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GENETIC ALGORITHM APPLIED TO FRACTIONAL OPTIMAL CONTROL OF A DIABETIC PATIENT

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Abstract. Diabetes is a dangerous disease that increases in incidence every year. The aim of this paper is to present and analyze the model of diabetes and its complications with the fractional derivative of Caputo, namely, we propose a mathematical model with a fractional derivative of the type 2 diabetes. The positivity and boundedness of the solutions is demonstrated by the Laplace transform method. We study the existence and uniqueness of the solution of the system. We use the genetic algorithm (GA) to solve the fractional differential equation model and to characterize the optimal control and this is an efficient and simple metaheuristic method to implement. Simulations of the total number of diabetics with the different values of a parameter α show that the combined control strategy leads to a significant decrease. The simulation results also show that the number of uncomplicated diabetics in the fractional model, for the different fractional values of α, decreases more rapidly than the integer derivative model.

Keywords: Diabetic population dynamic system, Optimal control, Fractional derivative, Genetic algorithm, Artificial intelligent

Mathematics Subject Classification: 34B45, 81Q15

1. Introduction

The development of technological mechanisms helps to solve most of the scientific challenges. For example, fractional derivatives are widely used in various fields such as engineering, economics, mathematics, physics and other scientific branches. The already done studies show that fractional derivatives give the behaviors of certain dynamic models in a very accurate way and they do this much better the derivatives of integer order, and this provided a better efficiency for engineering dynamic systems. In recent years, dynamic models of controlled fractions were developed, specifically diabetic population models, with the aim of studying the phenomenon of diabetes more precisely and accurately [1], [2], [3], [5]. The creators of dynamic models of this phenomenon considered several compartments, for example, the compartment of pre-diabetic people, the compartment of uncomplicated diabetic people, the compartment of people with complications, the compartment of healthy people and other types [6], [38].

Several studies were conducted on the type 2 diabetes in order to reduce the burden of this disease [4], [10], [11], [12], [13]. Among these studies, we can mention the dynamic studies that focus on the study of the phenomenon and the study of the stability of the proposed systems, and other studies that use optimal control strategies to reduce the negative effects of this disease [14], [9], [15], [16], [17].

In this work, we propose a fractional optimal control problem of a dynamic system of the diabetic population to reduce these negative socio-economic effects. We propose the genetic

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algorithm for the solution of this problem, as an accurate intelligent method that shows a very high efficiency.

In [7], the authors developed a multi-objective genetic algorithm (MOGA) to tune the gains of the fractional-order proportional-integral-derivative (FOPID) controller. Over the years, an interesting study of multi-objective optimal control was performed on dengue disease [8]. The authors reformulated a single objective mosquito population dynamic model into a multi-objective mosquito population dynamic model of minimization of several objective functions by the proposed multi-objective genetic algorithm (MGADENGUE), for more details see [8].

Diabetic is a global pandemic that has been ranked among the greatest health challenges facing humanity in the twenty-first century. In general, there are three well-known types of diabetes, but we focus on the type 2diabetes because this type is the most frequent and accounts for about 90% of all diabetes cases worldwide. Health professionals emphasize the need for early detection of diabetes in order to provide timely intervention, which has a positive impact on improving public health. Especially since people with pre-diabetes have no signs or symptoms according to the World Health Organization (WHO) reports.

Studies have been done for the phenomenon of the diabetes model in a fractional order dynamical system[19]. Let us mention them: Goharimanesh et al [20] proposed to study the nonlinear dynamic model of type 1 diabetes, they used the fractional order PID controller as one of the best dynamic system control. They also used the genetic algorithm to estimate the parameters and the control. Jagdev Singh et al [21] modified a diabetic population dynamics model by replacing the entire derivative with an arbitrary Caputo-Fabrizio order. They studied the existence and uniqueness of the solution using the fixed point and other methods and also used the homotopy analysis method, Laplace transform and Pad approximation to solve the proposed fractional model and to modify the convergence of the solution. Numerical simulation was done to enrich the efficiency of the proposed model to reduce the effect of the diabetes phenomenon. In 2019, a study was done on the dynamic model of a fractional order diabetic population by [22]. They introduced a fractionary model of diabetes that focused on two types of individuals, the diabetic person without and with complication. They also studied the existence and uniqueness of the solution and they studied the stability of the solution and they used the homotopy decomposition method (HDM) for the resolution of the proposed fractional order diabetic model. Another study was also conducted to investigate the impact of diabetes on the epidemiology of TB [23]. The authors proposed a dynamic fractional model. They considered two types of diabetic population, namely diabetic patients and non-diabetic patients. They solved the model numerically and studied the stability of the system, then proposed a fractional control model and an optimal control strategy to reduce the number of infected individuals. Andrew Omame et al [24] proposed a fractional optimal control model for the COVID-19 and diabetes co-infected population. The derivative used in this paper is the Atangana-Baleanu derivative. They proved positivity and boundedness of solutions, existence and uniqueness of solutions. They also studied the stability of the proposed fracional system. They proposed an optimal control strategy and the simulation results show that the number of patients co-infected with COVID-19 and diabetes reduced if it will have a compliance of policies and measures as COVID-19 vaccine and face mask.

Our paper is organized as follows. In the second section we present some preliminaries on the fractional derivative of Caputo that we have to use. In the third section, a new dynamic system of the diabetic population is proposed with a study on the existence, uniqueness, positivity and boundedness of the solution. A control strategy is also proposed in the fourth section, with the characterization of the objective function, and the determination of the adjoint values are all done. In the fifth section, we characterize our optimal control that minimizes the objective

function. In the sixth section, we present the different simulations found using the genetic algorithm and we end this work with a conclusion that also contains some perspectives.

2. Fractional calculus

We start with a brief preliminary of the fractional calculations we employ in this work, more precisely, some properties on the fractional derivatives of Riemann-Liouville and Caputo respectively.

