ISSN 2304-0122 Ufa Mathematical Journal. Vol. 15. No 3 (2023). P. 13-40.

doi:10.13108/2023-15-3-13

# NUMERICAL SOLUTION OF INITIAL-BOUNDARY VALUE PROBLEMS FOR A MULTI-DIMENSIONAL PSEUDOPARABOLIC EQUATION

**M.Kh. BESHTOKOV**

**Abstract.** We consider initial boundary value problems for a multi-dimensional pseudoparabolic equation with Dirichlet boundary conditions of a special form. For an approximate solution of the considered problems, the multi-dimensional pseudoparabolic equation is reduced to an integro-differential equation with a small parameter. It is shown that as the small parameter tends to zero, the solution of the corresponding modified problem converges to the solution of the original problem. For each of the problems we construct a locally one-dimensional difference scheme following A.A. Samarskii. The main idea is to reduce the transition from a layer to a layer to the sequential solving a number of one-dimensional problems in each of the coordinate directions. Using the maximum principle, we obtain apriori estimates, which imply the uniqueness, stability, and convergence of the solution of a locally one-dimensional difference scheme in the uniform metric. We construct an algorithm for numerical solving of the modified problem with conditions of a special form.

**Keywords:** pseudoparabolic equation, moisture transfer equation, integro-differential equation, initial boundary value problem, difference schemes, apriori estimates, stability and convergence

**Mathematics Subject Classification:** 35L35, 65N12

## 1. Introduction

It is well known that boundary value problems for pseudoparabolic equations arise in studying fluid filtration in fractured-porous media [1]–[3], soil moisture movements [4]–[6], when describing heat and mass transfer [7]–[10], wave processes and many other areas.

Boundary value problems for various classes of third order equations were studied in [11]–[16]. In [17], a wide range of results was obtained on initial and initial-boundary value problems for strongly nonlinear equations of pseudo-parabolic type, as well as issues of local solvability, conditions for the destruction of solutions, and global solvability in time. Boundary value problems with a general A.A. Samarskii nonlocal condition for high-order pseudoparabolic equations were studied in [18].

From the point of view of numerical implementation, the passage from the one-dimensional case to the multi-dimensional case causes significant difficulties. The difficulty lies in the essential increase in the amount of computation that occurs when moving from one-dimensional problems to the multi-dimensional ones. In this regard, the problem of constructing economical difference schemes for the numerical solution of multidimensional problems becomes topical.

In this paper, we consider initial-boundary value problems for a multi-dimensional pseudoparabolic equation with variable coefficients. The aim of the work is to construct and study

---

M.Kh. Beshtokov, Numerical solution of initial-boundary value problems for a multidimensional pseudoparabolic equation.



*Submitted July 26, 2022.*

the convergence of an approximate solution of each of the problems posed with time approximation based on locally one-dimensional splitting schemes [19], [20]. The main difficulty lies in the need to split not only the first operator, but also the operator at the time derivative; therefore, the construction of splitting schemes is achieved by passing to a time-nonlocal problem and its parabolic regularization. The study of stability and convergence is carried out according to the method of A.A. Samarsky [21]. Using the maximum principle for solving the corresponding problem, an apriori estimate in the uniform metric is obtained, from which the uniqueness and stability of the solution follow, and convergence is proved. An algorithm for the numerical solution of a modified problem with boundary conditions of a special type is constructed using a recursive formula for fast calculation in the multi-dimensional case.

This paper is a continuation of the author's series of papers [22]–[26] devoted to the study of local and nonlocal boundary value problems for generalized pseudoparabolic equations with variable coefficients.

## 2. Formulation of Dirichlet initial boundary value problem

In the cylinder $\overline{Q}_T = \overline{G} \times [0, T]$, the base of which is a $p$-dimensional rectangle parallelepiped

$$\overline{G} = \left\{x = (x_1, x_2, \dots, x_p) : \ 0 \leqslant x_k \leqslant l_k, \ k = 1, 2, \dots, p\right\}$$

with the boundary $\Gamma$, $\overline{G} = G + \Gamma$, we consider the following problem

$$\frac{\partial u}{\partial t} = Lu + \alpha \frac{\partial}{\partial t} Lu + f(x, t), \quad (x, t) \in Q_T, \tag{2.1}$$

$$u\Big|_{\Gamma} = \mu(x, t), \quad 0 \leqslant t \leqslant T, \tag{2.2}$$

$$u(x, 0) = u_0(x), \quad x \in \overline{G}, \quad \overline{G} = G + \Gamma, \tag{2.3}$$

where

$$\begin{gathered} Lu = \sum_{k=1}^{p} L_k u, \quad L_k u = \frac{\partial}{\partial x_k}\left(\Theta_k(x, t) \frac{\partial u}{\partial x_k}\right) + r_k(x, t) \frac{\partial u}{\partial x_k} - q_k(x, t) u, \quad k = 1, 2, \dots, p, \\ u(x, t) \in C^{4,2}\left(\overline{Q}_T\right), \quad \Theta_k(x, t) \in C^{3,1}\left(\overline{Q}_T\right), \quad r_k(x, t), q_k(x, t), f(x, t) \in C^{2,1}\left(\overline{Q}_T\right), \\ 0 < c_0 \leqslant \Theta_k(x, t), q_k(x, t) \leqslant c_1, \quad |r_k| < c_2, \quad \left|\frac{\Theta_k}{\partial t}, \frac{q_k}{\partial t}, \frac{r_k}{\partial t}\right| \leqslant c_3, \quad c_0, c_1, c_2 = const > 0. \end{gathered} \tag{2.4}$$

$C^{m,n}$ is the class of functions continuous together with its partial derivatives of order $m$ in $x$ and of order $n$ in $t$ and $Q_T = G \times (0, T]$, $\alpha > 0$.

We transform equation (2.1) by multiplying both sides of (2.1) by $\frac{1}{\alpha} e^{\frac{1}{\alpha} t}$, replacing then $t$ by $\xi$ and integrating the obtained expression in $\xi$ from 0 to $t$. This gives the equation

$$\mathcal{B}u = Lu + \widetilde{f}(x, t), \tag{2.5}$$

where

$$\mathcal{B}u = \frac{1}{\alpha} \int_0^t e^{-\frac{1}{\alpha}(t-\xi)} u_\xi d\xi, \qquad \widetilde{f}(x, t) = \frac{1}{\alpha} \int_0^t e^{-\frac{1}{\alpha}(t-\xi)} f(x, \xi) d\xi - e^{-\frac{1}{\alpha}} Lu_0(x), \qquad \alpha > 0.$$

In the same domain, instead of problem (2.2), (2.3), (2.5), we consider the following problem with a small parameter $\varepsilon$

$$\varepsilon \frac{\partial u^\varepsilon}{\partial t} + \mathcal{B}u^\varepsilon = Lu^\varepsilon + \widetilde{f}(x, t), \qquad (x, t) \in Q_T, \tag{2.6}$$

$$u^\varepsilon|_\Gamma = \mu(x, t), \qquad 0 \leqslant t \leqslant T, \tag{2.7}$$

$$u^\varepsilon(x, 0) = u_0(x), \quad x \in \overline{G}, \tag{2.8}$$

where $\varepsilon = const > 0$.

Since as $t = 0$ the initial conditions for equation (2.5) and (2.6) coincide, in the vicinity of $t = 0$ the derivative $u_t^\varepsilon$ has no singularity of boundary layer type [27], [28].

We are going to show that $u^\varepsilon \to u$ in some norm as $\varepsilon \to 0$. We denote $\tilde{z} = u^\varepsilon - u$ and substitute $u^\varepsilon = \tilde{z} + u$ into problem (2.6)–(2.8):

$$\varepsilon \frac{\partial \tilde{z}}{\partial t} + \mathcal{B}\tilde{z} = L\tilde{z} + \overline{f}(x,t), \qquad (x,t) \in Q_T, \tag{2.9}$$

$$\tilde{z}|_\Gamma = 0, \qquad 0 \leqslant t \leqslant T, \tag{2.10}$$

$$\tilde{z}(x,0) = 0, \qquad x \in \overline{G}, \qquad \overline{G} = G + \Gamma, \tag{2.11}$$

where $\overline{f}(x,t) = -\varepsilon \dfrac{\partial u}{\partial t}$.

**Lemma 2.1.** *For each absolutely continuous on $[0,T]$ function $v(t)$ the inequality holds:*

$$v(t)\mathcal{B}v(t) \geqslant \frac{1}{2}\mathcal{B}v^2(t), \tag{2.12}$$

*where*

$$\mathcal{B}u = \frac{1}{\alpha}\int_0^t e^{-\frac{1}{\alpha}(t-\xi)} u_\xi \, d\xi, \qquad \alpha > 0.$$

*Proof.* We rewrite inequality (2.12) in the form

$$\begin{aligned}
v(t)\mathcal{B}v - \frac{1}{2}\mathcal{B}v^2(t) =& v(t)\frac{1}{\alpha}\int_0^t e^{-\frac{1}{\alpha}(t-\tau)} v_\tau d\tau - \frac{1}{2\alpha}\int_0^t e^{-\frac{1}{\alpha}(t-\tau)} (v^2)_\tau \, d\tau \\
=& \frac{1}{\alpha}\int_0^t e^{-\frac{1}{\alpha}(t-\tau)} v_\tau(\tau) \left(v(t) - v(\tau)\right) d\tau \\
=& \frac{1}{\alpha}\int_0^t e^{-\frac{1}{\alpha}(t-\tau)} v_\tau(\tau) \left(\int_\tau^t v_\eta(\eta) d\eta\right) d\tau \\
=& \frac{1}{\alpha}\int_0^t v_\eta(\eta) d\eta \int_0^\eta e^{-\frac{1}{\alpha}(t-\tau)} v_\tau(\tau) \, d\tau \\
=& \frac{1}{\alpha}\int_0^t \frac{v_\eta(\eta) e^{-\frac{1}{\alpha}(t-\eta)}}{e^{-\frac{1}{\alpha}(t-\eta)}} d\eta \int_0^\eta e^{-\frac{1}{\alpha}(t-\tau)} v_\tau(\tau) \, d\tau \\
=& \frac{1}{2\alpha}\int_0^t e^{\frac{1}{\alpha}(t-\eta)} \frac{\partial}{\partial \eta}\left[\int_0^\eta e^{-\frac{1}{\alpha}(t-\tau)} v_\tau(\tau) d\tau\right]^2 d\eta \\
=& \frac{1}{2\alpha}\left[\int_0^t e^{-\frac{1}{\alpha}(t-\tau)} v_\tau(\tau) \, d\tau\right]^2 \\
&+ \frac{1}{2\alpha^2}\int_0^t e^{\frac{1}{\alpha}(t-\eta)} \left[\int_0^\eta e^{-\frac{1}{\alpha}(t-\tau)} v_\tau(\tau) d\tau\right]^2 d\eta \geqslant 0.
\end{aligned}$$

Thus,

$$v(t)\mathcal{B}v(t) \geqslant \frac{1}{2}\mathcal{B}v^2(t).$$

The proof is complete. □

In order to obtain an apriori estimate, we use the energy inequalities method. We calculate the scalar product of equation (2.9) with $\tilde{z}$:

$$\begin{aligned}\left(\varepsilon\frac{\partial \tilde{z}}{\partial t}, \tilde{z}\right) + \left(\mathcal{B}\tilde{z}, \tilde{z}\right) =& \left(\sum_{k=1}^{p} \frac{\partial}{\partial x_k}\left(\Theta_k(x,t)\frac{\partial \tilde{z}}{\partial x_k}\right), \tilde{z}\right) \\ &+ \left(\sum_{k=1}^{p} r_k(x,t)\frac{\partial \tilde{z}}{\partial x_k}, \tilde{z}\right) - \left(\sum_{k=1}^{p} q_k(x,t)\tilde{z}, \tilde{z}\right) + \left(\overline{f}(x,t), \tilde{z}\right),\end{aligned} \tag{2.13}$$

where the scalar product and the norm read as

$$(w,v) = \int_G wv dx, \qquad \|v(\cdot,t)\|_0^2 = \|v\|_0^2 = \int_G v^2 dx, \qquad \|v\|^2_{L_2(0,l_k)} = \int_0^{l_k} v^2(x,t)\, dx_k.$$

Then by $M_i$, $i = 1, 2, \dots$ we denote positive constants depending only on the input data in the considered problem.

