})$. We denote $T(z, r) := \{\zeta \in \mathbb{C} \colon |\zeta - z| = r\} \subset \mathbb{D}$. By the Cauchy formula

$$\psi_1(z) = \frac{1}{2\pi i} \int_{T(z,r)} \frac{\psi_1(\zeta) d\zeta}{\zeta - z}$$

we find

$$\partial\psi(z) = \psi_1'(z) = \frac{1}{2\pi i} \int_{T(z,r)} \frac{\psi_1(\zeta) d\zeta}{(\zeta - z)^2} = \frac{1}{2\pi i} \int_{T(z,r)} \frac{\psi_1(\zeta) - \psi_1(z)}{(\zeta - z)^2} d\zeta,$$

which implies the estimate

$$|\partial\psi(z)| \leqslant \frac{1}{2\pi i}\int_{T(z,r)} \frac{|\psi_1(\zeta)-\psi_1(z)|}{|\zeta-z|^2}|d\zeta| \leqslant \frac{1}{2\pi i}\int_{T(z,r)} \frac{c_\alpha|\zeta-z|^{2\alpha-1}}{|\zeta-z|^2}|d\zeta| = \frac{c_\alpha}{r^{2(1-\alpha)}},$$

where $c_\alpha = \sup_{\zeta\neq z}|\psi_1(\zeta)-\psi_1(z)|/|\zeta-z|^{2\alpha-1}$. Passing to the limit as $r \to (1-|z|)$, we obtain

$$|\partial\psi(z)| \leqslant c_\alpha(1-|z|)^{2(\alpha-1)},$$

see also [13] or [14]. This means that $\partial\psi \in L_p(\mathbb{D})$ if $2(1-\alpha)p < 1$. In the same way we establish the L_p-integrability of the derivative $\overline{\partial}\psi = \psi_2'$ under the same condition for p. The proof is complete. □

We consider the Beurling operator

$$\mathcal{B}\varphi(z) := \frac{1}{\pi}\int_{\mathbb{C}} \frac{\varphi(\zeta)d\mu}{(\zeta-z)^2}, \tag{3.1}$$

which by Calderon-Zygmund theorem [15] is a bounded mapping of the space $L_p(\mathbb{C})$ into itself for each $p \in (1,\infty)$. By $\|\mathcal{B}\|_p$ we denote its norm as of the mapping $L_p(\mathbb{C}) \to L_p(\mathbb{C})$ and in the same way we denote the norms of the operators acting in $L_p(U)$ for an arbitrary domain U. For further purposes an essential fact is that $\|\mathcal{B}\|_p \to 1$ as $p \to 2$, see [16], [17].

Proposition 3.1. *Operators* (2.10), (2.11) *possess the following properties:*
(i) $\mathcal{K}\colon L_p(\mathbb{D}) \to L_p(\mathbb{D})$ *is bounded as* $p > 1$*;*
(ii) $\mathcal{K}_\partial\colon L_p(\mathbb{D}) \to L_p(\mathbb{D})$ *is bounded as* $p > 1$ *and* $\|\mathcal{K}_\partial\|_p = \|\mathcal{B}\|_p \to 1$ *as* $p \to 2$*;*
(iii) $\mathcal{K}_{\overline{\partial}}\colon L_p(\mathbb{D}) \to L_p(\mathbb{D})$ *is bounded as* $p > 1$ *and* $\|\mathcal{K}_{\overline{\partial}}\|_p \to 1$ *as* $p \to 2$.

Proof. Property (i) is implied by the fact that the kernel of the integral operator $\mathcal{K}$ consists of the sum of two kernels with a weak singularity.

(ii) Let $\varphi \in L_p(\mathbb{D})$, $p > 1$. Then $\mathcal{K}_\partial\varphi = \mathcal{B}\varphi_1$, where the function φ_1 coincides with φ in the circle $\mathbb{D}$ and vanishes outside $\mathbb{D}$ so that $\|\varphi_1\|_{L_p(\mathbb{C})} = \|\varphi\|_{L_p(\mathbb{D})}$. Therefore, $\|\mathcal{K}_\partial\|_p = \|\mathcal{B}\|_p$.