Definition 2.1. *The Riemann-Liouville fractional integrals of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ are given by: the Left Riemann-Liouville Fractional Derivative*

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \tag{2.1} $$

and the Right Riemann-Liouville Fractional Derivative

$$ {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, d\tau. \tag{2.2} $$

Definition 2.2. *Let $f \in AC([a,b])$, where $AC([a,b])$ represents the space of absolutely continuous functions on $[a,b]$, $\alpha > 0$ and $n = [\alpha] + 1$. The Left Caputo Fractional Derivative is defined as*

$$ {}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-1-\alpha} f^{(n)}(\tau) d\tau \tag{2.3} $$

and the Right Caputo Fractional Derivative is defined as

$$ {}_t^cD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-1-\alpha} f^{(n)}(\tau)\, d\tau. \tag{2.4} $$

The Laplace transform of the caputo fractional derivative (CFD) is

$$ L[{}_0^cD_t^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} f^{(k)}(0) s^{\alpha-n-1}, \tag{2.5} $$

where $F(s)$ is the Laplace transform

$$ F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt. $$

Definition 2.3. *A two parameter Mittag-Leffler function is defined as*

$$ E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+\beta)}, \qquad \alpha > 0. \tag{2.6} $$

The Mittag-Leffler function solves the equation

$$ E_{\alpha,\beta}(t) = t \times E_{\alpha,\alpha+\beta}(t) + \frac{1}{\Gamma(\beta)}. \tag{2.7} $$

Definition 2.4. *The Laplace transform of Mittag-Leffler function $t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^\alpha))$ is defined as*

$$ L[t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^\alpha))] = \frac{s^{\alpha-\beta}}{s^\alpha \mp \lambda}. \tag{2.8} $$

3. Fractional Diabetes model

In [18] Boutayeb et al. have proposed an optimal control approach modeling the progression from pre-diabetes to diabetes with and without complications. Three types of diabetic patients were considered, patients who become diabetic for different reasons, which may be genetic or related to a negative lifestyle, this type is noted by E. The other two types are diabetic patients without complications and diabetic patients with complications, which are denoted by D and C respectively. We conclude that the diabetic population N is divided into three types such that $N = N(t)$ is the whole of the all compartments $E(t)$, $D(t)$ and $C(t)$ at a time t.

We propose a new fractional-order dynamic optimal control model (3.1) to determine the control u. Our objective is to minimize the cost associated with control to decrease the number of diabetic populations.

$$\begin{cases} {}_0^c\mathrm{D}_t^\alpha E(t) = I - (\mu + \beta_3 + \beta_1)E(t) \\ {}_0^c\mathrm{D}_t^\alpha D(t) = \beta_1 E(t) - (\mu + \beta_2)D(t) + \gamma C(t) \\ {}_0^c\mathrm{D}_t^\alpha C(t) = \beta_3 E(t) + \beta_2 D(t) - (\mu + \gamma + \nu + \delta)C(t), \end{cases} \tag{3.1}$$

where E, D and C represent the number of pre-diabetics, uncomplicated diabetics and complicated diabetics respectively, while $N = N(t) = E(t) + C(t) + D(t)$ stands for the size of the population of diabetics and pre-diabetics at a time t. In adition, I denotes the incidence of pre-diabetes, μ is the natural mortality rate, β_1 is the the probability of developing diabetes, β_3 is the the probability of developing diabetes at the stage of complications, γ is the rate at which complications are cured, ν is the rate at which patients with complications become severely disabled, and δ is the mortality rate due to complications, see Figure 1.

Theorem 3.1. *Let*

$$\Omega = \left\{ (E, D, C) \in \mathbb{R}^3 \mid 0 \leqslant C, D, E \leqslant \frac{I}{\mu} \right\}.$$

The region Ω is a positive invariant set for system (3.1).

Proof. The sum of all equations of (3.1) gives

$${}_0^c\mathrm{D}_t^\alpha N(t) =_0^c \mathrm{D}_t^\alpha (E(t) + D(t) + C(t)) = I - \mu E(t) - \mu D(t) - \mu C(t) - (\nu + \delta)C(t) \leqslant I - \mu N(t).$$

Therefore,

$${}_0^c\mathrm{D}_t^\alpha N(t) \leqslant I - \mu N(t).$$

Then by applying the Laplace transform we find

$$s^\alpha F(N) - s^{\alpha-1} N(0) \leqslant \frac{I}{s} - \mu F(N)$$

and we obtain

$$F(N)(s^\alpha + \mu) \leqslant \frac{I}{s} + s^{\alpha-1} N(0),$$

which can be written as

$$F(N) \leqslant I \frac{s^{-1}}{s^\alpha + \mu} + \frac{s^{\alpha-1}}{s^\alpha + \mu} N(0).$$

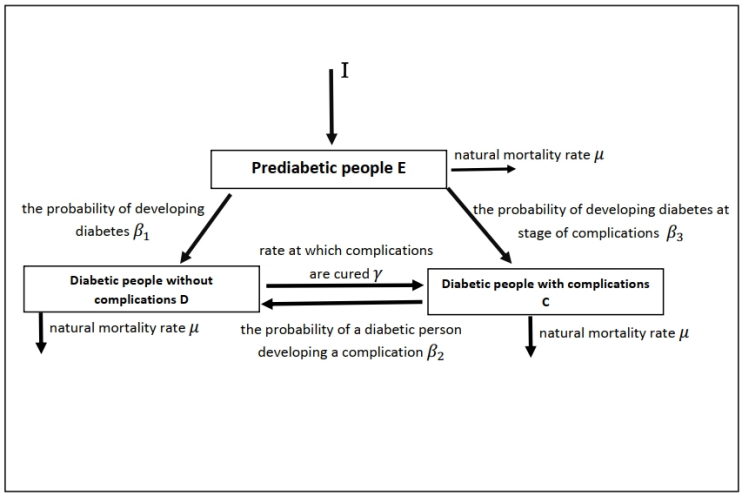


Figure 1: Description of the model

As we have by (2.7) and (2.8), if $N(0) \in \Omega$, then

$$\begin{aligned} N(t) &\leqslant It^{\alpha} E_{\alpha,\alpha+1}\left(-\mu t^{\alpha}\right) + E_{\alpha,1}\left(-\mu t^{\alpha}\right) N(0) \\ &\leqslant \frac{I}{\mu} \mu t^{\alpha} E_{\alpha,\alpha+1}\left(-\mu t^{\alpha}\right) + E_{\alpha,1}\left(-\mu t^{\alpha}\right) N(0) \\ &\leqslant \frac{I}{\mu}\left(\mu t^{\alpha} E_{\alpha,\alpha+1}\left(-\mu t^{\alpha}\right) + E_{\alpha,1}\left(-\mu t^{\alpha}\right)\right) \leqslant \frac{I}{\mu} \frac{1}{\Gamma(1)} = \frac{I}{\mu} \end{aligned}$$

Finally, we have $0 < N(t) \leqslant \dfrac{I}{\mu}$. Then, Ω is a positive invariant set. The proof is complete. □

Corollary 3.1. *The solution of the fractional order differential system (3.1) with the initial condition X_0 is bounded.*

3.1. Existence and uniqueness.

Let

$$ {}_0^c\mathrm{D}_t^{\alpha} X(t) = \chi(t, X), $$

where $X(t) = (E, D, C)$, $\chi(X, t) = (\chi_1(X, t), \chi_2(X, t), \chi_3(X, t))$ and

$$\begin{aligned} \chi_1(X, t) &= I - (\mu + \beta_3 + \beta_1) E(t), \\ \chi_2(X, t) &= \beta_1 E(t) - (\mu + \beta_2) D(t) + \gamma C(t), \\ \chi_3(X, t) &= \beta_3 E(t) + \beta_2 D(t) - (\mu + \gamma + \nu + \delta) C(t). \end{aligned}$$

The following result prove the unique solvability of (3.1).