Using Lemma 2.1, we transform the integrals involved in identity (2.13):

$$\left(\varepsilon\frac{\partial \tilde{z}}{\partial t}, \tilde{z}\right) = \frac{\varepsilon}{2}\frac{\partial}{\partial t}\|\tilde{z}\|_0^2, \tag{2.14}$$

$$\begin{aligned}\left(\mathcal{B}\tilde{z}, \tilde{z}\right) =& \left(\frac{1}{p}\sum_{k=1}^{p}\mathcal{B}\tilde{z}, \tilde{z}\right) = \frac{1}{p}\int_G \sum_{k=1}^{p} \tilde{z}\mathcal{B}\tilde{z} dx = \frac{1}{p}\sum_{k=1}^{p}\int_G \tilde{z}\mathcal{B}\tilde{z}\, dx \\ =& \frac{1}{p}\sum_{k=1}^{p}\int_{G'}\left(\int_0^{l_k} \tilde{z}\mathcal{B}\tilde{z} dx_k\right) dx' \geqslant \frac{1}{2p}\sum_{k=1}^{p}\int_{G'}\left(\int_0^{l_k} \mathcal{B}\tilde{z}^2 dx_k\right) dx' \\ =& \frac{1}{2p}\sum_{k=1}^{p}\int_{G'} \mathcal{B}\|\tilde{z}\|^2_{L_2(0,l_k)}\, dx' = \frac{1}{2p}\sum_{k=1}^{p}\mathcal{B}\|\tilde{z}\|_0^2 = \frac{1}{2}\mathcal{B}\|\tilde{z}\|_0^2,\end{aligned} \tag{2.15}$$

$$\left(\sum_{k=1}^{p} \frac{\partial}{\partial x_k}\left(\Theta_k(x,t)\frac{\partial \tilde{z}}{\partial x_k}\right), \tilde{z}\right) = -\sum_{k=1}^{p}\int_G \Theta_k(x,t)\left(\frac{\partial \tilde{z}}{\partial x_k}\right)^2 dx \leqslant -c_0\left\|\frac{\partial \tilde{z}}{\partial x}\right\|_0^2, \tag{2.16}$$

where

$$G' = \left\{x' = (x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_p) : 0 < x_k < l_k\right\},$$
$$dx' = dx_1 dx_2 \dots dx_{k-1} dx_{k+1} \dots dx_p.$$

We estimate the terms in the right hand side by means of Cauchy $\varepsilon$-inequality:

$$\begin{aligned}\left(\sum_{k=1}^{p} r_k(x,t)\frac{\partial \tilde{z}}{\partial x_k}, \tilde{z}\right) =& \int_G \sum_{k=1}^{p} r_k(x,t)\frac{\partial \tilde{z}}{\partial x_k}\tilde{z} dx = \sum_{k=1}^{p}\int_G r_k(x,t)\frac{\partial \tilde{z}}{\partial x_k}\tilde{z}\, dx \\ \leqslant& \varepsilon_0\sum_{k=1}^{p}\int_G\left(\frac{\partial \tilde{z}}{\partial x_k}\right)^2 dx + \frac{c_2^2}{4\varepsilon_0}\sum_{k=1}^{p}\int_G \tilde{z}^2\, dx,\end{aligned} \tag{2.17}$$

$$\left(\overline{f}(x,t), \tilde{z}\right) \leqslant \frac{1}{4\varepsilon_1}\|\overline{f}\|_0^2 + \varepsilon_1\|\tilde{z}\|_0^2. \tag{2.18}$$

Taking into consideration transformations (2.14)–(2.18), by (2.13) we obtain the inequality

$$\frac{\varepsilon}{2}\frac{\partial}{\partial t}\|\tilde{z}\|_0^2 + \frac{1}{2}\mathcal{B}\|\tilde{z}\|_0^2 + (c_0 - \varepsilon_0)\left\|\frac{\partial \tilde{z}}{\partial x}\right\|_0^2 + \left(c_0 - \varepsilon_1 - \frac{c_2^2}{4\varepsilon_0}\right)\|\tilde{z}\|_0^2 \leqslant \frac{1}{4\varepsilon_1}\|\overline{f}\|_0^2. \tag{2.19}$$

We transform the second term in the left hand side of (2.19) as

$$\mathcal{B}\|\tilde{z}\|_0^2 = \frac{1}{\alpha}\int\limits_0^t e^{-\frac{1}{\alpha}(t-\xi)}\left(\|\tilde{z}\|_0^2\right)_\xi d\xi = \frac{1}{\alpha}\|\tilde{z}\|_0^2 - \frac{1}{\alpha^2}\int\limits_0^t e^{-\frac{1}{\alpha}(t-\xi)}\|\tilde{z}\|_0^2\, d\xi. \tag{2.20}$$

Choosing $\varepsilon_0 = \varepsilon_1 = \dfrac{c_0}{2}$ and $c_2 < c_0$, by (2.19) and in view of (2.20) we find:

$$\varepsilon\frac{\partial}{\partial t}\|\tilde{z}\|_0^2 + \left\|\frac{\partial \tilde{z}}{\partial x}\right\|_0^2 + \|\tilde{z}\|_0^2 \leqslant M_1\int\limits_0^t \|\tilde{z}\|_0^2\, d\xi + M_2\|\overline{f}\|_0^2. \tag{2.21}$$

We integrate (2.21) in $\tau$ from 0 to $t$:

$$\varepsilon\|\tilde{z}\|_0^2 + \int_0^t\left(\|\tilde{z}\|_0^2 + \left\|\frac{\partial \tilde{z}}{\partial x}\right\|_0^2\right)d\tau \leqslant M_1\int\limits_0^t d\tau\int\limits_0^\tau \|\tilde{z}\|_0^2\, d\xi + M_2\int_0^t \|\overline{f}\|_0^2\, d\tau. \tag{2.22}$$

We estimate the first term in the right hand side of (2.22). Then we rewrite (2.22) as

$$Y \leqslant M_1\int_0^t Y\,d\tau + M_2 F, \tag{2.23}$$

where

$$Y = \int_0^t \|\tilde{z}\|_0^2\, d\tau, \qquad F = \int_0^t \|\overline{f}\|_0^2\, d\tau.$$

Owing to Grönwall's lemma [29, Ch. III, Sect. 1, Lm. 1.1], by (2.23) we obtain the estimate

$$\varepsilon\|\tilde{z}\|_0^2 + \int_0^t\left(\|\tilde{z}\|_0^2 + \left\|\frac{\partial \tilde{z}}{\partial x}\right\|_0^2\right)d\tau \leqslant M_3\int_0^t \|\overline{f}\|_0^2\, d\tau = \varepsilon^2 M_3\int_0^t \left\|\frac{\partial u}{\partial \tau}\right\|_0^2 d\tau = O(\varepsilon^2), \tag{2.24}$$

where $M$ depends only on the input data of problem (2.1)–(2.3).

Apriori estimate (2.24) implies the convergence of $u^\varepsilon$ to $u$ as $\varepsilon \to 0$ in the norm

$$\|\tilde{z}\|_1^2 = \varepsilon\|\tilde{z}\|_0^2 + \|\tilde{z}\|_{2,Q_t}^2 + \left\|\frac{\partial \tilde{z}}{\partial x}\right\|_{2,Q_t}^2,$$

where

$$\|\tilde{z}\|_{2,Q_t}^2 = \int\limits_0^t \|\tilde{z}\|_0^2\, d\tau$$

if $u_t$ is a bounded sufficiently smooth function. This is why for small $\varepsilon$ the solution of problem (2.6)–(2.8) is treated as an approximate solution of the Dirichlet initial boundary value problem for a multi-dimensional pseudoparabolic equation (2.1)–(2.3).

## 3. Locally one-dimensional difference scheme

We choose a spatial grid uniformly in each direction $Ox_k$ with steps $h_k = \frac{l_k}{N_k}$, $k = 1, 2, \ldots, p$ :

$$\bar{\omega}_{h_k} = \left\{x_k^{(i_k)} = i_k h_k : i_k = 0, 1, \ldots, N_k, \quad h_k = \frac{l_k}{N_k}, \quad k = 1, 2, \ldots, p\right\}, \quad \bar{\omega} = \prod_{k=1}^p \bar{\omega}_{h_k}.$$

On the segment $[0, T]$ we introduce the grid

$$\bar{\omega}'_\tau = \left\{0, t_{j+\frac{k}{p}} = \left(j + \frac{k}{p}\right)\tau, \qquad \tau = \frac{T}{j_0}, \qquad j = 0, 1, \ldots, j_0 - 1, \qquad k = 1, 2, \ldots, p\right\},$$

involving, together with the nodes $t_j = j\tau$, fictitious nodes $t_{j+\frac{k}{p}}$, $k = 1, 2, \dots, p-1$. We denote by $\omega'_\tau$ the set of the nodes in the grid $\bar{\omega}'_\tau$, for which $t > 0$.

With equation (2.6) we associate a chain of $p$ "one-dimensional" equations; in order to do this, we rewrite equation (2.6) in the form

$$\sum_{k=1}^{p} \pounds_k u^\varepsilon = 0, \qquad \pounds_k u^\varepsilon = \frac{\varepsilon}{p}\frac{\partial u^\varepsilon}{\partial t} + \frac{1}{p}\mathcal{B}u^\varepsilon - L_k u^\varepsilon - f_k,$$

where $f_k(x,t)$, $k = 1, 2, \dots, p$, are arbitrary functions possessing the same smoothness as $f(x,t)$ and satisfying the condition $\sum\limits_{k=1}^{p} f_k = f$.

On each semi-interval $\Delta_k = \left(t_{j+\frac{k-1}{p}}, t_{j+\frac{k}{p}}\right]$, $k = 1, 2, \dots, p$, we subsequently solve the problems

$$\pounds_k \vartheta_{(k)} = 0, \quad x \in G, \quad t \in \Delta_k, \qquad k = 1, 2, \dots, p, \qquad \vartheta_{(k)} = \mu(x,t) \quad \text{as} \quad x \in \Gamma_k, \tag{3.1}$$

letting

$$\vartheta_{(1)}(x,0) = u_0(x), \qquad \vartheta_{(1)}(x,t_j) = \vartheta_{(p)}(x,t_j),$$
$$\vartheta_{(k)}\left(x, t_{j+\frac{k-1}{p}}\right) = \vartheta_{(k-1)}\left(x, t_{j+\frac{k-1}{p}}\right), \qquad k = 2, 3, \dots, p, \qquad j = 0, 1, \dots, j_0 - 1,$$

$\Gamma_k$ is the set of boundary points in the direction $x_k$. The function $\vartheta(t_{j+1}) = \vartheta_{(p)}(t_{j+1})$ is called a solution of this problem as $t = t_{j+1}$.

Let us find a discrete analogue of $\mathcal{B}u$:

$$\begin{aligned}
\frac{1}{\alpha}\int\limits_0^{t_{j+\frac{k}{p}}} e^{-\frac{1}{\alpha}\left(t_{j+\frac{k}{p}}-\xi\right)} u_\xi \, d\xi =& \frac{1}{\alpha}\sum_{s=1}^{pj+k}\int_{t_{\frac{s-1}{p}}}^{t_{\frac{s}{p}}} e^{-\frac{1}{\alpha}\left(t_{j+\frac{k}{p}}-\xi\right)} \dot{u}(x,\xi)\, d\xi \\
=& \frac{1}{\alpha}\sum_{s=1}^{pj+k}\int_{t_{\frac{s-1}{p}}}^{t_{\frac{s}{p}}} e^{-\frac{1}{\alpha}\left(t_{j+\frac{k}{p}}-\xi\right)} \left(\dot{u}(x,\tilde{t}) + \ddot{u}(x,\bar{\xi})(\xi - \tilde{t})\right) d\xi \\
=& \sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right) u_{\bar{t}}^{\frac{s}{p}} \\
&+ \frac{1}{\alpha}\sum_{s=1}^{pj+k}\int_{t_{\frac{s-1}{p}}}^{t_{\frac{s}{p}}} e^{-\frac{1}{\alpha}\left(t_{j+\frac{k}{p}}-\xi\right)} \ddot{u}(x,\bar{\xi})(\xi - \tilde{t})\, d\xi,
\end{aligned} \tag{3.2}$$

where

$$u_{\bar{t}}^{\frac{s}{p}} = \frac{u^{\frac{s}{p}} - u^{\frac{s-1}{p}}}{\frac{\tau}{p}}, \qquad \tilde{t} = t_{\frac{s}{p}-\frac{1}{2p}}, \qquad \dot{u} = \frac{\partial u}{\partial t}, \qquad \ddot{u} = \frac{\partial^2 u}{\partial t^2}, \qquad t_{\frac{s-1}{p}} < \bar{\xi} < \xi.$$

We estimate the second term in the right hand side of (3.2):

$$\begin{aligned}
\frac{1}{\alpha}\sum_{s=1}^{pj+k}\int_{t_{\frac{s-1}{p}}}^{t_{\frac{s}{p}}} e^{-\frac{1}{\alpha}\left(t_{j+\frac{k}{p}}-\xi\right)} \ddot{u}(x,\bar{\xi})(\xi - \tilde{t}) d\xi \leqslant& \frac{1}{\alpha}\frac{\tau}{p}M\sum_{s=1}^{pj+k}\int_{t_{\frac{s-1}{p}}}^{t_{\frac{s}{p}}} e^{-\frac{1}{\alpha}\left(t_{j+\frac{k}{p}}-\xi\right)} d\xi \\
=& \frac{M\tau}{p}\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right) \\
=& \frac{M\tau}{p}\left(1 - e^{-\frac{1}{\alpha}t_{j+\frac{k}{p}}}\right) \leqslant \frac{M\tau}{p} = O\left(\frac{\tau}{p}\right),
\end{aligned}$$

where $|\ddot{u}(x,\xi)| \leqslant M$.