(iii) In the integral for $\mathcal{K}_{\overline{\partial}}$ from formula (2.11) we make the change $\xi = 1/\overline{\zeta}$ and we obtain

$$\mathcal{K}_{\overline{\partial}}[\varphi(z)] = \frac{1}{\pi}\int_{\mathbb{C}\setminus\overline{\mathbb{D}}} \frac{\varphi(1/\overline{\xi})}{\xi^2}\cdot\frac{d\mu}{(\overline{\xi}-\overline{z})^2} - \varphi(z) = \overline{\mathcal{B}[\varphi_2(z)]} - \varphi(z), \tag{3.2}$$

where $\varphi_2(z) = \overline{\varphi(1/\overline{z})}/\overline{z}^2$ as $z \in \mathbb{C}\setminus\overline{\mathbb{D}}$ and $\varphi_2(z) = 0$ as $z \in \mathbb{D}$. Making the inverse change $\zeta = 1/\overline{\xi}$, we find

$$\|\varphi_2\|_{L_p(\mathbb{D})} = \left(\int_{\mathbb{C}\setminus\overline{\mathbb{D}}} |\varphi_2(\xi)|^p d\mu\right)^{\frac{1}{p}} = \left(\int_{\mathbb{D}} |\xi|^{2p-4}\cdot|\varphi(\zeta)|^p d\mu\right)^{\frac{1}{p}} \leqslant \|\varphi\|_{L_p(\mathbb{D})}$$

as $p \geqslant 2$ with the identity as $p = 2$. Then it follows from (3.2) that

$$\|\mathcal{K}_{\overline{\partial}}\varphi\|_{L_p(\mathbb{D})} \leqslant (\|\mathcal{B}\|_p + 1)\|\varphi\|_{L_p(\mathbb{D})},$$

that is, $\mathcal{K}_{\overline{\partial}}\colon L_p(\mathbb{D}) \to L_p(\mathbb{D})$ is bounded as $p > 1$.

We are going to find $\|\mathcal{K}_{\overline{\partial}}\|_2$. First let φ be an arbitrary test function from the class $C_0^2(\mathbb{D})$ of twice continuously differentiable in $\mathbb{C}$ compactly supported functions $\operatorname{supp}\varphi \subset D_r := \{|z| < r\}$, where $r \in (0,1)$. Then $\mathcal{K}[\varphi(z)] \in C(\overline{\mathbb{D}})$, see [16]. By (2.10) we have:

$$\mathcal{K}[\varphi(z)] = \frac{1-|z|^2}{\pi}\int_{\operatorname{supp}(\varphi)} \frac{\varphi(\zeta)d\mu(\zeta)}{(\zeta-z)(1-\zeta\overline{z})},$$

and for $|z| > r$ we find

$$|\mathcal{K}[\varphi(z)]| \leqslant \frac{1-|z|^2}{\pi(|z|-r)(1-r|z|)}\int_{D_r} |\varphi(\zeta)|d\mu(\zeta) \to 0 \tag{3.3}$$

as $|z| \to 1$. Since $\varphi(z) = 0$ and $\mathcal{K}[\varphi(z)] = 0$ as $z \in \mathbb{T}$, integrating by parts several times, we obtain