Theorem 3.2. [36] *Fractional-order system of differential (3.1) with initial condition $X_0 = (E(0), D(0), C(0))$ possesses a unique solution $X(t) = (E, D, C) \in \Omega$ for all $t > 0$.*

Proof. In order to prove the unique solvability of system (3.1), we need to prove that the function χ is Lipschitz. We have

$$\|\chi(X_1) - \chi(X_2)\| = |\chi_1(X_1) - \chi_1(X_2)| + |\chi_2(X_1) - \chi_2(X_2)| + |\chi_3(X_1) - \chi_3(X_2)|$$

and

$$
\begin{aligned}
|\chi_1(X_1) - \chi_1(X_2)| &= |-(\mu+\beta_3+\beta_1)E_1 + (\mu+\beta_3+\beta_1)E_2|\,,\\
|\chi_2(X_1) - \chi_2(X_2)| &= |\beta_1 E_1 - (\mu+\beta_2)D_1 + \gamma C_1 - \beta_1 E_2 + (\mu+\beta_2)D_2 - \gamma C_2|\,,\\
|\chi_3(X_1) - \chi_3(X_2)| &= |\beta_3 E_1 + \beta_2 D_1 - (\mu+\gamma+\nu+\delta)C_1 - \beta_3 E_2 - \beta_2 D_2 + (\mu+\gamma+\nu+\delta)C_2|\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\chi_1(X_1) - \chi_1(X_2)| &= |(\mu+\beta_3+\beta_1)(E_1 - E_2)|\,,\\
|\chi_2(X_1) - \chi_2(X_2)| &= |\beta_1(E_1 - E_2) + (\mu+\beta_2)(D_2 - D_1) + \gamma(C_1 - C_2)|\,,\\
|\chi_3(X_1) - \chi_3(X_2)| &= |\beta_3(E_1 - E_2) + \beta_2(D_1 - D_2) + (\mu+\gamma+\nu+\delta)(C_2 - C_1)|\,,
\end{aligned}
$$

and

$$
\begin{aligned}
|\chi_1(X_1) - \chi_1(X_2)| &= (\mu+\beta_3+\beta_1)\,|E_1 - E_2|\,,\\
|\chi_2(X_1) - \chi_2(X_2)| &\leqslant \beta_1\,|E_1 - E_2| + (\mu+\beta_2)\,|D_1 - D_2| + \gamma\,|C_1 - C_2|\,,\\
|\chi_3(X_1) - \chi_3(X_2)| &\leqslant \beta_3\,|E_1 - E_2| + \beta_2\,|D_1 - D_2| + (\mu+\gamma+\nu+\delta)\,|C_2 - C_1|\,.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\|\chi(X_1) - \chi(X_2)\| \leqslant & (\mu+2\beta_3+2\beta_1)\,|E_1 - E_2| + (\mu+2\beta_2)\,|D_1 - D_2|\\
& + (\mu+2\gamma+\nu+\delta)\,|C_2 - C_1|\\
\leqslant & L\,\|X_1 - X_2\|
\end{aligned}
$$

with $L := \max\{\mu+2\beta_3+2\beta_1, \mu+2\beta_2, \mu+2\gamma+\nu+\delta\}$. Hence, the function χ is uniformly Lipshitz and consequently, there exists a unique solution to system (3.1) with the initial state X_0. The proof is complete. □

4. Fractional optimal control problem

In this section we propose a first fractional optimal dynamic control system for diabetics. In the following, we provide various theorems on the conditions of minimization of the objective function, in particular, the conditions of optimality.

$$
\begin{cases}
{}_0^c\mathrm{D}_t^\alpha E(t) = I - (\mu + (\beta_3+\beta_1)(1-\omega(t)))E(t)\\
{}_0^c\mathrm{D}_t^\alpha D(t) = \beta_1(1-\omega(t))E(t) - (\mu+\beta_2(1-\omega(t)))D(t) + \gamma C(t)\\
{}_0^c\mathrm{D}_t^\alpha C(t) = \beta_3(1-\omega(t))E(t) + \beta_2(1-\omega(t))D(t) - (\mu+\gamma+\nu+\delta)C(t),
\end{cases}
\tag{4.1}
$$

where ω is a control. The objective function is defined as

$$
J(\omega) = \int_0^T \left(D(t) + C(t) + \frac{A}{2}\omega^2(t) \right) dt \qquad A > 0. \tag{4.2}
$$

We define the control set U as

$$
U = \big\{\omega :\ \omega \text{ is measurable},\ \omega(t) \in [0,1] \text{ for all } t \in [0,T]\big\}.
$$

We are going to solve the constrained optimization problem, that is, we need a control ω^* which belongs to the set of controls U defined earlier and which minimizes the objective function $J(\omega)$, that is,

$$
J(w^*) = \min_{w \in U} J(w).
$$

Theorem 4.1. *If (x, ω) is a minimizer of*

$$\begin{cases} \min\ J(w), \\ \textit{under dynamic constraint } (4.1) \\ x(0) = x_0, \end{cases}$$

then there exist a function λ in class C^1 in $[0, b]$ such that (x, w, λ) satisfies the following properties:

the state and co-state systems:

$$ {}_t^c\mathrm{D}_b^\alpha x(t) = \frac{\partial H}{\partial \lambda}(t, x(t), \omega(t), \lambda(t)) \tag{4.3}$$

$$ {}_0^c\mathrm{D}_t^\alpha \lambda(t) = \frac{\partial H}{\partial t}(t, x(t), \omega(t), \lambda(t)); \tag{4.4}$$

the stationary condition:

$$\frac{\partial H}{\partial u}(t, x(t), \omega(t), \lambda(t)) = 0 \tag{4.5}$$

and the transversality condition

$$\lambda(b) = 0 \tag{4.6}$$

with H is the Hamiltonian defined as

$$H(t, x, w, \lambda) = f(t, x, w) + \lambda g(t, x, w),$$

where $x(t) = (E(t), D(t), C(t))$ and $g(t, x, w) =_0^c \mathrm{D}_t^\alpha x(t)$.