Thus,

$$\frac{1}{\alpha}\int\limits_0^{t_{j+\frac{k}{p}}} e^{-\frac{1}{\alpha}\left(t_{j+\frac{k}{p}}-\xi\right)} u_\xi\, d\xi = \sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right) u_{\bar{t}}^{\frac{s}{p}} + O\left(\frac{\tau}{p}\right). \tag{3.3}$$

Similarly to [20, Ch. VII, Sect. 1, Item 10], for equation (3.1) with index $k$ we obtain a monotone scheme of second order of approximation in $h_k$, for which the maximum principle holds for all $\tau$ and $h_k$, $k=1,2,\dots,p$. In order to do this, we consider equation (3.1) for a fixed $k$ with a perturbed operator $\widetilde{L}_k$:

$$\frac{\varepsilon}{p}\frac{\partial\vartheta}{\partial t} + \frac{1}{p}\mathcal{B}\vartheta_{(k)} = \widetilde{L}_k\vartheta_{(k)} + f_k, \qquad t\in\Delta_k, \qquad k=1,2,\dots,p, \tag{3.4}$$

where

$$\widetilde{L}_k\vartheta_{(k)} = \chi_k\frac{\partial}{\partial x_k}\left(\Theta_k(x,t)\frac{\partial\vartheta_{(k)}}{\partial x_k}\right) + r_k(x,t)\frac{\partial\vartheta_{(k)}}{\partial x_k} - q_k(x,t)\vartheta_{(k)}, \qquad \chi_k = \frac{1}{1+R_k},$$

and $R_k = 0.5h_k\dfrac{|r_k|}{\Theta_k}$ is a Raynolds difference number.

In view of (3.3), we replace each of equations (3.4) by a scheme on $\Delta_k$, $k=1,2,\dots,p$:

$$\begin{aligned}
&\frac{\varepsilon}{p}y_{\bar{t}}^{j+\frac{k}{p}} + \frac{1}{p}\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right) y_{\bar{t}}^{\frac{s}{p}}\\
&\qquad = \widetilde{\Lambda}_k\left(\sigma_k y^{j+\frac{k}{p}} + (1-\sigma_k)y^{j+\frac{k-1}{p}}\right) + \varphi_k^{j+\frac{k}{p}}, \quad x\in\omega_h,\\
&y^{j+\frac{k}{p}}|_{\gamma_{h,k}} = \mu^{j+\frac{k}{p}}, \quad j=0,1,\dots,j_0-1,\\
&y(x,0) = u_0(x),
\end{aligned} \tag{3.5}$$

where $\sigma_k$ are arbitrary parameters, $\gamma_{h,k}$ is the set of all boundary nodes in the direction of $x_k$,

$$x\in\bar{\omega}_h = \left\{x_i = (i_1h_1,\dots,i_ph_p)\in\overline{G}, \qquad i_k=0,1,\dots,N_k, \quad h_k=\frac{l_k}{N_k}\right\}, \qquad y_{\bar{t}}^{\frac{s}{p}} = \frac{y^{\frac{s}{p}} - y^{\frac{s-1}{p}}}{\frac{\tau}{p}},$$

$$\widetilde{\Lambda}_k y^{j+\frac{k}{p}} = \chi_k\left(a_k y_{\bar{x}}^{j+\frac{k}{p}}\right)_{x_k} + b_k^+ a_k^{(+1)} y_{x_k}^{j+\frac{k}{p}} + b_k^- a_k y_{\bar{x}_k}^{j+\frac{k}{p}} - d_k y^{j+\frac{k}{p}}, \qquad a_k = \Theta_k(x_{i-1/2},\bar{t}),$$

$$r_k^+ = 0.5(r_k+|r_k|)\geqslant 0, \qquad r_k^- = 0.5(r_k-|r_k|)\leqslant 0, \qquad b_k^+ = \frac{r_k^+}{\Theta_k},$$

$$b_k^- = \frac{r_k^-}{\Theta_k}, \qquad r_k = r_k^+ + r_k^-, d_k = q_k(x,\bar{t}), \qquad \varphi^{j+\frac{k}{p}} = f_k(x,\bar{t}),$$

$$\mu^{j+\frac{k}{p}} = \mu(x,\bar{t}), \qquad \bar{t} = t^{j+1/2}, \qquad k=1,2,\dots,p.$$

## 4. Error in approximation by locally one-dimensional scheme

We proceed to studying the error of approximation by locally one-dimensional scheme and we are going to make sure that each equation (3.5) with index $k$ does not approximate equation (2.6), but the sum of errors of approximations $\psi = \psi_1 + \psi_2 + \ldots + \psi_p$ tends to zero as $\tau$ and $|h|$ tend to zero.

We shall calculate $\sigma_k = 1$, $k=1,2,\dots,p$. Let $u=u(x,t)$ be a solution of problem (2.6), and $y^{j+\frac{k}{p}}$ is a solution of difference scheme (3.5). A characteristics of sharpness of locally one-dimensional scheme is the difference $y^{j+1} - u^{j+1} = z^{j+1}$. We compare intermediate values $y^{j+\frac{k}{p}}$

with $u^{j+\frac{k}{p}}=u\left(x,t_{j+\frac{k}{p}}\right)$ letting $z^{j+\frac{k}{p}}=y^{j+\frac{k}{p}}-u^{j+\frac{k}{p}}$. Substituting $y^{j+\frac{k}{p}}=z^{j+\frac{k}{p}}+u^{j+\frac{k}{p}}$ into difference equation (3.5), we get:

$$\frac{\varepsilon}{p}z_{\bar{t}}^{j+\frac{k}{p}}+\frac{1}{p}B_\tau z^{j+\frac{k}{p}}=\widetilde{\Lambda}_k z^{j+\frac{k}{p}}+\psi_k^{j+\frac{k}{p}}, \tag{4.1}$$

$$z^{j+\frac{k}{p}}|_{\gamma_{h,k}}=0, \quad z(x,0)=0, \tag{4.2}$$

where

$$\psi_k^{j+\frac{k}{p}}=\widetilde{\Lambda}_k u^{j+\frac{k}{p}}+\varphi_k^{j+\frac{k}{p}}-\frac{1}{p}\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}}-e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right)u_{\bar{t}}^{\frac{s}{p}}-\frac{\varepsilon}{p}u_{\bar{t}}^{j+\frac{k}{p}},$$

$$B_\tau z^{j+\frac{k}{p}}=\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}}-e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right)z_{\bar{t}}^{\frac{s}{p}}.$$

Introducing the notation $\overset{\circ}{\psi}_k=\left(L_k u+f_k-\frac{\varepsilon}{p}\frac{\partial u}{\partial t}-\frac{1}{p}\mathcal{B}u\right)^{j+\frac{1}{2}}$ and observing that $\sum\limits_{k=1}^{p}\overset{\circ}{\psi}_k=0$ if $\sum\limits_{k=1}^{p}f_k=f$, we represent $\psi_k=\psi_k^{j+\frac{k}{p}}$ in the form $\psi_k=\overset{\circ}{\psi}_k+\overset{*}{\psi}_k$, where

$$\overset{*}{\psi}_k=\left(\widetilde{\Lambda}_k u^{j+\frac{k}{p}}-L_k u^{j+\frac{1}{2}}\right)+\left(\varphi_k^{j+\frac{k}{p}}-f_k^{j+\frac{1}{2}}\right)-\left(\frac{1}{p}B_\tau u^{j+\frac{k}{p}}-\frac{1}{p}\mathcal{B}u^{j+\frac{1}{2}}\right)-\left(\frac{\varepsilon}{p}u_{\bar{t}}^{j+\frac{k}{p}}-\frac{\varepsilon}{p}\frac{\partial u^{j+\frac{1}{2}}}{\partial t}\right).$$

It is clear that $\overset{*}{\psi}_k=O(h_k^2+\tau)$ since each of the schemes (3.5) of the index $k$ approximates a corresponding equation (3.4) in the usual sense. Thus, locally one-dimensional scheme (3.5) possesses a total approximation

$$\overset{*}{\psi}_k=O(h_k^2+\tau), \qquad \overset{\circ}{\psi}_k=O(1), \qquad \sum_{k=1}^{p}\overset{\circ}{\psi}_k=0,$$

$$\psi=\sum_{k=1}^{p}\psi_k=\sum_{k=1}^{p}\left(\overset{\circ}{\psi}_k+\overset{*}{\psi}_k\right)=\sum_{k=1}^{p}\overset{*}{\psi}_k=O(|h|^2+\tau).$$

## 5. Stability of locally one-dimensional scheme

To solve difference scheme (3.5) by means of the maximum principle [20, Ch. IV, Sect. 2, Item 1], we obtain an apriori estimate in the uniform metrics expressing the stability of locally one-dimensional scheme by initial data and the right hand side. In order to do this, we represent the solution of problem (3.5) as the sum

$$y=\overline{y}+v,$$

where $\overline{y}$ is the solution of homogeneous equations (3.5) with inhomogeneous boundary and initial conditions

$$\begin{aligned}&\frac{\varepsilon}{p}\overline{y}_{\bar{t}}^{j+\frac{k}{p}}+\frac{1}{p}\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}}-e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right)\overline{y}_{\bar{t}}^{\frac{s}{p}}=\widetilde{\Lambda}_k\overline{y}^{j+\frac{k}{p}},\\&\overline{y}^{j+\frac{k}{p}}|_{\gamma_{h,k}}=\mu^{j+\frac{k}{p}}, \qquad \overline{y}(x,0)=u_0(x),\end{aligned} \tag{5.1}$$

and $v$ is the solution of inhomogeneous equations subject to boundary and initial conditions:

$$\begin{aligned}
&\frac{\varepsilon}{p} v_{\bar t}^{j+\frac{k}{p}} + \frac{1}{p}\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right) v_{\bar t}^{\frac{s}{p}} = \widetilde{\Lambda}_k v^{j+\frac{k}{p}} + \varphi_k^{j+\frac{k}{p}},\\
&v^{j+\frac{k}{p}}|_{\gamma_{h,k}} = 0, \qquad v(x,0) = 0.
\end{aligned} \tag{5.2}$$

We are going to obtain an estimate for $\bar y$. In order to do this, we write equation (5.1) in a canonical form. At the point $P = P(x_{i_k}, t_{j+\frac{k}{p}})$ we have

$$\begin{aligned}
&\left[\frac{1}{\tau}\left(\varepsilon + 1 - e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right) + \frac{\chi_{i_k} a_{k,i_k+1}}{h_k^2} + \frac{\chi_{i_k} a_{k,i_k}}{h_k^2} + \frac{b_k^+ a_{k,i_k+1}}{h_k} - \frac{b_k^- a_{k,i_k}}{h_k} + d_k\right] \bar y_{i_k}^{j+\frac{k}{p}}\\
&= \frac{\chi_{i_k} a_{k,i_k+1}}{h_k^2} \bar y_{i_k+1}^{j+\frac{k}{p}} + \frac{\chi_{i_k} a_{k,i_k}}{h_k^2} \bar y_{i_k-1}^{j+\frac{k}{p}} + \frac{b_k^+ a_{k,i_k+1}}{h_k} \bar y_{i_k+1}^{j+\frac{k}{p}} - \frac{b_k^- a_{k,i_k}}{h_k} \bar y_{i_k-1}^{j+\frac{k}{p}}\\
&+ \frac{1}{\tau}\left[\varepsilon + 1 - 2e^{-\frac{1}{\alpha}t_{\frac{1}{p}}} + e^{-\frac{1}{\alpha}t_{\frac{2}{p}}}\right] \bar y_{i_k}^{j+\frac{k-1}{p}} + \frac{1}{\tau}\left[e^{-\frac{1}{\alpha}t_{\frac{1}{p}}} - 2e^{-\frac{1}{\alpha}t_{\frac{2}{p}}} + e^{-\frac{1}{\alpha}t_{\frac{3}{p}}}\right] \bar y_{i_k}^{j+\frac{k-2}{p}}\\
&+ \dots + \frac{1}{\tau}\left[e^{-\frac{1}{\alpha}t_{j+\frac{k-2}{p}}} - 2e^{-\frac{1}{\alpha}t_{j+\frac{k-1}{p}}} + e^{-\frac{1}{\alpha}t_{j+\frac{k}{p}}}\right] \bar y_{i_k}^{\frac{1}{p}} + \frac{1}{\tau}\left[e^{-\frac{1}{\alpha}t_{j+\frac{k-1}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k}{p}}}\right] \bar y_{i_k}^{0}.
\end{aligned} \tag{5.3}$$

In [20, Ch. IV, Sect. 2, Item 1] the maximum principle was proved and apriori estimates for the solution of grid equation of general form

$$A(P)y(P) = \sum_{Q\in\Upsilon'(P)} B(P,Q)y(Q) + F(P), \qquad P\in\Omega, \qquad y(P) = \mu(P) \quad \text{as} \quad P\in S, \tag{5.4}$$

where $P$, $Q$ are the nodes of the grid $\Omega + S$, $\Upsilon'(P)$ is the neighbourhood of the node $P$ not containing the node $P$ itself. The coefficients $A(P)$, $B(P,Q)$ in (5.4) satisfy the conditions

$$A(P) > 0, \qquad B(P,Q) > 0, \qquad D(P) = A(P) - \sum_{Q\in\Upsilon'(P)} B(P,Q) \geqslant 0. \tag{5.5}$$

For a given $x\in\omega_h$, $t'\in\omega'_\tau$ by $P(x,t')$ we denote a node in the $(p+1)$-dimensional grid $\Omega = \omega_h\times\omega'_\tau$, by $S$ we denote the boundary of $\Omega$ consisting of the nodes $P(x,0)$ as $x\in\overline{\omega}_h$ and the nodes $P\left(x, t_{j+\frac{k}{p}}\right)$ as $t_{j+\frac{k}{p}}\in\omega'_\tau$ and $x\in\gamma_{h,k}$ for all $k = 1,2,\dots,p$, $j = 0,1,\dots,j_0$.