$$\begin{aligned}
\|\mathcal{K}_{\overline{\partial}}\varphi\|^2_{L_2(\mathbb{D})} &= \int_{\mathbb{D}} \frac{\partial}{\partial \overline{z}} \mathcal{K}[\varphi(z)] \cdot \frac{\partial}{\partial z} \overline{\mathcal{K}[\varphi(z)]} d\mu(z) = -\int_{\mathbb{D}} \overline{\mathcal{K}[\varphi(z)]} \frac{\partial^2}{\partial z \partial \overline{z}} \mathcal{K}[\varphi(z)] d\mu(z) \\
&= \int_{\mathbb{D}} \overline{\mathcal{K}[\varphi(z)]} \frac{\partial \varphi(z)}{\partial z} d\mu(z) = -\int_{\mathbb{D}} \varphi(z) \frac{\partial}{\partial z} \overline{\mathcal{K}[\varphi(z)]} d\mu(z) \\
&= -\int_{\mathbb{D}} \varphi(z) \left(\frac{1}{\pi} \int_{\mathbb{D}} \frac{\overline{\varphi(\zeta)} d\mu(\zeta)}{(1-\overline{\zeta} z)^2} - \overline{\varphi(z)} \right) d\mu(z) \\
&= \|\varphi\|^2_{L_2(\mathbb{D})} - \frac{1}{\pi} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{\varphi(z)\overline{\varphi(\zeta)}}{(1-\overline{\zeta} z)^2} d\mu(\zeta) d\mu(z).
\end{aligned}$$

The quantity deducted from $\|\varphi\|^2_{L_2(\mathbb{D})}$ is equal to

$$\begin{aligned}
\frac{1}{\pi} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{\varphi(z)\overline{\varphi(\zeta)}}{(1-\overline{\zeta} z)^2} d\mu(\zeta) d\mu(z) &= \sum_{n=0}^{\infty} (n+1) \frac{1}{\pi} \int_{\mathbb{D}} \int_{\mathbb{D}} \varphi(z) \overline{\varphi(\zeta)} z^n \overline{\zeta}^n d\mu(\zeta) d\mu(z) \\
&= \sum_{n=0}^{\infty} (n+1) \frac{1}{\pi} \left| \int_{\mathbb{D}} \varphi(z) z^n d\mu(z) \right|^2 \geqslant 0,
\end{aligned}$$

and this is why

$$\|\mathcal{K}_{\overline{\partial}}\varphi\|_{L_2(\mathbb{D})} \leqslant \|\varphi\|_{L_2(\mathbb{D})}$$

and the identity is achieved at the functions $\varphi \in C^2_0(\mathbb{D})$, for which $\int_{\mathbb{D}} \varphi(z) z^n d\mu(z) = 0$ as $n = 0, 1, \dots$. Approximating the functions from $L_p(\mathbb{D})$ by the functions from the class $C^2_0(\mathbb{D})$, we obtain the same estimate. Thus, $\|\mathcal{K}_{\overline{\partial}}\|_2 = 1$. The norm $\|\mathcal{K}_{\overline{\partial}}\|_p$ is well-defined for all $p > 1$. Then the Riesz-Thorin theorem [16], which states that the quantity $\log \|\mathcal{K}_{\overline{\partial}}\|_p$ is a convex function of the variable $1/p$, implies the continuity of this quantity with respect to p. Therefore, $\|\mathcal{K}_{\overline{\partial}}\|_p \to 1$ as $p \to 2$. The proof is complete. □

Proof of Theorem 2.1. **Step 1.** First we are going to establish the convergence of series (2.5) with the first partial derivatives in the norm of the space $L_p(\mathbb{D})$. It follows from Lemma 3.1 that the function $F_0 = \mathcal{P}[h]$ belongs to the Sobolev space $W^1_p(\mathbb{D})$ as $p < (2(1-\alpha))^{-1}$. Suppose that $F_{n-1} \in L_p(\mathbb{D})$, $p > 2$, for some index n. Then

$$\partial F_n = \mathcal{K}_{\partial}[(\overline{\omega'}/\omega')(\alpha_0 \partial F_{n-1} + \beta_0 \partial \overline{F_{n-1}})], \qquad \overline{\partial} F_n = \mathcal{K}_{\overline{\partial}}[(\overline{\omega'}/\omega')(\alpha_0 \partial F_{n-1} + \beta_0 \partial \overline{F_{n-1}})]$$