Proof. We start by rewriting our cost function given by the equation (4.2) in the following form:

$$J(u) = \int_0^b \left[H(t, x(t), \omega(t), \lambda(t)) - \lambda {}_0^c\mathrm{D}_t^\alpha x(t)\right] dt, \tag{4.7}$$

where λ is the adjoint variable. Taking a variation of (4.7), we obtain:

$$\delta J(u) = \int_0^b \left[\delta x \frac{\partial H}{\partial x} + \delta u \frac{\partial H}{\partial u} + \delta\lambda \frac{\partial H}{\partial \lambda} - \delta\lambda {}_0^c\mathrm{D}_t^\alpha x - \lambda\delta({}_0^c\mathrm{D}_t^\alpha x)\right] dt, \tag{4.8}$$

where δx, δu and $\delta\lambda$ are the variations of x, u and λ, respectively. Then by integration by parts, for Caputo fractional derivatives we have the following result.

$$\int_0^b \lambda(t)\delta({}_0^c\mathrm{D}_t^\alpha x)\, dt = \left({}_t\mathrm{I}_b^{1-\alpha}\lambda(t)\right)\left[\delta x(t)\right]_0^b - \int_0^b \delta x.({}_t^c\mathrm{D}_b^\alpha \lambda(t))\, dt. \tag{4.9}$$

Since $x(0)$ is specified, then $\delta x(0) = 0$. Therefore,

$$\delta J = \int_0^b \left[\delta x \left(\frac{\partial H}{\partial x} - {}_t\,\mathrm{D}_b^\alpha \lambda(t)\right) + \delta u \frac{\partial H}{\partial u} + \delta\lambda \left(\frac{\partial H}{\partial \lambda} - {}_0^c\,\mathrm{D}_t^\alpha\right)\right] dt - \left({}_t\mathrm{I}_b^{1-\alpha}\lambda(t)\right)\left[\delta x(t)\right]_{t=b}. \tag{4.10}$$

The necessary condition for the optimality of the FOCP is $\delta J = 0$. This requires that the coefficients of δx, δu and $\delta\lambda$ in (4.10) be zero. Therefore,

$$
\begin{aligned}
&{}_{t}^{c}\mathrm{D}_{b}^{\alpha}x(t) = \frac{\partial H}{\partial \lambda}(t, x(t), \omega(t), \lambda(t)), \qquad (4.11)\\
&{}_{0}^{c}\mathrm{D}_{t}^{\alpha}\lambda(t) = \frac{\partial H}{\partial t}(t, x(t), \omega(t), \lambda(t)),\\
&\frac{\partial H}{\partial u}(t, x(t), \omega(t), \lambda(t)) = 0,\\
&\left[{}_{t}\mathrm{I}_{b}^{1-\alpha}\lambda(t)\right]_{t=b} = 0. \qquad (4.12)
\end{aligned}
$$

Since we require λ to be continuous, the condition $\left[{}_{t}\mathrm{I}_{b}^{1-\alpha}\lambda(t)\right]_{t=b} = 0$ yields $\lambda(b) = 0$. Therefore, equations (4.11) and (4.12) are equivalent to the following ones:

$$
\begin{aligned}
&{}_{t}^{c}\mathrm{D}_{b}^{\alpha}x(t) = \frac{\partial H}{\partial \lambda}(t, x(t), \omega(t), \lambda(t)),\\
&{}_{0}^{c}\mathrm{D}_{t}^{\alpha}\lambda(t) = \frac{\partial H}{\partial t}(t, x(t), \omega(t), \lambda(t)),\\
&\frac{\partial H}{\partial u}(t, x(t), \omega(t), \lambda(t)) = 0,\\
&\lambda(b) = 0.
\end{aligned}
$$

The proof is complete. □

We are going to rewrite the conditions necessary for the optimality of our fractional problem. To achieve this goal, we shall convert the equation (4.4) into a left fractional derivative equation. We need the following lemma.

Lemma 4.1. *Two following equations are equivalent:*

$$
{}_{t}^{c}\mathrm{D}_{b}^{\alpha}\lambda(t) = \frac{\partial H}{\partial x}(t, x(t), \omega(t), \lambda(t)), \qquad (4.13)
$$

and

$$
{}_{0}^{c}\mathrm{D}_{t}^{\alpha}\lambda(b-t) = \frac{\partial H}{\partial x}(b-t, x(b-t), w(b-t), \lambda(b-t)). \qquad (4.14)
$$

Proof. By (2.3) and (2.4) and some simply calculations we show that

$$
{}_{b-t}^{c}\mathrm{D}_{b}^{\alpha}\lambda(b-t) = {}_{0}^{c}\mathrm{D}_{t}^{\alpha}\lambda(b-t).
$$

Indeed, by changing t to $b-t$ in

$$
{}_{t}^{c}\mathrm{D}_{b}^{\alpha}\lambda(t) = \frac{-1}{\Gamma(1-\alpha)}\int_{t}^{b}(\tau - t)^{-\alpha}\lambda'(\tau)\,d\tau,
$$

we have

$$
{}_{b-t}^{c}\mathrm{D}_{b}^{\alpha}\lambda(b-t) = \frac{-1}{\Gamma(1-\alpha)}\int_{b-t}^{b}(\tau - (b-t))^{-\alpha}\lambda'(\tau)\,d\tau.
$$

We let $z = b - \tau$, then

$$\begin{aligned}
{}^{c}_{b-t}\mathrm{D}^{\alpha}_{b}\lambda(b-t) &= \frac{-1}{\Gamma(1-\alpha)}\int_{t}^{0}(t-z)^{-\alpha}\lambda'(b-z)\,(-dz)\\
&= \frac{-1}{\Gamma(1-\alpha)}\int_{0}^{t}(t-z)^{-\alpha}(\lambda(b-z))'\,dz = {}^{c}_{0}\mathrm{D}^{\alpha}_{t}\lambda(b-t).
\end{aligned}$$

Changing t to $b-t$ in (4.13), we arrive at the desired result. The proof is complete. □

Replacing (4.4) by (4.14), we can rewrite equations (4.3) and (4.6) as follows:

$$\begin{aligned}
&{}^{c}_{0}\mathrm{D}^{\alpha}_{t}x(t) = \frac{\partial H}{\partial \lambda}(t, x(t), \omega(t), \lambda(t)),\\
&{}^{c}_{0}\mathrm{D}^{\alpha}_{t}\lambda(b-t) = \frac{\partial H}{\partial x}(b-t, x(b-t), w(b-t), \lambda(b-t)),\\
&\frac{\partial H}{\partial u}(t, x(t), \omega(t), \lambda(t)) = 0,\\
&x(0) = x_0, \qquad \lambda(b) = 0.
\end{aligned}$$