We see that the coefficients of equation (5.3) at the point $P = P\left(x_{i_k}, t_{j+\frac{k}{p}}\right)$ satisfy the conditions (5.5) and $D(P) = 0$.

It follows from Theorem 3 [21, Ch. V, Addendum, Sect. 2, Item 2] that the solution of problem (5.1) satisfies the estimate

$$\|\overline{y}^j\|_C \leqslant \|u_0\|_C + \max_{0<t'\leqslant j\tau} \|\mu(x,t')\|_{C_\gamma}, \tag{5.6}$$

where $\|y\|_C = \max\limits_{x\in\overline{\omega}_h}|y|$, $\|y\|_{C_\gamma} = \max\limits_{x\in\gamma_h}|y|$.

We proceed to estimating the function $v$. We rewrite equation (5.2) in the form

$$\begin{aligned}
&\frac{1}{p}\left(\varepsilon + 1 - e^{-\frac{1}{\alpha}t_{\frac{1}{p}}}\right) v_{\bar t}^{j+\frac{k}{p}} = \widetilde{\Lambda}_k v^{j+\frac{k}{p}} + \widetilde{\varphi}_k^{j+\frac{k}{p}},\\
&v^{j+\frac{k}{p}}|_{\gamma_{h,k}} = 0, \qquad v(x,0) = 0,
\end{aligned} \tag{5.7}$$

where

$$\widetilde{\varphi}_k^{j+\frac{k}{p}} = \varphi^{j+\frac{k}{p}} - \frac{1}{p}\sum_{s=1}^{pj+k-1}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right) v_{\bar t}^{\frac{s}{p}}.$$

We transform equation (5.7) to a canonical form

$$\left[\frac{1}{\tau}\left(\varepsilon+1-e^{-\frac{1}{\alpha}t_{\frac{1}{p}}}\right)+\frac{\chi_{i_k}a_{k,i_k+1}}{h_k^2}+\frac{\chi_{i_k}a_{k,i_k}}{h_k^2}+\frac{b_k^+a_{k,i_k+1}}{h_k}-\frac{b_k^-a_{k,i_k}}{h_k}+d_k\right]v_{i_k}^{j+\frac{k}{p}}$$
$$=\frac{1}{h_k^2}\left[\chi_{i_k}a_{k,i_k+1}v_{i_k+1}^{j+\frac{k}{p}}+\chi_{i_k}a_{k,i_k}v_{i_k-1}^{j+\frac{k}{p}}\right]+\frac{b_k^+a_{k,i_k+1}}{h_k}v_{i_k+1}^{j+\frac{k}{p}}-\frac{b_k^-a_{k,i_k}}{h_k}v_{i_k-1}^{j+\frac{k}{p}}+\Phi(P_{j+\frac{k}{p}}),$$

where

$$\Phi(P_{j+\frac{k}{p}})=\frac{1}{\tau}\left(\varepsilon+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2\right)v_{i_k}^{j+\frac{k-1}{p}}+\overline{\varphi}_k^{j+\frac{k}{p}},$$
$$\overline{\varphi}_k^{j+\frac{k}{p}}=\varphi_k^{j+\frac{k}{p}}+\frac{1}{\tau}\left(e^{-\frac{1}{\alpha}t_{\frac{1}{p}}}-e^{-\frac{1}{\alpha}t_{\frac{2}{p}}}\right)v_{i_k}^{j+\frac{k-2}{p}}$$
$$-\frac{1}{\tau}\sum_{s=1}^{pj+k-2}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}}-e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right)\left(v_{i_k}^{\frac{s}{p}}-v_{i_k}^{\frac{s-1}{p}}\right).$$

Let us check the assumptions of Theorem 4 [21, Ch. V, Addendum, Sect. 2, Item 2]:

$$D'(P_{(k)})=A(P_{(k)})-\sum_{Q\in\Upsilon'_k(P)}B(P_{(k)},Q)=\frac{1}{\tau}\left(\varepsilon+1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)+d_k$$
$$\geqslant\frac{1}{\tau}\left(\varepsilon+1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)>0, \tag{5.8}$$

where

$$P_{(k)}=P(x,t_{j+\frac{k}{p}}),\qquad A(P_{(k)})>0,\qquad B(P_{(k)},Q)>0\qquad \text{for all}\qquad Q\in\Upsilon''_{k-1},\quad Q\in\Upsilon'_k.$$

Then we obtain

$$\sum_{Q\in\Upsilon''_{k-1}}B(P_{(k)},Q)=\frac{1}{\tau}\left(\varepsilon+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2\right)>0,$$
$$\frac{1}{D'(P_{(k)})}\sum_{Q\in\Upsilon''_{k-1}}B(P_{(k)},Q)=\frac{\varepsilon+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2}{\varepsilon+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)}\leqslant 1,$$

where

$$\Upsilon'_{\left(P\left(x,t_{j+\frac{k}{p}}\right)\right)}=\Upsilon'_k+\Upsilon'_{k-1},$$

$\Upsilon'_k$ is the set of nodes $Q=Q(\xi,t_k)\in\Upsilon'_{(P(x,t_k))}$, $\Upsilon'_{k-1}$ is the set of nodes $Q=Q(\xi,t_{k-1})\in\Upsilon'_{(P(x,t_{k-1}))}$.

On the base of Theorem 3 [21, Ch. V, Addendum, Sect. 2, Item 2], by (5.8) we obtain

$$\|v^{j+\frac{k}{p}}\|_C\leqslant\frac{\tau}{\varepsilon+1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}}\|\overline{\varphi}_k^{j+\frac{k}{p}}\|_C+\frac{\varepsilon+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2}{\varepsilon+1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}}\|v^{j+\frac{k-1}{p}}\|_C. \tag{5.9}$$

Let us estimate $\|\overline{\varphi}_k^{j+\frac{k}{p}}\|_C$, where

$$\begin{aligned}\overline{\varphi}_k^{j+\frac{k}{p}} =&\varphi_k^{j+\frac{k}{p}} + \frac{1}{\tau}\left(e^{-\frac{1}{\alpha}t_{\frac{1}{p}}} - e^{-\frac{1}{\alpha}t_{\frac{2}{p}}}\right) v_{i_k}^{j+\frac{k-2}{p}} \\ &- \frac{1}{\tau}\sum_{s=1}^{pj+k-2}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right)\left(v_{i_k}^{\frac{s}{p}} - v_{i_k}^{\frac{s-1}{p}}\right) \\ =&\varphi_k^{j+\frac{k}{p}} + \frac{1}{\tau}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-1}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k}{p}}}\right) v_{i_k}^0 \\ &+ \frac{1}{\tau}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-2}{p}}} - 2e^{-\frac{1}{\alpha}t_{j+\frac{k-1}{p}}} + e^{-\frac{1}{\alpha}t_{j+\frac{k}{p}}}\right) v_{i_k}^{\frac{1}{p}} + \cdots \\ &+ \frac{1}{\tau}\left(e^{-\frac{1}{\alpha}t_{\frac{1}{p}}} - 2e^{-\frac{1}{\alpha}t_{\frac{2}{p}}} + e^{-\frac{1}{\alpha}t_{\frac{3}{p}}}\right) v_{i_k}^{j+\frac{k-2}{p}}.\end{aligned} \tag{5.10}$$

Since the expression in round brackets are positive, by (5.10) we obtain the estimate

$$\|\overline{\varphi}_k^{j+\frac{k}{p}}\|_C \leqslant \|\varphi_k^{j+\frac{k}{p}}\|_C + \frac{1}{\tau}\left(e^{-\frac{1}{\alpha}t_{\frac{1}{p}}} - e^{-\frac{1}{\alpha}t_{\frac{2}{p}}}\right) \max_{0\leqslant j'<j}\max_{0\leqslant s\leqslant k-2}\|v^{j'+\frac{s}{p}}\|_C. \tag{5.11}$$

By means of (5.11) and (5.9) we find

$$\max_{0\leqslant j'<j}\max_{0\leqslant s\leqslant k}\|v^{j'+\frac{s}{p}}\|_C \leqslant \max_{0\leqslant j'<j}\max_{0\leqslant s\leqslant k-1}\|v^{j'+\frac{s}{p}}\|_C + \frac{\tau}{\varepsilon + 1 - e^{-\frac{1}{\alpha}\frac{\tau}{p}}}\max_{0\leqslant j'<j}\max_{0\leqslant s\leqslant k}\|\varphi_k^{j'+\frac{s}{p}}\|_C. \tag{5.12}$$

Summing up (5.12) first over $k = 1, 2, \ldots, p$ and then over $j' = 0, 1, \ldots, j$, we obtain the estimate

$$\|v^{j+1}\|_C \leqslant \|v^0\|_C + \sum_{j'=0}^{j}\frac{\tau}{\varepsilon + \gamma\tau}\sum_{k=1}^{p}\max_{0\leqslant s\leqslant k}\|\varphi^{j'+\frac{s}{p}}\|_C, \tag{5.13}$$

where $\gamma = \frac{1}{\alpha p}$.

Estimates (5.6) and (5.13) imply a final one:

$$\|y^{j+1}\|_C \leqslant \|y^0\|_C + \max_{0<t'\leqslant t_{j+1}}\|\mu(x,t')\|_{C_\gamma} + \sum_{j'=0}^{j}\frac{\tau}{\varepsilon + \gamma\tau}\sum_{k=1}^{p}\max_{0\leqslant s\leqslant k}\|\varphi^{j'+\frac{s}{p}}\|_C. \tag{5.14}$$

Thus, the following theorem holds true.