in the sense of distributions, see [16]. Employing these relations and applying Proposition 3.1 as well as the identity $|\alpha_0| + |\beta_0| = 1$, by formula (2.12) we deduce estimates

$$\begin{aligned}
\|\partial F_n\|_{L_p(\mathbb{D})} &\leqslant \|\mathcal{K}_{\partial}\|_p \max\{\|\partial F_{n-1}\|_{L_p(\mathbb{D})}, \|\overline{\partial} F_{n-1}\|_{L_p(\mathbb{D})}\}, \\
\|\overline{\partial} F_n\|_{L_p(\mathbb{D})} &\leqslant \|\mathcal{K}_{\overline{\partial}}\|_p \max\{\|\partial F_{n-1}\|_{L_p(\mathbb{D})}, \|\overline{\partial} F_{n-1}\|_{L_p(\mathbb{D})}\},
\end{aligned}$$

which imply

$$\|DF_n\|_{L_p(\mathbb{D})} \leqslant \|\mathcal{D}\mathcal{K}\|_p \cdot \|DF_{n-1}\|_{L_p(\mathbb{D})}, \tag{3.4}$$

where

$$\|DF_n\|_{L_p(\mathbb{D})} := \max\{\|\partial F_n\|_{L_p(\mathbb{D})}, \|\overline{\partial} F_n\|_{L_p(\mathbb{D})}\}, \qquad \|\mathcal{D}\mathcal{K}\|_p := \max\{\|\mathcal{K}_{\partial}\|_p, \|\mathcal{K}_{\overline{\partial}}\|_p\}.$$

By (2.12) this yields

$$\|F_n\|_{L_p(\mathbb{D})} \leqslant \|\mathcal{K}\|_p \cdot \|DF_{n-1}\|_{L_p(\mathbb{D})} \leqslant \|\mathcal{K}\|_p \cdot \|\mathcal{D}\mathcal{K}\|^{n-1}_p \cdot \|DF_0\|_{L_p(\mathbb{D})}. \tag{3.5}$$

Estimate (3.5) proves that under the condition $\|\mathcal{DK}\|_p \cdot \|T\| < 1$ series (2.5) converges to its sum $F \in L_p(\mathbb{D})$ and

$$\begin{aligned} \|F\|_{L_p(\mathbb{D})} &= \left\| \sum_{n=0}^{\infty} F_n \|T\|^n \right\|_{L_p(\mathbb{D})} \leqslant \|F_0\|_{L_p(\mathbb{D})} + \sum_{n=1}^{\infty} \|F_n\|_{L_p(\mathbb{D})} \cdot \|T\|^n \\ &\leqslant \|F_0\|_{L_p(\mathbb{D})} + \sum_{n=1}^{\infty} \|\mathcal{K}\|_p \cdot \|\mathcal{DK}\|_p^{n-1} \cdot \|DF_0\|_{L_p(\mathbb{D})} \cdot \|T\|^n \\ &= \|F_0\|_{L_p(\mathbb{D})} + \frac{\|\mathcal{K}\|_p \cdot \|T\|}{1 - \|\mathcal{DK}\|_p \cdot \|T\|} \|DF_0\|_{L_p(\mathbb{D})}. \end{aligned} \tag{3.6}$$

Estimate (3.4) also proves that the first partial derivatives of series (2.5) converge to corresponding derivatives of the function F in the same norm:

$$\|DF\|_{L_p(\mathbb{D})} \leqslant \frac{\|DF_0\|_{L_p(\mathbb{D})}}{1 - \|\mathcal{DK}\|_p \cdot \|T\|}, \tag{3.7}$$

where

$$\|DF\|_{L_p(\mathbb{D})} := \max\{\|\partial F\|_{L_p(\mathbb{D})}, \|\overline{\partial} F\|_{L_p(\mathbb{D})}\}.$$