5. Characterization of optimal control

Now we are in position to characterize the control associated to our fractional problem. In order to do this, we emply the Pontryagin maximum principle [37]. The ideal theory of this principle consists in specifying the Hamiltonian and in our case (4.2) it reads as

$$\begin{aligned}
H =& D + C + \frac{A}{2}w^2 + \lambda_1\left[I - (\mu + (\beta_3 + \beta_1)(1-\omega(t)))E(t)\right]\\
&+ \lambda_2\left[\beta_1(1-\omega(t))E(t) - (\mu + \beta_2(1-\omega(t)))D(t) + \gamma C(t)\right]\\
&+ \lambda_3\left[\beta_3(1-\omega(t))E(t) + \beta_2(1-\omega(t))D(t) - (\mu+\gamma+\nu+\delta)C(t)\right].
\end{aligned}$$

Then we have:

$$\begin{aligned}
{}^{c}_{b-t}\mathrm{D}^{\alpha}_{b}\lambda_1(b-t) =& -\lambda_1(b-t)\,(\mu + (\beta_3+\beta_1)(1-w(b-t)))\\
&+ \lambda_2\beta_1(1-w(b-t)) + \lambda_3\beta_3(1-w(b-t))\\
=& -\lambda_1\mu - \lambda_1\beta_3(1-w(b-t)) - \lambda_1\beta_1(1-w(b-t)) + \lambda_2\beta_1(1-w(b-t))\\
&+ \lambda_3\beta_3(1-w(b-t))\\
=&(\lambda_2-\lambda_1)\beta_1(1-\omega(t)) + (\lambda_3-\lambda_1)\beta_3(1-\omega(t)) - \lambda_1\mu
\end{aligned}$$

and

$$\begin{aligned}
{}^{c}_{b-t}\mathrm{D}^{\alpha}_{b}\lambda_2(b-t) =& 1 - \lambda_2(\mu + \beta_2(1-w(b-t))) + \lambda_3\beta_2(1-\omega(t))\\
=& 1 + \beta_2(1-\omega(t))(\lambda_3-\lambda_2) - \mu\lambda_2
\end{aligned}$$

as well as

$$\begin{aligned}
{}^{c}_{b-t}\mathrm{D}^{\alpha}_{b}\lambda_3(b-t) &= 1 + \lambda_2\gamma - \lambda_3(\mu+\gamma+\nu+\delta)\\
&= 1 + (\lambda_2-\lambda_3)\gamma - \lambda_3(\mu+\nu+\delta).
\end{aligned}$$

Then finally we get:

$$\begin{aligned}
{}^{c}_{b-t}\mathrm{D}^{\alpha}_{b}\lambda_1(b-t) &= (\lambda_2-\lambda_1)\beta_1(1-\omega(t)) + (\lambda_3-\lambda_1)\beta_3(1-\omega(t)) - \lambda_1\mu^{c}_{b-t}\mathrm{D}^{\alpha}_{b}\lambda_2(b-t)\\
&= 1 + \beta_2(1-\omega(t))(\lambda_3-\lambda_2) - \lambda_2\mu^{c}_{b-t}\mathrm{D}^{\alpha}_{b}\lambda_3(b-t)\\
&= 1 + (\lambda_2-\lambda_3)\gamma - \lambda_3(\mu+\nu+\delta)
\end{aligned}$$

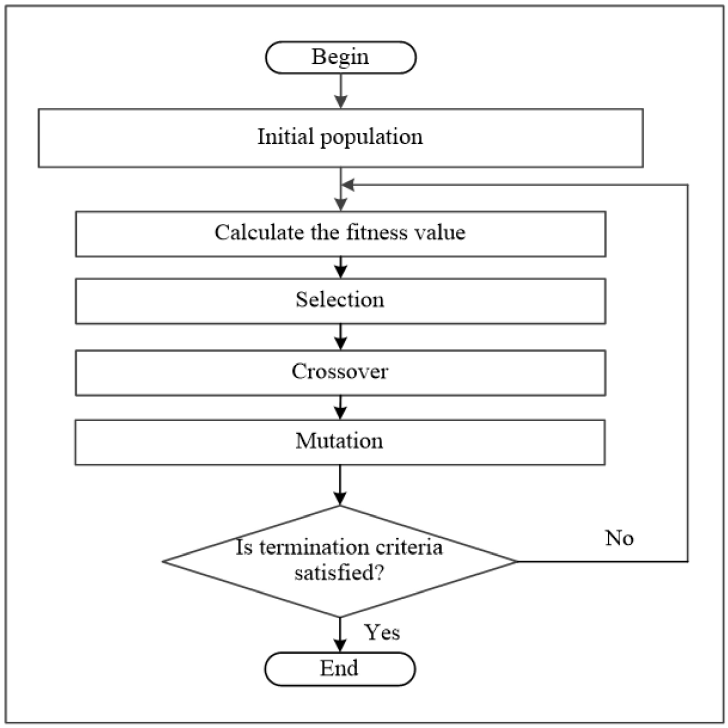


Figure 2: Genetic algorithm flowchart

with the transversality conditions: $\lambda_1(b) = \lambda_2(b) = \lambda_3(b) = 0$. On the other hand,

$$\begin{aligned}\frac{\partial H}{\partial w} = 0 &\Rightarrow Aw + \lambda_1 E(\beta_3 + \beta_1) - \lambda_2\beta_1 E + \lambda_2\beta_2 D - \lambda_3\beta_3 E - \lambda_3\beta_2 D = 0 \\ &\Rightarrow Aw = -E\beta_1(\lambda_1 - \lambda_2) - E\beta_3(\lambda_1 - \lambda_3) - D\beta_2(\lambda_2 - \lambda_3)\end{aligned}$$

Moreover, the optimal control is given by

$$w^* = \min\left(1, \max\left(0, \frac{1}{A}\left[E^*\beta_1(\lambda_2 - \lambda_1) + E^*\beta_3(\lambda_3 - \lambda_1) + D^*\beta_2(\lambda_3 - \lambda_2)\right]\right)\right). \tag{5.1}$$

6. Experimental results

6.1. Genetic algorithm. Genetic algorithms are artificial intelligence methods and are heuristic search techniques that are very simple to handle. Genetic algorithms are search optimization algorithms based on three essential operators, selection, crossover and mutation [29],[30]. In general, genetic algorithms were first developed by Holland and are derived from Darwin's theory of evolution [25]. At the first step, a population of initial solutions, called chromosomes, is randomly selected, then these solutions are evaluated by the objective function or fitness function, if the solution has a very high performance then this is the solution we are looking for. Otherwise, we move to the crossover and mutation step, which generates a new solution from the initial solution, likewise this solution generated and evaluated by the objective function. This procedure is repeated until we find the best solution. Genetic algorithms are very effective in complex optimization problems and are better than simple methods, which sometimes fail to achieve a certain problem due to its complexity. Figure 2 shows the diagramme of the genetic algorithm [26].