**Theorem 5.1.** *Let conditions (2.4) be satisfied, then locally one-dimensional scheme (3.5) is stable in the initial data and the right hand side and the solution of problem (3.5) obeys estimate (5.14).*

## 6. Uniform convergence of locally one-dimensional scheme

In order to use the property

$$\sum_{k=1}^{p}\mathring{\psi}_k = 0, \qquad \mathring{\psi}_k = O(1),$$

similar to [20, Ch. IX, Sect. 3, Item 8], we represent the solutions of problems for errors (4.1)–(4.2) as the sum

$$z_{(k)} = v_{(k)} + \eta_{(k)}, \qquad z_{(k)} = z^{j+\frac{k}{p}},$$

where $\eta_{(k)}$, $k = 1, 2, \ldots, p$, are determined by the conditions

$$\frac{\varepsilon}{p}\eta_{\bar t}^{j+\frac{k}{p}} + \frac{1}{p}\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right)\eta_{\bar t}^{\frac{s}{p}} = \mathring{\psi}_k, \quad x \in \omega_h + \gamma_{h,k}, \qquad \eta(x,0) = 0. \tag{6.1}$$

The function $v_{(k)}$ is determined by the conditions

$$\begin{aligned} &\frac{\varepsilon}{p} v_{\bar{t}}^{j+\frac{k}{p}} + \frac{1}{p}\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right) v_{\bar{t}}^{\frac{s}{p}} = \widetilde{\Lambda}_k v_{(k)} + \widetilde{\psi}_k, \\ &v_{(k)}|_{\gamma_{h,k}} = -\eta_{(k)}, \quad v(x,0) = 0, \end{aligned} \tag{6.2}$$

where

$$\widetilde{\psi}_k = \overset{*}{\psi}_k + \widetilde{\Lambda}_k \eta_{(k)}, \qquad \overset{*}{\psi}_k = 0(h_k^2 + \tau).$$

Let us show that

$$\eta^{j+\frac{k}{p}} = O\left(\frac{\tau}{\varepsilon + \gamma\tau}\right), \quad k = 1, 2, \dots, p, \quad j = 0, 1, 2, \dots, j_0 - 1.$$

For the sake of simplicity we consider a two-dimensional case ($p = 2$). We first let $j = 0$ and consider the first layer $(0, t_1]$. Then problem (6.1) becomes

$$\frac{\varepsilon}{2}\eta_{\bar{t}}^{\frac{k}{2}} + \frac{1}{2}\sum_{s=1}^{k}\left(e^{-\frac{1}{\alpha}t_{\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{\frac{k-s+1}{p}}}\right)\eta_{\bar{t}}^{\frac{s}{2}} = \overset{\circ}{\psi}_k, \qquad k = 1, 2.$$

Letting $k = 1$, we obtain

$$\frac{\varepsilon}{2}\eta_{\bar{t}}^{\frac{1}{2}} + \frac{1}{2}\left(1 - e^{-\frac{1}{\alpha}\frac{\tau}{2}}\right)\eta_{\bar{t}}^{\frac{1}{2}} = \overset{\circ}{\psi}_1. \tag{6.3}$$

For $k = 2$ we get

$$\frac{\varepsilon}{2}\eta_{\bar{t}}^{1} + \frac{1}{2}\left(e^{-\frac{1}{\alpha}t_{\frac{1}{2}}} - e^{-\frac{1}{\alpha}t_1}\right)\eta_{\bar{t}}^{\frac{1}{2}} + \frac{1}{2}\left(1 - e^{-\frac{1}{\alpha}t_{\frac{1}{2}}}\right)\eta_{\bar{t}}^{1} = \overset{\circ}{\psi}_2. \tag{6.4}$$

Summing expressions (6.3) and (6.4), we find:

$$\frac{\varepsilon}{2}\eta_{\bar{t}}^{\frac{1}{2}} + \frac{\varepsilon}{2}\eta_{\bar{t}}^{1} + \frac{1}{2}\left(1 - e^{-\frac{1}{\alpha}\frac{\tau}{2}}\right)\eta_{\bar{t}}^{\frac{1}{2}} + \frac{1}{2}\left(e^{-\frac{1}{\alpha}\frac{\tau}{2}} - e^{-\frac{1}{\alpha}\tau}\right)\eta_{\bar{t}}^{\frac{1}{2}} + \frac{1}{2}\left(1 - e^{-\frac{1}{\alpha}\frac{\tau}{2}}\right)\eta_{\bar{t}}^{1} = 0.$$

This yields

$$\eta^1 = \frac{e^{-\frac{1}{\alpha}\frac{\tau}{2}}\left(e^{-\frac{1}{\alpha}\frac{\tau}{2}} - 1\right)\eta^{\frac{1}{2}}}{\varepsilon + 1 - e^{-\frac{1}{\alpha}\frac{\tau}{2}}} = -\frac{e^{-\frac{1}{\alpha}\frac{\tau}{2}}\gamma\tau\eta^{\frac{1}{2}}}{\varepsilon + \gamma\tau}. \tag{6.5}$$

By (6.3) we find

$$\eta^{\frac{1}{2}} = \frac{\tau}{\varepsilon + 1 - e^{-\frac{1}{\alpha}\frac{\tau}{2}}}\overset{\circ}{\psi}_1 = \frac{\tau}{\varepsilon + \gamma\tau}\,\psi_1 = -\frac{\tau}{\varepsilon + \gamma\tau}\,\psi_2\,. \tag{6.6}$$

Taking into consideration (6.6), by (6.5) have

$$\eta^{\frac{1}{2}} = O\left(\frac{\tau}{\varepsilon + \gamma\tau}\right), \qquad \eta^1 = O\left(\frac{\tau^2}{(\varepsilon + \gamma\tau)^2}\right).$$

Suppose that as $j = n$, the condition

$$\eta^{\frac{1}{2}}, \eta^1, \eta^{1+\frac{1}{2}}, \dots, \eta^{n+1} = O\left(\frac{\tau}{\varepsilon + \gamma\tau}\right). \tag{6.7}$$

holds. Basing on this condition, we are going to show a similar condition holds also for $j = n+1$. In order to do this we write equation (6.1) for $j = n + 1$, $p = 2$ :

$$\frac{\varepsilon}{2}\eta_{\bar{t}}^{n+1+\frac{k}{2}} + \frac{1}{2}\sum_{s=1}^{2(n+1)+k}\left(e^{-\frac{1}{\alpha}t_{n+1+\frac{k-s}{2}}} - e^{-\frac{1}{\alpha}t_{n+1+\frac{k-s+1}{2}}}\right)\eta_{\bar{t}}^{\frac{s}{2}} = \overset{\circ}{\psi}_k, \quad k = 1, 2. \tag{6.8}$$

Letting $k = 1$ in (6.8), we find

$$\frac{\varepsilon}{2}\eta_{\bar{t}}^{n+\frac{3}{2}} + \frac{1}{2}\sum_{s=1}^{2(n+1)+1}\left(e^{-\frac{1}{\alpha}t_{n+1+\frac{1-s}{2}}} - e^{-\frac{1}{\alpha}t_{n+1+\frac{2-s}{2}}}\right)\eta_{\bar{t}}^{\frac{s}{2}} = \overset{\circ}{\psi}_1.$$

We transform this identity as

$$-\frac{1}{\tau}\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}\right)\left[\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}\right)e^{-\frac{1}{\alpha}(n+\frac{1}{2})\tau}\eta^{\frac{1}{2}}+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}\right)e^{-\frac{1}{\alpha}n\tau}\eta^{1}+\ldots\right.$$
$$\left.+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}+\frac{\varepsilon}{1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}}\right)\eta^{n+1}-\left(1+\frac{\varepsilon}{1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}}\right)\eta^{n+\frac{3}{2}}\right]=\mathring{\psi}_1. \tag{6.9}$$

We consider separately the expressions in square brackets and rewrite as

$$\left(1+\frac{\varepsilon}{1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}}\right)|\eta^{n+\frac{3}{2}}|\leqslant\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}+\frac{\varepsilon}{1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}}\right)\max_{\frac{1}{2}\leqslant s\leqslant n+1}|\eta^s|\sum_{s=0}^{n+\frac{1}{2}}e^{-\frac{1}{\alpha}s\tau}$$

and therefore,

$$|\eta^{n+\frac{3}{2}}|\leqslant\frac{1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}+\frac{\varepsilon}{1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}}}{1+\frac{\varepsilon}{1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}}}\max_{\frac{1}{2}\leqslant s\leqslant n+1}|\eta^s|\leqslant\max_{\frac{1}{2}\leqslant s\leqslant n+1}|\eta^s|=O\left(\frac{\tau}{\varepsilon+\gamma\tau}\right). \tag{6.10}$$

Taking into consideration (6.7), (6.10) and a sufficient boundedness of the coefficients as $\eta^{\frac{1}{2}},\eta^1$ ,..., $\eta^{n+\frac{3}{2}}$, we find: $\eta^{n+\frac{3}{2}}=O\left(\dfrac{\tau}{\varepsilon+\gamma\tau}\right)$.

Now we let $k=2$ in (6.8):

$$\frac{\varepsilon}{2}\eta_{\bar t}^{n+2}+\frac{1}{2}\sum_{s=1}^{2n+4}\left(e^{-\frac{1}{\alpha}t_{n+1+\frac{2-s}{2}}}-e^{-\frac{1}{\alpha}t_{n+1+\frac{3-s}{2}}}\right)\eta_{\bar t}^{\frac{s}{2}}=\mathring{\psi}_2,$$

then

$$-\frac{1}{\tau}\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}\right)\left[(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}})e^{-\frac{1}{\alpha}(n+1)\tau}\eta^{\frac{1}{2}}\right.$$
$$+(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}})e^{-\frac{1}{\alpha}(n+\frac{1}{2})\tau}\eta^{1}+\ldots+(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}})e^{-\frac{1}{\alpha}\frac{\tau}{2}}\eta^{n+1}$$
$$\left.+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}+\frac{\varepsilon}{1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}}\right)\eta^{n+\frac{3}{2}}-\left(1+\frac{\varepsilon}{1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}}\right)\eta^{n+2}\right]=\mathring{\psi}_2. \tag{6.11}$$

Summing (6.9) and (6.11), in view of identity $\mathring{\psi}_1+\mathring{\psi}_2=0$ we obtain

$$-\frac{1}{\tau}\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}\right)\left[\left(1-e^{-\frac{1}{\alpha}\tau}\right)e^{-\frac{1}{\alpha}(n+\frac{1}{2})\tau}\eta^{\frac{1}{2}}+\left(1-e^{-\frac{1}{\alpha}\tau}\right)e^{-\frac{1}{\alpha}n\tau}\eta^{1}+\ldots\right.$$
$$\left.+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}+\frac{\varepsilon}{1-e^{-\frac{1}{\alpha}\tau}}\right)\eta^{n+1}-e^{\frac{1}{\alpha}\tau}\eta^{n+\frac{3}{2}}-\left(1+\frac{\varepsilon}{1-e^{-\frac{1}{\alpha}\frac{\tau}{2}}}\right)\eta^{n+2}\right]=0. \tag{6.12}$$

Then by (6.12) we obtain

$$\eta^{\frac{1}{2}},\eta^{1},\eta^{n+1},\eta^{n+\frac{3}{2}},\eta^{n+2}=O\left(\frac{\tau}{\varepsilon+\gamma\tau}\right). \tag{6.13}$$

Hence, identity (6.13) is satisfied for each value $j$. In the same way we also show that

$$\eta^{j+\frac{k}{p}}=O\left(\frac{\tau}{\varepsilon+\gamma\tau}\right),\qquad k=1,2,\ldots,p,\qquad j=0,1,\ldots,j_0-1.$$

To estimate the solution of problem (6.2) we employ Theorem 5.1:

$$\|v^{j+1}\|_C \leqslant \max_{0<j'+\frac{k}{p}\leqslant j+1} \|\eta^{j'+\frac{k}{p}}\|_{C_\gamma} + \sum_{j'=0}^{j} \frac{\tau}{\varepsilon+\gamma\tau} \sum_{k=1}^{p} \max_{0\leqslant s\leqslant k} \|\widetilde{\psi}^{j'+\frac{s}{p}}\|_C. \tag{6.14}$$

If there exist continuous in $\overline{Q}_T$ derivatives $\dfrac{\partial^4 u}{\partial x_k^2 \partial x_\nu^2}$, $k \neq \nu$, then

$$\widetilde{\Lambda}_k \eta_{(k)} = -\frac{\tau}{\tau+\varepsilon} a_k \widetilde{\Lambda}_k (\mathring{\psi}_{k+1} + \ldots + \mathring{\psi}_p) = O\left(\frac{\tau}{\varepsilon+\gamma\tau}\right),$$

$a_k$ are known constants. Then by estimate (6.14) we find:

$$\begin{aligned}\|v^{j+1}\|_C &= M\left(\frac{\tau}{\varepsilon+\gamma\tau} + \sum_{j'=0}^{j} \frac{\tau}{\varepsilon+\gamma\tau} \sum_{k=1}^{p} \left(h_k^2 + \frac{\tau}{\varepsilon+\gamma\tau}\right)\right) \\ &= M\left(\frac{\tau}{\varepsilon+\gamma\tau} + \frac{pt_j}{\varepsilon+\gamma\tau}\left(h^2 + \frac{\tau}{\varepsilon+\gamma\tau}\right)\right) \\ &\leqslant M\left(\frac{h^2}{\varepsilon+\tau} + \frac{\tau}{(\varepsilon+\tau)^2}\right), \quad h = \max_{1\leqslant k\leqslant p} h_k.\end{aligned}$$

This yields

$$\|z^{j+1}\|_C \leqslant \|\eta^{j+1}\|_C + \|v^{j+1}\|_C = O\left(\frac{h^2}{\varepsilon+\tau} + \frac{\tau}{(\varepsilon+\tau)^2}\right).$$

Thus, the following theorem is true.