Estimates (3.6) and (3.7) yield the convergence of series (2.5) in the norm of the Sobolev space $W_p^1(\mathbb{D})$:

$$\lim_{m\to\infty} \|F - S_m\|_{W_p^1(\mathbb{D})} = 0. \tag{3.8}$$

By the Sobolev embedding theorem [18], $W_p^1(\mathbb{D}) \subset C(\overline{\mathbb{D}})$ as $1 - 2/p > 0$, or $p > 2$, and this embedding is compact. Hence, since $F \in W_p^1(\mathbb{D})$, then $F \in C(\overline{\mathbb{D}})$ and $f = F \circ \omega^{-1} \in C(\overline{\Omega})$. By the compactness of the embedding, series (2.5), and hence series (2.1) converge uniformly in $\overline{\mathbb{D}}$ and $\overline{\Omega}$, respectively.

Step 2. Now we are going to show that the function $f = F \circ \omega^{-1}$ solves equation $\mathcal{L}f = 0$ in the domain Ω. First we shall show this identity in the generalized sense.

Let ϕ be an arbitrary test function from the class $C_0^2(\mathbb{D})$ and

$$\langle g|\phi\rangle := \int_{\mathbf{C}} g(z)\phi(z)d\mu$$

is the action of the distribution g on a function ϕ. We have:

$$\begin{aligned} \langle F_n|\partial\overline{\partial}\phi\rangle &= \int_{\mathbb{D}} \partial\overline{\partial}\phi(z)d\mu(z) \int_{\mathbb{D}} \partial_\zeta G(\zeta, z)\frac{\overline{\omega'(\zeta)}}{\omega'(\zeta)}\partial(T_0F_{n-1}(\zeta))d\mu(\zeta) \\ &= \int_{\mathbb{D}} \frac{\overline{\omega'(\zeta)}}{\omega'(\zeta)}\partial(T_0F_{n-1}(\zeta))d\mu(\zeta)\partial_\zeta \int_{\mathbb{D}} G(\zeta, z)\partial\overline{\partial}\phi(z)d\mu(z) \\ &= \int_{\mathbb{D}} \frac{\overline{\omega'(\zeta)}}{\omega'(\zeta)}\partial(T_0F_{n-1}(\zeta))\partial\phi(\zeta)d\mu(\zeta) = \langle(\overline{\omega'}/\omega')\partial(T_0F_{n-1})|\partial\phi\rangle. \end{aligned}$$

This implies the identity $\partial\overline{\partial}F_n = -\partial[(\overline{\omega'}/\omega')\partial(T_0F_{n-1})]$ of the generalized derivatives in $\mathbb{D}$. By (2.4) this implies the following chain of identities for the generalized functions

$$\begin{aligned}|\omega'|^2\mathcal{L}s_m &= \sum_{n=0}^{m}\left(\partial\overline{\partial}F_n + \|T\|\partial\left(\frac{\overline{\omega'}}{\omega'}\partial(T_0F_n)\right)\right)\|T\|^n\\ &= \partial\overline{\partial}F_0 + \sum_{n=1}^{m}\left(\partial\overline{\partial}F_n + \partial\left(\frac{\overline{\omega'}}{\omega'}\partial(T_0F_{n-1})\right)\right)\|T\|^n + \partial\left(\frac{\overline{\omega'}}{\omega'}\partial(T_0F_m)\right)\|T\|^{m+1}\\ &= \partial\left(\frac{\overline{\omega'}}{\omega'}\partial(T_0F_m)\right)\|T\|^{m+1},\end{aligned}$$

that is, for each function $\varphi \in C_0^2(\Omega)$, using the function $\phi := \varphi\circ\omega \in C_0^2(\mathbb{D})$, we can write

$$\langle\mathcal{L}s_m\,|\,\varphi\rangle = \langle\partial\left[(\overline{\omega'}/\omega')\partial(T_0F_m)\right]\,|\,\phi\rangle\cdot\|T\|^{m+1} = -\langle\partial(T_0F_m)\,|\,(\overline{\omega'}/\omega')\partial\phi\rangle\cdot\|T\|^{m+1}.$$