They are used in many research areas and are very useful in real world applications, because they are very simple and give the best solutions. Classical optimization methods, which are

Table 1: Configuartion of GA

Option and description	**Configuration**
Crossover	multiple
Initialization	random
Number of iteration	100*dim
Mutation	gaussian
Population size	200
Selection function	stochastic(uniform)

purely computational methods, start with a single initial solution and then search for the optimal solution, but the genetic algorithm starts at an initial population of candidates and then searches for the best optimal solution in the search space [34].

Optimal control problems are also among the optimization problems that were solved by the genetic algorithm. The real beginning of the use of genetic algorithm for dynamic control systems was in 1992 by Krishnakumar and Goldberg [27], who gave a start to GA in a very important discipline of applied mathematics [28], [31], [33]. Among the most recent works of optimal control that have studied the phenomenon with metaheuristic methods is that by El Moutaouakil et al [32], which used artificial intelligence methods for a general study of the phenomenon of diabetes, for more details see [32]. The major objective of this work is to achieve a significant reduction in the number of diabetics. And among the three categories we have considered, we focus on reducing the number of people without complications and diseases that have treatable complications, which implies the search for an optimal control to achieve this goal. Also, in this work, we consider the linear mathematical model that is the core of other linear and non-linear models that have been devoted to the diabetes disease. This is the dynamic model EDC that represents the evolution of the diabetic population and the influence of uncomplicated diabetic patients on pre-diabetics.

We use the configurations in the table below (see 1) for our genetic algorithm

6.2. Discussion. Diabetes is a serious disease and requires very careful study, and for this reason the following data shows precisely how it developed over ten years in patients. We will present different numerical results whose purpose is to follow cases over ten years.

We solve the problem of several cases the first which was represented by Boutayeb, the second is like a problem of fractional equations and lately like an optimal control problem

First we solve the problems of ordinary differential equations (1) with the MATLAB program. And the initial conditions are presented in Table 3.

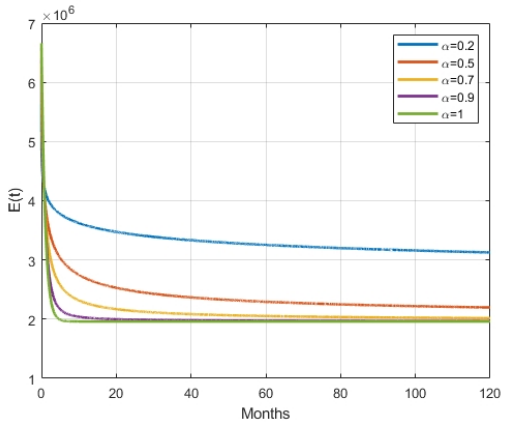
Table 2: Parameters values

Parameter	μ	ν	γ	δ	I	β_1	β_2	β_3	A
value	0.02	0.05	0.08	0.05	2000000	0.5	0.1	0.5	3550000

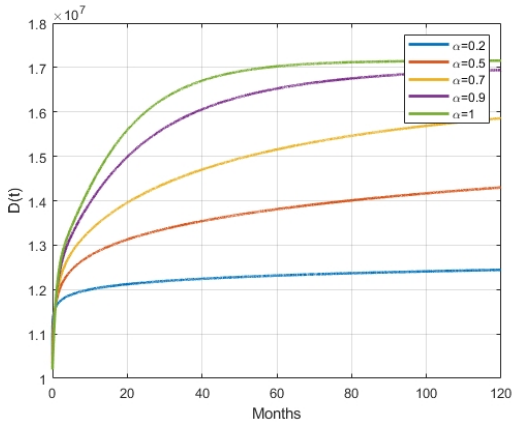
First, we solved our fractional optimal control problem using Matlab and then we give numerical simulations. Of course, we used different fractional values of α. For this, we used the file fde12 (which has inside matlab software). The values of the parameters that we used are in Table 2. The graphic presentation of different compartments $E(t)$, $D(t)$ and $C(t)$ for the fractional model with different α values is given by diagrams 3a, 3b and 3c.

Table 3: The initial conditions

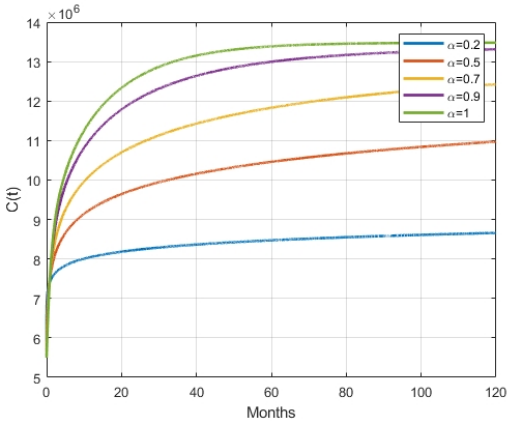
Initial compartment	E(0)	D(0)	C(0)
value	660000	10200000	5500000



(a) compartment E(t) with different α values



(b) compartment D(t) with different α valuese



(c) compartment C(t) with different α values

Figure 3: The three compartments with different values

Now we pass to what interests us, this is the solution of fractional problem of optimal control applied to diabetic people with the most popular method genetic algorithm. The GA intillegent method that we have used for solving the fractional mono-objective problem shows a very high efficiency, and the simulation results above justified the succiency of the method.

Figures 4, 5, 6 and 7 show the different performance curves of the GA applied to the described control model for different values of α. We notice that the optimization process converges very quickly.

We implemented the GA intelligent method to estimate the control of the fractional dynamical system 4.1 for different values of α. To do so, we discretized the interval $[0, T]$ into 200 points at which we estimate our control by the genetic algorithm method. Figure 8 gives the curve of the controls obtained by the different values of α that we considered.

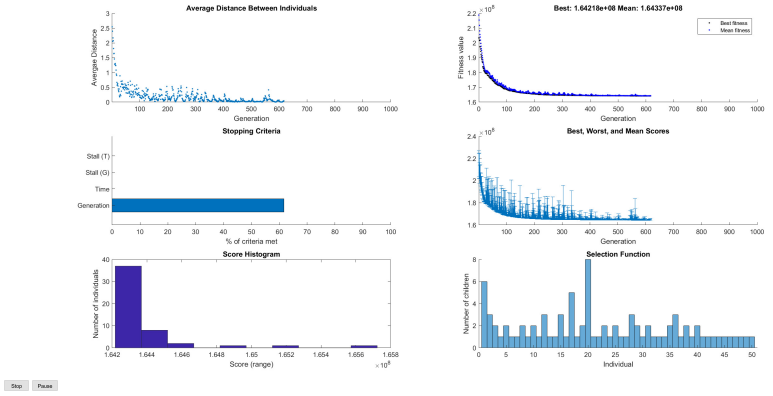


Figure 4: Performance curves of GA applied to descritised control model for $\alpha = 0.25$