**Theorem 6.1.** *Let problem (2.6)–(2.8) possesses a unique continuous solution $u(x,t)$ in $\overline{Q}_T$ for all values $\varepsilon$ and there exist continuous in $\overline{Q}_T$ derivatives*

$$\frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial^4 u}{\partial x_k^2 \partial x_\nu^2}, \quad \frac{\partial^3 u}{\partial x_k^2 \partial t}, \quad \frac{\partial^2 f}{\partial x_k^2}, \quad 1 \leqslant k, \quad \nu \leqslant p, \quad k \neq \nu, \quad \alpha > 0,$$

*and conditions (2.4) hold. Then the solution of difference scheme (3.5) uniformly converges to the solution of differential problem (2.6)–(2.8) at the rate*

$$O\left(\frac{h^2}{\varepsilon+\tau} + \frac{\tau}{(\varepsilon+\tau)^2}\right), \qquad h^2 = o(\varepsilon+\tau), \qquad \tau = o\left((\varepsilon+\tau)^2\right),$$

*where $\varepsilon$ is a small parameter, $\alpha > 0$.*

## 7. Initial boundary value problem with special conditions

Instead of boundary conditions (2.2) we consider ones of form

$$\begin{cases} \Theta_k u_{x_k} + \alpha \dfrac{\partial\left(\Theta_k u_{x_k}\right)}{\partial t} = \beta_{-k}(x)\dfrac{\partial u}{\partial t} - \mu_{-k}(x,t), & x_k = 0, \quad 0 \leqslant t \leqslant T, \\ -\left(\Theta_k u_{x_k} + \alpha \dfrac{\partial\left(\Theta_k u_{x_k}\right)}{\partial t}\right) = \beta_{+k}(x)\dfrac{\partial u}{\partial t} - \mu_{+k}(x,t), & x_k = l_k, \quad 0 \leqslant t \leqslant T, \end{cases} \tag{7.1}$$

where

$$0 < c_0 \leqslant \beta_{\pm k}(x) \leqslant c_1, \qquad r_k(0,t) \geqslant 0, \qquad r_k(l_k,t) \leqslant 0, \tag{7.2}$$

$\beta_{-k} = \beta(0,x')$, $\beta_{+k} = \beta(l_k,x')$, $\mu_{-k} = \mu(0,x',t)$, $\mu_{+k} = \mu(l_k,x',t)$ are continuous functions.

Similar problems for the pseudoparabolic equation arise while regulating the salt regime of soils, when the desalinization of the upper layer is achieved by draining the water layer from the surface of the flooded for some time plot [8]. If there is a water layer of a constant thickness

$h$ on the field surface, then at the upper boundary in view of the fractality of the soil medium, the condition

$$h\frac{\partial c}{\partial t} = D\frac{\partial c}{\partial x} + A\frac{\partial^2 c}{\partial t \partial x}$$

should be imposed, where $c$ is the salt concentration in soil solution, $D$ is the diffusion coefficient, $A$, $h = const > 0$.

We transform conditions (7.1) by multiplying both sides of (7.1) by $\frac{1}{\alpha}e^{\frac{1}{\alpha}t}$, replacing $t$ by $\xi$ and integrating the obtained expression in $\xi$ from 0 to $t$:

$$\begin{cases} \Theta_k(x,t)\dfrac{\partial u}{\partial x_k} = \mathcal{B}_{-k}u - \widetilde{\mu}_{-k}(x,t), & x_k = 0, \quad 0 \leqslant t \leqslant T, \\ -\Theta_k(x,t)\dfrac{\partial u}{\partial x_k} = \mathcal{B}_{+k}u - \widetilde{\mu}_{+k}(x,t), & x_k = l_k, \quad 0 \leqslant t \leqslant T, \end{cases} \tag{7.3}$$

where

$$\mathcal{B}_{-k}u(0,x',\xi) = \frac{\beta_{-k}(0,x')}{\alpha}\int_0^t e^{-\frac{1}{\alpha}(t-\xi)}\frac{\partial u}{\partial t}(0,x',\xi)\,d\xi,$$

$$\mathcal{B}_{+k}u(l_k,x',\xi) = \frac{\beta_{+k}(l_k,x')}{\alpha}\int_0^t e^{-\frac{1}{\alpha}(t-\xi)}\frac{\partial u}{\partial t}(l_k,x',\xi)\,d\xi,$$

$$\widetilde{\mu}_{-k}(0,x',t) = \frac{1}{\alpha}\int_0^t e^{-\frac{1}{\alpha}(t-\xi)}\mu_{-\alpha}(0,x',\tau)d\xi + e^{-\frac{t}{\alpha}}\Theta_k(x,0)u_0'(0,x'),$$

$$\widetilde{\mu}_{+k}(1,x',t) = \frac{1}{\alpha}\int_0^t e^{-\frac{1}{\alpha}(t-\xi)}\mu_{+\alpha}(1,x',\xi)\,d\xi - e^{-\frac{t}{\alpha}}\Theta_k(x,0)u_0'(1,x').$$

In the same domain, instead of problem (2.3), (2.5), (7.3) we consider problem (2.6), (2.8) with the small parameter $\varepsilon$ with boundary conditions

$$\begin{cases} \Theta_k(x,t)\dfrac{\partial u^\varepsilon}{\partial x_k} = \mathcal{B}_{-k}u^\varepsilon - \widetilde{\mu}_{-k}(x,t), & x_k = 0, \quad 0 \leqslant t \leqslant T, \\ -\Theta_k(x,t)\dfrac{\partial u^\varepsilon}{\partial x_k} = \mathcal{B}_{+k}u^\varepsilon - \widetilde{\mu}_{+k}(x,t), & x_k = l_k, \quad 0 \leqslant t \leqslant T, \end{cases} \tag{7.4}$$

where $\varepsilon = const > 0$.

We are going to show that $u^\varepsilon \to u$ in some norm as $\varepsilon \to 0$. We denote $\tilde{z} = u^\varepsilon - u$ and substitute $u^\varepsilon = \tilde{z} + u$ into problem (2.6), (2.8), (7.4). Then we obtain problem (2.9), (2.11) with boundary conditions

$$\begin{cases} \Theta_k(x,t)\dfrac{\partial \tilde{z}}{\partial x_k} = \mathcal{B}_{-k}\tilde{z}, & x_k = 0, \quad 0 \leqslant t \leqslant T, \\ -\Theta_k(x,t)\dfrac{\partial u}{\partial x_k} = \mathcal{B}_{+k}\tilde{z}, & x_k = l_k, \quad 0 \leqslant t \leqslant T. \end{cases} \tag{7.5}$$

Since

$$\sum_{k=1}^p \int_{G'} \Theta_k(x,t)\tilde{z}\frac{\partial \tilde{z}}{\partial x_k}\bigg|_0^{l_k} dx' = \sum_{k=1}^p \int_{G'} \Big(\tilde{z}(l_k,x',t)\mathcal{B}_{+k}\tilde{z}(l_k,x',t) - \tilde{z}(0,x',t)\mathcal{B}_{-k}\tilde{z}(0,x',t)\Big)\,dx'$$
$$\leqslant -\sum_{k=1}^p \int_{G'} \left(\frac{\beta_{+k}(l_k,x')}{\alpha}\tilde{z}^2(l_k,x',t) + \frac{\beta_{-k}(0,x')}{\alpha}\tilde{z}^2(0,x',t)\right)\,dx'$$

$$+\varepsilon_2 M_1\left(\left\|\frac{\partial\tilde z}{\partial x}\right\|_0^2+\|\tilde z\|_0^2\right)+M_2^{\varepsilon_2}\int_0^t\left(\left\|\frac{\partial\tilde z}{\partial x}\right\|_0^2+\|\tilde z\|_0^2\right)d\tau,$$

reproducing arguing (2.13)–(2.24), by (2.13) for the solution of problem (2.9), (2.11), (7.5) we obtain the estimate

$$\varepsilon\|\tilde z\|_0^2+\int_0^t\left(\|\tilde z\|_0^2+\left\|\frac{\partial\tilde z}{\partial x}\right\|_0^2\right)d\tau\leqslant M_3\int_0^t\|\overline{f}\|_0^2 d\tau=\varepsilon^2 M_3\int_0^t\left\|\frac{\partial u}{\partial\tau}\right\|_0^2 d\tau. \tag{7.6}$$

This apriori estimate implies the convergence of $u^\varepsilon$ to $u$ as $\varepsilon\to 0$ in the norm

$$\|\tilde z\|_1^2=\varepsilon\|\tilde z\|_0^2+\|\tilde z\|_{2,Q_t}^2+\left\|\frac{\partial\tilde z}{\partial x}\right\|_{2,Q_t}^2$$

if $u_t$ is a bounded sufficiently smooth function. This is why for a small $\varepsilon$, the solution of problem (2.6), (2.8), (7.4) is taken as an approximate solution of initial boundary value problem of a multi-dimensional pseudoparabolic equation (2.1), (7.1), (2.3) .

## 8. Construction of locally one-dimensional scheme

On each semi-interval $\Delta_k=\left(t_{j+\frac{k-1}{p}},t_{j+\frac{k}{p}}\right]$, $k=1,2,\ldots,p$, we subsequently solve problems

$$\pounds_k\vartheta_{(k)}=\frac{\varepsilon}{p}\vartheta_t+\frac{1}{p}\mathcal{B}\vartheta_{(k)}-\widetilde{L}_k\vartheta_{(k)}-f_k=0,\qquad x\in G,\quad t\in\Delta_k,\qquad k=1,2,\ldots,p, \tag{8.1}$$

$$\begin{cases}\Theta_k(x,t)\dfrac{\partial\vartheta_{(k)}}{\partial x_k}=\mathcal{B}_{-k}(x,t)\vartheta_{(k)}-\mu_{-k}(x,t), & x_k=0,\\ -\Theta_k(x,t)\dfrac{\partial\vartheta_{(k)}}{\partial x_k}=\mathcal{B}_{+k}(x,t)\vartheta_{(k)}-\mu_{+k}(x,t), & x_k=l_k,\end{cases} \tag{8.2}$$

letting at the same time

$$\begin{aligned}&\vartheta_{(1)}(x,0)=u_0(x),\qquad \vartheta_{(1)}(x,t_j)=\vartheta_{(p)}(x,t_j),\qquad j=0,1,\ldots,j_0-1,\\ &\vartheta_{(k)}(x,t_{j+\frac{k-1}{p}})=\vartheta_{(k-1)}(x,t_{j+\frac{k-1}{p}}),q\quad k=2,3,\ldots,p.\end{aligned} \tag{8.3}$$

We call the function $\vartheta(t_{j+1})=\vartheta_{(p)}(t_{j+1})$ a solution of this problem as $t=t_{j+1}$.

We replace problem (8.1)–(8.3) by the following difference scheme on $\Delta_k$ :

$$\begin{aligned}&\frac{\varepsilon}{p}y_{\bar t}^{j+\frac{k}{p}}+\frac{1}{p}\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}}-e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right)y_{\bar t}^{\frac{s}{p}}\\ &=\widetilde{\Lambda}_k\left(\sigma_k y^{j+\frac{k}{p}}+(1-\sigma_k)y^{j+\frac{k-1}{p}}\right)+\varphi_k^{j+\frac{k}{p}},\quad x\in\omega_h,\quad k=1,2,\ldots,p,\end{aligned} \tag{8.4}$$

$$\begin{cases}a_k^{(1_k)}y_{x_k,0}^{j+\frac{k}{p}}=B_{-k}y_0^{j+\frac{k}{p}}-\widetilde{\mu}_{-k}, & x_k=0,\\ -a_k^{(N_k)}y_{\bar x_k,N_k}^{j+\frac{k}{p}}=B_{+k}y_{N_k}^{j+\frac{k}{p}}-\widetilde{\mu}_{+k}, & x_k=l_k,\end{cases} \tag{8.5}$$

$$y(x,0)=u_0(x),\quad k=1,2,\ldots,p, \tag{8.6}$$

where

$$B_{-k}y_0^{j+\frac{k}{p}}=\frac{\beta_{-k}(0,x')}{p}\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}}-e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right)y_{\bar t,0}^{\frac{s}{p}},$$

$$B_{+k}y_{N_k}^{j+\frac{k}{p}}=\frac{\beta_{+k}(l_k,x')}{p}\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}}-e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right)y_{\bar t,N_k}^{\frac{s}{p}},$$

$$\widetilde{\mu}_{\mp k}^{j+\frac{k}{p}} = \sum_{s=0}^{pj+k} \left( e^{-\frac{1}{\alpha} t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha} t_{j+\frac{k-s+1}{p}}} \right) \mu_{\mp k} \left( x, t_{\frac{s}{p}} \right) \mp e^{-\frac{1}{\alpha} t_{\frac{s}{p}}} \Theta_k^0 u_0'(x).$$

Conditions (8.5) have approximation order $O(h_k)$. Let us increase the approximation order to $O(h_k^2)$ on solutions of equation (8.1) for some $k$.