Then

$$\begin{aligned}\langle\mathcal{L}f\,|\,\varphi\rangle &:= \langle f|\mathcal{L}\varphi\rangle = \langle f - s_m|\mathcal{L}\varphi\rangle + \langle s_m|\mathcal{L}\varphi\rangle\\ &= \langle F - S_m|\mathcal{M}\phi\rangle - \langle\partial(T_0F_m)|(\overline{\omega'}/\omega')\partial\phi\rangle\cdot\|T\|^{m+1}.\end{aligned}$$

We let $1/p + 1/q = 1$. Applying the Hölder inequality and taking into consideration relations (3.4) and (3.8), we get:

$$\begin{aligned}|\langle\mathcal{L}f\,|\,\varphi\rangle| &\leqslant \|F - S_m\|_{L_p(\mathbb{D})}\cdot\|\mathcal{M}\phi\|_{L_q(\mathbb{D})} + \|DF_m\|_{L_p(\mathbb{D})}\cdot\|\partial\phi\|_{L_q(\mathbb{D})}\cdot\|T\|^{m+1}\\ &\leqslant \|F - S_m\|_{L_p(\mathbb{D})}\cdot\|\mathcal{M}\phi\|_{L_q(\mathbb{D})} + \|DF_0\|_{L_p(\mathbb{D})}\cdot\|\partial\phi\|_{L_q(\mathbb{D})}\cdot\|\mathcal{DK}\|_p^m\cdot\|T\|^{m+1} \to 0\end{aligned}$$

as $m\to\infty$ and $\|\mathcal{DK}\|_p\cdot\|T\| < 1$. Thus, $\langle\mathcal{L}f\,|\,\varphi\rangle = 0$, that is, the function f solves the equation $\mathcal{L}f = 0$ in Ω in the generalized sense. By the ellipticity of this equation and the Weyl lemma, it is satisfied also in the classical sense.

Step 3. It remains to show that $f|_\Gamma = h$. It follows from estimates (3.4) and (3.5) that $F_n \in W_p^1(\mathbb{D})$. By the Sobolev embedding theorem as $p > 2$ this implies that $F_n \in C(\overline{\mathbb{D}})$. Since the function F_0 is a harmonic continuation of the boundary function $H \in C^\alpha(\mathbb{T})$, we have $F_0|_\mathbb{T} = H$. Other functions F_n calculated by the second formula in (2.12) vanish on $\mathbb{T}$: this fact can be shown by approximating the function $(\overline{\omega'}/\omega')\partial(T_0F_{n-1}) \in L_p(\mathbb{D})$, $p\in(1,\infty)$, by compactly supported functions from $C_0^2(\mathbb{D})$ for $n \geqslant 1$ and applying then estimate (3.3).

Thus, $S_m|_\mathbb{T} = H$ for all m. The compact embedding $W_p^1(\mathbb{D}) \subset C(\overline{\mathbb{D}})$ and convergence (3.8) yield the uniform convergence $\|F - S_m\|_{C(\overline{\mathbb{D}})} \to 0$ and hence $F|_\mathbb{T} = S_m|_\mathbb{T} = H$. Therefore, $f|_\Gamma = h$.

The above arguing is true under the inequalities $2 < p < (2(1-\alpha))^{-1}$, which are compatible owing to $\alpha \in (1/2, 1)$. Since $\|\mathcal{DK}\|_p \to 1$ as $p \to 2$, for each value $\|T\| < 1$ we can find a sufficiently close to 2 value of p, under which $\|\mathcal{DK}\|_p\cdot\|T\| < 1$. The proof is complete. □

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