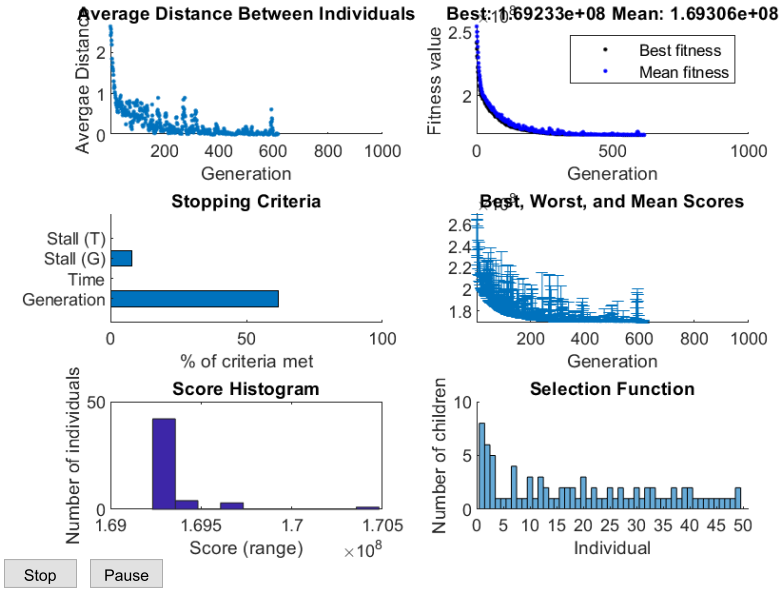


Figure 5: Performance curves of GA applied to descritised control model for $\alpha = 0.5$

Figures 9, 10 and 11 give the compartments of the fractional dynamic model of the diabetic population, proposed in this research work, by the different values of α. Figure 9 shows the behavior $E(t)$ of the number of pre-diabetic persons for different values of α as a function of the time t. The considered values of α are α=0.25, α=0.5, α=0.75 and α=0.95. This figure shows a monotonic decrease of the prediabetic population $E(t)$ each time the fractional order α increases. It can be deduced from figure 9 that the number of pre-diabetic people for fractional values of α decreases more rapidly. It can be seen that with the proposed control strategy, the number of prediabetic individuals continued to decrease during the 120-month period. The uncontrolled compartment E shows a very high growth, so it is very clear that the combined strategy leads to a significant decrease, and this for different values of α even with a small difference on the way of decrease. This shows, according to the given results, that the fractional dynamic model of the diabetic population is more efficient compared to the basic model proposed by Boutayeb [18].

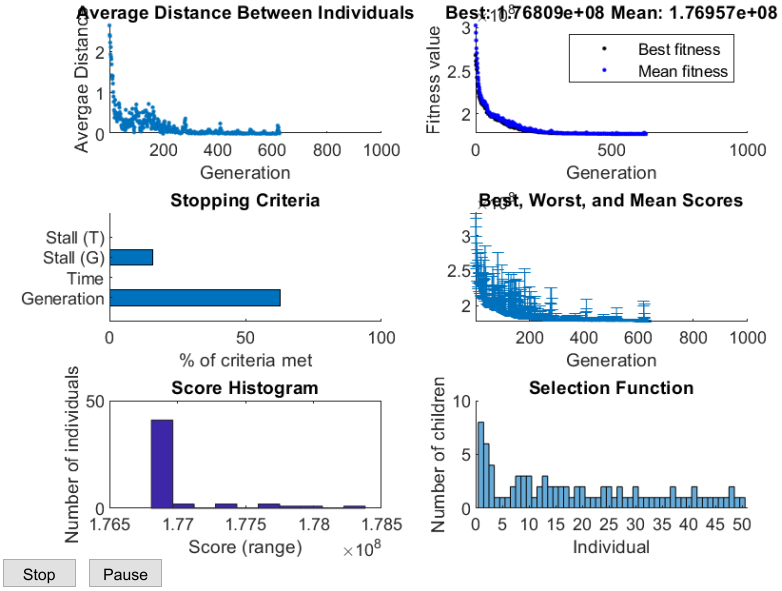


Figure 6: Performance curves of GA applied to descritized control model for $\alpha = 0.75$

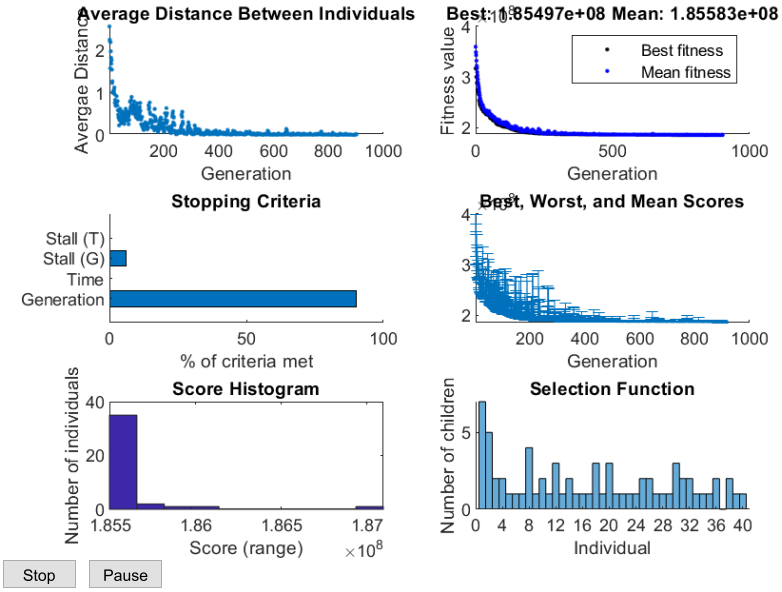


Figure 7: Performance curves of GA applied to descritized control model for $\alpha = 0.95$

Figure 10 shows the compartment D obtained with the control produced by GA in the fractional order derived dynamic model. Using the same compartmentalization strategy as before, we consider values of α, which are α=0.25, α=0.5, α=0.75 and α=0.95. Figure 10 shows the behavior $D(t)$ of the number of uncomplicated diabetics for different values of α versus the time t. The GA intelligent method that we have used for fractional problem solving, allows us to follow the patients all the time of control in a very precise way and this allows a doctor to treat the cases. The conclusion is that we have obtained very positive results, which allow us to say that the number of diabetics without complication can be controlled using the proposed strategy. We notice this time that the most decreased representation is that of