Since

$$\Theta_k \frac{\partial \vartheta_{(k)}}{\partial x_k} = a_k^{(1_k)} \vartheta_{(k)x_k,0} - 0.5h_k \left( \frac{\varepsilon}{p} \vartheta_t + \frac{1}{p} B\vartheta_{(k)} - r_k(x,t) \frac{\partial \vartheta_{(k)}}{\partial x_k} + q_k(x,t)\vartheta_{(k)} - f_k \right)_0 + O(h_k^2),$$

then

$$\begin{aligned} &\left( a_k^{(1_k)} + 0.5h_k r_{k,0} \right) \vartheta_{x_k,0}^{j+\frac{k}{p}} \\ &\quad - 0.5h_k \left( \frac{1}{p} \sum_{s=1}^{pj+k} \left( e^{-\frac{1}{\alpha} t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha} t_{j+\frac{k-s+1}{p}}} \right) \vartheta_{\bar t}^{\frac{s}{p}} - 0.5h_k \frac{\varepsilon}{p} \vartheta_t + q_k(x,t)\vartheta^{j+\frac{k}{p}} - f_k \right)_0 \qquad (8.7) \\ &= B_{-k} \vartheta_0^{j+\frac{k}{p}} - \widetilde{\mu}_{-k} + O(h_k^2) + O(h_k \tau). \end{aligned}$$

In (8.7) we neglect the terms of smallness order $O(h_k^2)$, $O(h_k\tau)$, then replacing $\vartheta_{(k)}$ by $y$, we obtain

$$0.5h_k \frac{\varepsilon}{p} y_{t,0} + \frac{0.5h_k}{p} B_\tau y_0^{j+\frac{k}{p}} = \bar{a}_k^{(1_k)} y_{x_k,0}^{j+\frac{k}{p}} - \bar{d}_{-k} y_0^{j+\frac{k}{p}} - B_{-k} y_0^{j+\frac{k}{p}} + \bar{\mu}_{-k}. \qquad (8.8)$$

Similarly for $x_k = l_k$ we obtain:

$$0.5h_k \frac{\varepsilon}{p} y_{t_N} + \frac{0.5h_k}{p} B_\tau y_{N_k}^{j+\frac{k}{p}} = -\bar{a}_k^{(N_k)} y_{\bar{x}_k,N_k}^{j+\frac{k}{p}} - \bar{d}_{+k} y_{N_k}^{j+\frac{k}{p}} - B_{+k} y_{N_k}^{j+\frac{k}{p}} + \bar{\mu}_{+k}, \qquad (8.9)$$

where

$$\bar{a}_k^{(1_k)} = a_k^{(1_k)} + 0.5h_k r_{k,0}, \qquad \bar{a}_k^{(N_k)} = a_k^{(N_k)} - 0.5h_k r_{k,N_k}, \qquad \bar{d}_{-k} = 0.5h_k d_k^{(0)},$$
$$\bar{d}_{+k} = 0.5h_k d_k^{(N_k)}, \qquad \bar{\mu}_{-k} = \widetilde{\mu}_{-k} + 0.5h_k f_{k,0}, \qquad \bar{\mu}_{+k} = \widetilde{\mu}_{+k} + 0.5h_k f_{k,N_k}.$$

Thus, a difference analogue of problems (2.6), (2.8), (7.4) reads as

$$\frac{\varepsilon}{p} y_t + \frac{1}{p} B_\tau y^{j+\frac{k}{p}} = \bar{\Lambda}_k y^{j+\frac{k}{p}} + \Phi_k^{j+\frac{k}{p}}, \qquad k = 1, 2, \ldots, p, \qquad (8.10)$$

$$y(x,0) = u_0(x), \qquad (8.11)$$

where

$$\bar{\Lambda}_k y = \begin{cases} \widetilde{\Lambda}_k y = \chi_k \left( a_k y_{\bar{x}_k} \right)_{x_k} + b_k^+ a_k^{(+1)} y_{x_k} + b_k^- a_k y_{\bar{x}_k} - d_k y, & x_k \in \omega_{h_k}, \\ \dfrac{1}{0.5h_k} \Lambda_k^- y = \dfrac{\bar{a}_k^{(1_k)} y_{x_k,0} - \bar{d}_{-k} y_0 - B_{-k} y_0}{0.5h_k}, & x_k = 0, \\ \dfrac{1}{0.5h_k} \Lambda_k^+ y = -\dfrac{\bar{a}_k^{(N_k)} y_{\bar{x}_k,N_k} + \bar{d}_{+k} y_{N_k} + B_{+k} y_{N_k}}{0.5h_k}, & x_k = l_k, \end{cases}$$

$$\Phi_{(k)} = \begin{cases} \varphi_k, & x_k \in \omega_{h_k}, \\ \dfrac{\bar{\mu}_{-k}}{0.5h_k}, & x_k = 0, \\ \dfrac{\bar{\mu}_{+k}}{0.5h_k}, & x_k = l_k. \end{cases}$$

## 9. Approximation error of locally one-dimensional scheme

We consider the error of boundary conditions in difference scheme (8.8), (8.9) denoting $z^{j+\frac{k}{p}} = y^{j+\frac{k}{p}} - u^{j+\frac{k}{p}}$. We write boundary condition at $x_k = 0$ as

$$0.5h_k\frac{\varepsilon}{p}y_{t,0} + \frac{0.5h_k}{p}B_\tau y_0^{j+\frac{k}{p}} = \bar{a}_k^{(1_k)}y_{x_k,0}^{j+\frac{k}{p}} - \bar{d}_{-k}y_0^{j+\frac{k}{p}} - B_{-k}y_0^{j+\frac{k}{p}} + 0.5h_k f_{k,0} + \mu_{-k}. \tag{9.1}$$

Then substituting $y^{j+\frac{k}{p}} = z^{j+\frac{k}{p}} + u^{j+\frac{k}{p}}$ into (9.1), we obtain

$$\begin{aligned}0.5h_k\frac{\varepsilon}{p}z_{t,0} + \frac{0.5h_k}{p}B_\tau z_0^{j+\frac{k}{p}} =&\bar{a}_k^{(1_k)}z_{x_k,0}^{j+\frac{k}{p}} - \bar{d}_{-k}z_0^{j+\frac{k}{p}} - B_{-k}z_0^{j+\frac{k}{p}} - \bar{d}_{-k}u_0^{j+\frac{k}{p}} - B_{-k}u_0^{j+\frac{k}{p}}\\ &- 0.5h_k\frac{\varepsilon}{p}u_{t,0} - \frac{0.5h_k}{p}B_\tau u_0^{j+\frac{k}{p}} + \bar{a}_k^{(1_k)}u_{x_k,0}^{j+\frac{k}{p}} + 0.5h_k f_{k,0} + \mu_{-k}.\end{aligned}$$

To the right hand side of the obtained expression we add and deduct the expression

$$0.5h_k\mathring{\psi}_{-k} = 0.5h_k\left(L_k u + f_k - \frac{\varepsilon}{p}u_t - \frac{1}{p}\mathcal{B}u\right)_0^{j+\frac{1}{2}}.$$

Then

$$\begin{aligned}\psi_{-k} =&0.5h_k\left(f_{k,0} - \frac{\varepsilon}{p}u_{t,0}^{j+\frac{k}{p}} - \frac{1}{p}B_\tau u_0^{j+\frac{k}{p}}\right) + \bar{a}_k^{(1_k)}u_{x_k,0}^{j+\frac{k}{p}} - \bar{d}_{-k}u_0^{j+\frac{k}{p}}\\ &- B_{-k}u_0^{j+\frac{k}{p}} + \mu_{-k} - 0.5h_k\left(L_k u + f_k - \frac{\varepsilon}{p}u_t - \frac{1}{p}\mathcal{B}u\right)_0^{j+\frac{1}{2}} + 0.5h_k\mathring{\psi}_{-k}\\ =&\bar{a}_k^{(1_k)}u_{x_k,0}^{j+\frac{k}{p}} - \bar{d}_{-k}u_0^{j+\frac{k}{p}} - B_{-k}u_0^{j+\frac{k}{p}} + \mu_{-k} - 0.5h_k\left(L_k u\right)_0^{j+\frac{1}{2}} + 0.5h_k\mathring{\psi}_{-k} + O(h_k\tau)\\ =&\Theta_k\frac{\partial u^{j+\frac{k}{p}}}{\partial x_k} + 0.5h_k\frac{\partial}{\partial x_k}\left(\Theta_k\frac{\partial u}{\partial x_k}\right)^{j+\frac{k}{p}} + 0.5h_k r_k^{(0)}u_{x_k,0}^{j+\frac{k}{p}} - 0.5h_k d_{k,0}u_0^{j+\frac{k}{p}} - B_{-k}u_0^{j+\frac{k}{p}}\\ &+ \mu_{-k} - 0.5h_k\left(\frac{\partial}{\partial x_k}\left(\Theta_k\frac{\partial u}{\partial x_k}\right) + r_k\frac{\partial u}{\partial x_k} - q_k u\right)_0^{j+\frac{1}{2}} + 0.5h_k\mathring{\psi}_{-k} + O(h_k^2) + O(h_k\tau)\\ =&\left[\Theta_k\frac{\partial u^{j+\frac{k}{p}}}{\partial x_k} - B_{-k}u_0^{j+\frac{k}{p}} + \mu_{-k}\right]\Bigg|_{x_k=0} + 0.5h_k\mathring{\psi}_{-k} + O(h_k^2) + O(h_k\tau).\end{aligned}$$

By boundary conditions (7.5) the expression in the square brackets vanishes. This is why

$$\psi_{-k} = 0.5h_k\mathring{\psi}_{-k} + \overset{*}{\psi}_{-k}, \qquad \overset{*}{\psi}_{-k} = O(h_k^2 + \tau) + O(h_k\tau).$$

Thus,

$$0.5h_k\frac{\varepsilon}{p}z_{t,0} + \frac{0.5h_k}{p}B_\tau z_0^{j+\frac{k}{p}} = \Lambda_k^- z^{j+\frac{k}{p}} + \psi_{-k}, \qquad \psi_{-k} = 0.5h_k\mathring{\psi}_{-k} + \overset{*}{\psi}_{-k}. \tag{9.2}$$

Similarly as $x_\alpha = l_\alpha$ we have

$$\begin{aligned}&0.5h_k\frac{\varepsilon}{p}z_{t,N_k} + \frac{0.5h_k}{p}B_\tau z_{N_k}^{j+\frac{k}{p}} = \Lambda_k^+ z^{j+\frac{k}{p}} + \psi_{+k}, \qquad \psi_{+k} = 0.5h_k\mathring{\psi}_{+k} + \overset{*}{\psi}_{+k},\\ &\mathring{\psi}_{\pm k} = O(1), \qquad \sum_{k=1}^{p}\mathring{\psi}_{\pm k} = 0, \qquad \overset{*}{\psi}_{\pm k} = O(h_k^2 + \tau) + O(h_k\tau).\end{aligned} \tag{9.3}$$

Thus, for the error $z^{j+\frac{k}{p}}$ we obtain the problem

$$\frac{\varepsilon}{p}z_{\bar{t}}^{j+\frac{k}{p}} + \frac{1}{p}B_\tau z^{j+\frac{k}{p}} = \bar{\Lambda}_k z^{j+\frac{k}{p}} + \Psi_k^{j+\frac{k}{p}}, \tag{9.4}$$

$$z(x, 0) = 0, \tag{9.5}$$

where

$$\bar{\Lambda}_k = \begin{cases} \widetilde{\Lambda}_k, & x_k \in \omega_{h_k}, \\ \dfrac{1}{0.5h_k}\Lambda_k^-, & x_k = 0 \\ \dfrac{1}{0.5h_k}\Lambda_k^+, & x_k = l_k, \end{cases} \qquad \Psi_k = \begin{cases} \psi_k, & x_k \in \omega_{h_k}, \\ \psi_{-k}, & x_k = 0, \\ \psi_{+k}, & x_k = l_k, \end{cases}$$

$$\overset{\circ}{\psi}_k = O(1), \qquad \sum_{k=1}^{p} \overset{\circ}{\psi}_k = 0, \qquad \psi = \sum_{k=1}^{p} \psi_k = \sum_{k=1}^{p} \overset{*}{\psi}_k = O(|h|^2 + \tau), \qquad \overset{*}{\psi}_k = O\left(h_k^2 + \tau\right),$$

$$\overset{\circ}{\psi}_{\pm k} = O(1), \qquad \psi_{\pm k} = 0.5h_k \overset{\circ}{\psi}_{\pm k} + \overset{*}{\psi}_{\pm k}, \qquad \psi_{\pm k} = O(h_k^2 + \tau), \qquad \sum_{k=1}^{p} \overset{\circ}{\psi}_{\pm k} = 0.$$