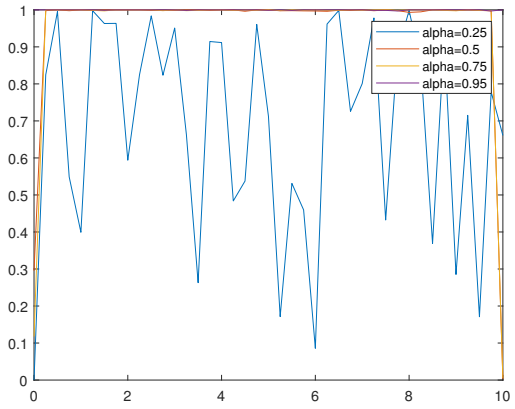


Figure 8: Controls associated with different values of α

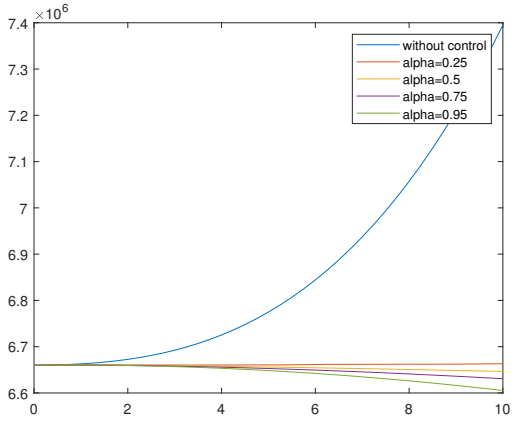


Figure 9: Compartment E behavior associated with different values of α

α=0.25, and the highest is that of α=0.95. As for the first compartment E, we can notice from Figure 10 that the number of diabetics without complication for fractional values of α decreases more rapidly, and the uncontrolled compartment D shows a very high growth, so it is very clear that the combined strategy leads to a significant decrease, and this for different values of α even with a small difference on the way of decrease.

Figure 11 represents the most difficult case to treat, that of diabetic patients with complications, because several factors influence it. This category is exposed to many dangers and must be monitored very carefully. Also from the same strategy as the two previous compartments, we studied the compartment of diabetic patients with complication with different values of α. We note each time a success of the control strategy that we proposed, and the difference between the controlled compartment and the uncontrolled compartment was very clear on Figure 11. We also notice this time that despite the different values of α, the representations with control remain closer to each other. The figure 11 also shows a monotonic decrease in the diabetic population with complication $C(t)$ each time the fractional order α decreased, with a smaller difference.

Thanks to the referee's remark, we also make the following comparison: the Gumel method produces controls that are very costly in terms of human and material resources, in Figure 12. We have compared the control produced with the genetic algorithm and with the Gumel,

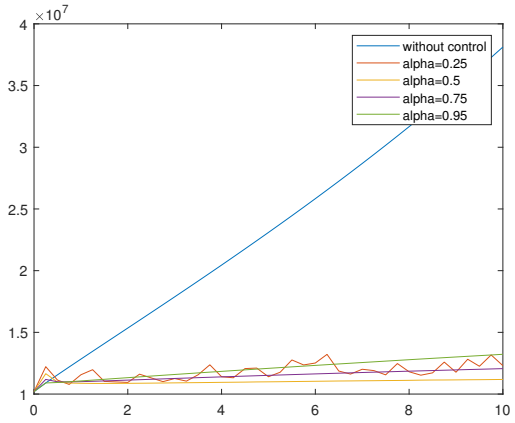


Figure 10: Compartment D behavior associated with different values of α

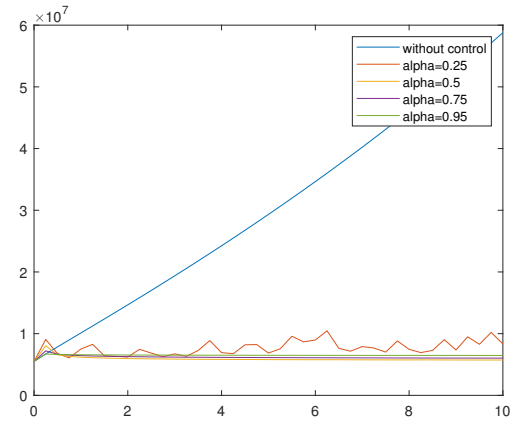


Figure 11: Compartment C behavior associated with different values of α

and as we have mentioned earlier, the Gumel control is very expansive. In previous work, we have used meta-heuristic methods in the same system, with very satisfactory results. However, the introduction of the fractional derivative allows us to improve the quality of the control by testing all possible values of α ($\alpha = 0.1$; $\alpha = 0.5$; $\alpha = 0.75$; $\alpha = 0.95$). In term of mobilized resources, we find that model with $\alpha = 0.95$ produces the best control.

Figure 13 shows the comparison the total required resources used between the different fractional values of α, and we see that with $\alpha = 0.95$ we have the lowest requirements in terms of material and human resources.

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7. Conclusion

In this paper we have proposed a fractional order diabetic population dynamic model to track different cases of diabetic patients in a very accurate and efficient way. We studied

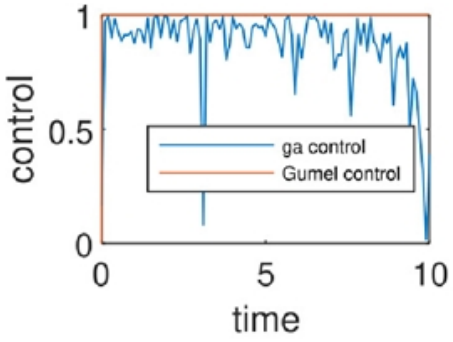

Figure 12: Comparison between the control produced by Gumel and GA in the case where $\alpha = 1$

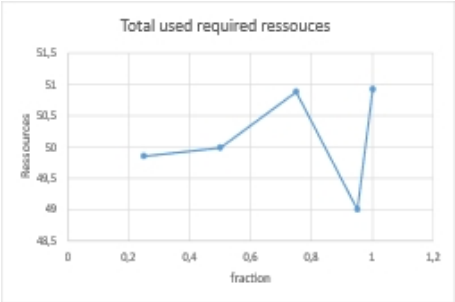

Figure 13: Total required resources

the existence, uniqueness, positivity and boundedness of the solution of the system. We have also proposed a new fractional control dynamic model that we inspired from the integer order model of Deriouch et al [18]. We have defined the different parameters of this model and we have determined the adjoints and characterized the control. For the numerical solution of this fractional order controlled dynamic model, we used the genetic algorithm (GA), as an efficient and easy to implement metaheuristic method. The different simulations were performed by using MATLAB. We treated several cases and the found simulation showed the efficiency of the proposed fractional model. The simulations show for each compartments that the proposed strategy leads to a significant decrease for different values of α, even with a small difference in the way of decrease. This proves that the fractional order model gives more accurate results than the integer order models. In a future work we shall improve other dynamic diabetic population models that contains more population types, deal with different control strategies and study different environmental effects by using artificial intelligence methods, especially, metaheuristic methods.

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