## 10. Stability of locally one-dimensional scheme

In order to solve scheme (8.10), (8.11), we are going to obtain an apriori estimate in the grid norm $C$. We rewrite (8.10), (8.11) as

$$\begin{aligned} &\frac{\varepsilon}{p} y_{\bar{t}}^{j+\frac{k}{p}} + \frac{1}{p} \sum_{s=1}^{pj+k} \left( e^{-\frac{1}{\alpha} t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha} t_{j+\frac{k-s+1}{p}}} \right) y_{\bar{t}}^{\frac{s}{p}} \\ &\quad = \chi_k \left( a_k y_{\bar{x}_k}^{j+\frac{k}{p}} \right)_{x_k} + b_k^+ a_k^{(+1)} y_{x_k}^{j+\frac{k}{p}} + b_k^- a_k y_{\bar{x}_k}^{j+\frac{k}{p}} - d_k y^{j+\frac{k}{p}} + \varphi_k^{j+\frac{k}{p}}, \quad x_k \in \omega_{h_k}, \end{aligned} \tag{10.1}$$

$$\begin{aligned} &\frac{\varepsilon}{p} y_{t,0}^{j+\frac{k}{p}} + \frac{1}{p} \sum_{s=1}^{pj+k} \left( e^{-\frac{1}{\alpha} t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha} t_{j+\frac{k-s+1}{p}}} \right) y_{t,0}^{\frac{s}{p}} \\ &\quad = \frac{\bar{a}_k^{(1_k)} y_{x_k,0}^{j+\frac{k}{p}} - \bar{d}_{-k} y_0^{j+\frac{k}{p}} - B_{-k} y_0^{j+\frac{k}{p}}}{0.5h_k} + \frac{\bar{\mu}_{-k}}{0.5h_k}, \end{aligned} \tag{10.2}$$

$$\begin{aligned} &\frac{\varepsilon}{p} y_{\bar{t},N}^{j+\frac{k}{p}} + \frac{1}{p} \sum_{s=1}^{pj+k} \left( e^{-\frac{1}{\alpha} t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha} t_{j+\frac{k-s+1}{p}}} \right) y_{\bar{t},N_k}^{\frac{s}{p}} \\ &\quad = -\frac{\bar{a}_k^{(N_k)} y_{\bar{x}_k,N_k}^{j+\frac{k}{p}} + \bar{d}_{+k} y_{N_k}^{j+\frac{k}{p}} + B_{+k} y_{N_k}^{j+\frac{k}{p}}}{0.5h_k} + \frac{\bar{\mu}_{+k}}{0.5h_k}, \end{aligned} \tag{10.3}$$

$$y(x,0) = u_0(x). \tag{10.4}$$

We study the stability of difference scheme (10.1)–(10.4) by means of the maximum principle [21, Ch. V, Addendum, Sect. 2, Item 2]. We first obtain an apriori estimate for (10.1)–(10.4) and for this we represent the solution of problem (10.1)–(10.4) as the sum $y = \overline{y} + v$, where $\overline{y}$ is the solution of homogeneous equations (10.1) with inhomogeneous boundary conditions (10.2)–(10.3) and homogeneous initial conditions (10.4):

$$\begin{aligned} &\frac{\varepsilon}{p} \overline{y}_{\bar{t}}^{j+\frac{k}{p}} + \frac{1}{p} \sum_{s=1}^{pj+k} \left( e^{-\frac{1}{\alpha} t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha} t_{j+\frac{k-s+1}{p}}} \right) \overline{y}_{\bar{t}}^{\frac{s}{p}} \\ &\quad = \chi_k \left( a_k \overline{y}_{\bar{x}_k}^{j+\frac{k}{p}} \right)_{x_k} + b_k^+ a_k^{(+1)} \overline{y}_{x_k}^{j+\frac{k}{p}} + b_k^- a_k \overline{y}_{\bar{x}_k}^{j+\frac{k}{p}} - d_k \overline{y}^{j+\frac{k}{p}}, \quad x_k \in \omega_{h_k}, \end{aligned} \tag{10.5}$$

and $v$ is a solution of inhomogeneous equations (10.1) with homogeneous boundary conditions (10.2)–(10.3) and inhomogeneous initial conditions (10.4):

$$\begin{aligned}&\frac{\varepsilon}{p}v_t^{j+\frac{k}{p}}+\frac{1}{p}\sum_{s=1}^{pj+k}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}}-e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}}\right)v_t^{\frac{s}{p}}\\&=\chi_k\left(a_k v_{\bar{x}_k}^{j+\frac{k}{p}}\right)_{x_k}+b_k^+a_k^{(+1)}v_{x_k}^{j+\frac{k}{p}}+b_k^-a_kv_{\bar{x}_k}^{j+\frac{k}{p}}-d_kv^{j+\frac{k}{p}}+\varphi_k^{j+\frac{k}{p}},\quad x_k\in\omega_{h_k}.\end{aligned}\tag{10.6}$$

Let us estimate $\bar{y}$. In order to do this, we reduce boundary conditions for equation (10.5) to canonical form at the points $P=P(x_0,t_{j+\frac{k}{p}})$, $P=P(x_{N_k},t_{j+\frac{k}{p}})$ and check the assumptions of Theorem 3 in [21, Ch. V, Addendum, Sect. 2, Item 2] taking into consideration the positivity of the expressions in round brackets.

At the point $P=P(x_{i_k},t_{j+\frac{k}{p}})$ we have

$$\begin{aligned}A(P)&=\left[\frac{1}{\tau}\left(\varepsilon+1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)+\frac{\chi_{i_k}a_{k,i_k+1}}{h_k^2}+\frac{\chi_{i_k}a_{k,i_k}}{h_k^2}+\frac{b_{i_k}^+a_{k,i_k+1}}{h_k}-\frac{b_{i_k}^-a_{k,i_k}}{h_k}+d_k\right]>0,\\B(P,Q)&=\Bigg\{\frac{\chi_{i_k}a_{k,i_k+1}}{h_k^2}+\frac{b_{i_k}^+a_{k,i_k+1}}{h_k};\frac{\chi_{i_k}a_{k,i_k}}{h_k^2}-\frac{b_{i_k}^-a_{k,i_k}}{h_k};\\&\quad\frac{1}{\tau}\left[\varepsilon+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2\right];\frac{1}{\tau}\left[e^{-\frac{1}{\alpha}t_{\frac{1}{p}}}-2e^{-\frac{1}{\alpha}t_{\frac{2}{p}}}+e^{-\frac{1}{\alpha}t_{\frac{3}{p}}}\right];\cdots;\\&\quad\frac{1}{\tau}\left[e^{-\frac{1}{\alpha}t_{j+\frac{k-2}{p}}}-2e^{-\frac{1}{\alpha}t_{j+\frac{k-1}{p}}}+e^{-\frac{1}{\alpha}t_{j+\frac{k}{p}}}\right];\frac{1}{\tau}\left[e^{-\frac{1}{\alpha}t_{j+\frac{k-1}{p}}}-e^{-\frac{1}{\alpha}t_{j+\frac{k}{p}}}\right]\Bigg\}>0,\\D(P)&=d_k\geqslant c_0>0,\end{aligned}$$

while at the point $P=P(x_0,t_{j+\frac{k}{p}})$ we get

$$\begin{aligned}A(P)&=\left[\frac{1}{\tau}\left(\varepsilon+1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)+\frac{\bar{a}_k^{(1_k)}}{0.5h_k^2}+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)\frac{\beta_{-k}}{0.5h_k\tau}+d_k^{(0)}\right]>0,\\B(P,Q)&=\Bigg\{\frac{\bar{a}_k^{(1_k)}}{0.5h_k^2};\left[\frac{\varepsilon}{\tau}+\frac{1}{\tau}\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2+\frac{\beta_{-k}}{0.5h_k\tau}\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2\right];\\&\quad\left[\frac{1}{\tau}e^{-\frac{1}{\alpha}\frac{\tau}{p}}\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2+\frac{\beta_{-k}}{0.5h_k\tau}e^{-\frac{1}{\alpha}\frac{\tau}{p}}\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2\right];\cdots;\\&\quad\left[\frac{1}{\tau}e^{-\frac{1}{\alpha}t_{j+\frac{k-2}{p}}}\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2+\frac{\beta_{-k}}{0.5h_k\tau}e^{-\frac{1}{\alpha}t_{j+\frac{k-2}{p}}}\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2\right];\\&\quad\left[\frac{1}{\tau}e^{-\frac{1}{\alpha}t_{j+\frac{k-1}{p}}}\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2+\frac{\beta_{-k}}{0.5h_k\tau}e^{-\frac{1}{\alpha}t_{j+\frac{k-1}{p}}}\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2\right]\Bigg\}>0,\\D(P)&=\frac{\beta_{-k}}{0.5h_k\alpha p}+d_k^{(0)}>\frac{\beta_{-k}}{0.5h_k\alpha p}\geqslant\frac{2c_0}{l_kp}>0;\end{aligned}$$

and, at the point $P=P(x_{N_k},t_{j+\frac{k}{p}})$,

$$A(P)=\left[\frac{1}{\tau}\left(\varepsilon+1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)+\frac{\bar{a}_k^{(N_k)}}{0.5h_k^2}+\left(1-e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)\frac{\beta_{+k}}{0.5h_k\tau}+d_k^{(N_k)}\right]>0,$$

$$B(P,Q) = \left\{ \frac{\bar{a}_k^{(N_k)}}{0.5h_k^2}; \left[ \frac{\varepsilon}{\tau} + \frac{1}{\tau}\left(1 - e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2 + \frac{\beta_{+k}}{0.5h_k\tau}\left(1 - e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2 \right]; \right.$$
$$\left[ \frac{1}{\tau} e^{-\frac{1}{\alpha}\frac{\tau}{p}}\left(1 - e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2 + \frac{\beta_{+k}}{0.5h_k\tau} e^{-\frac{1}{\alpha}\frac{\tau}{p}}\left(1 - e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2 \right]; \cdots;$$
$$\left[ \frac{1}{\tau} e^{-\frac{1}{\alpha}t_{j+\frac{k-2}{p}}}\left(1 - e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2 + \frac{\beta_{+k}}{0.5h_k\tau} e^{-\frac{1}{\alpha}t_{j+\frac{k-2}{p}}}\left(1 - e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2 \right];$$
$$\left. \left[ \frac{1}{\tau} e^{-\frac{1}{\alpha}t_{j+\frac{k-1}{p}}}\left(1 - e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2 + \frac{\beta_{+k}}{0.5h_k\tau} e^{-\frac{1}{\alpha}t_{j+\frac{k-1}{p}}}\left(1 - e^{-\frac{1}{\alpha}\frac{\tau}{p}}\right)^2 \right] \right\} > 0,$$

$$D(P) = \frac{\beta_{+k}}{0.5h_k\alpha p} + d_k^{(N_k)} > \frac{\beta_{+k}}{0.5h_k\alpha p} \geqslant \frac{2c_0}{l_k p} > 0.$$

Thus, on the base of Theorem 3 in [21, Ch. V, Addendum, Sect. 2, Item 2], for $\bar{y}$ we get the estimate:

$$\|\bar{y}^{j+1}\|_C \leqslant M \max_{0<t'\leqslant t_j} \left( \|\bar{\mu}_{-k}(x,t')\|_{C_\gamma} + \|\bar{\mu}_{+k}(x,t')\|_{C_\gamma} \right), \tag{10.7}$$

where

$$M = const > 0, \qquad h = \max_{1\leqslant k\leqslant p} h_k, \qquad \|y\|_C = \max_{x\in\bar{\omega}_h} |y|, \quad \|y\|_{C_\gamma} = \max_{x\in\gamma_h} |y|.$$

We proceed to estimating the function $v$. Reproducing arguing (5.7)–(5.13), we easily confirm that $v$ satisfies estimate (5.14).

Thus, estimates (5.14) and (10.7) imply a final one:

$$\|y^{j+1}\|_C \leqslant \|u^0\|_C + M \max_{0<t'\leqslant j\tau} \left( \|\bar{\mu}_{-k}(x,t')\|_{C_\gamma} + \|\bar{\mu}_{+k}(x,t')\|_{C_\gamma} \right) + \sum_{j'=0}^{j} \frac{\tau}{\varepsilon+\gamma\tau} \sum_{k=1}^{p} \max_{0\leqslant s\leqslant k} \|\varphi_k^{j'+\frac{s}{p}}\|_C. \tag{10.8}$$

Thus, the following theorem holds true.

**Theorem 10.1.** *Let conditions (2.4) be satisfied. Then locally one-dimensional scheme (8.10), (8.11) is stable in the initial data and the right hand side and the solution of problem (8.10), (8.11) obeys estimate (10.8).*

## 11. Uniform convergence of locally one-dimensional scheme

We represent a solution for error (9.4), (9.5) as the sum

$$z_{(k)} = v_{(k)} + \eta_{(k)}, \qquad z_{(k)} = z^{j+\frac{k}{p}}, \tag{11.1}$$

where $\eta_{(k)}$ are determined by conditions (6.1), while the function $v_{(k)}$ is determined by the conditions

$$\frac{\varepsilon}{p} v_{\bar{t}}^{j+\frac{k}{p}} + \frac{1}{p}\sum_{s=1}^{pj+k} \left( e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}} \right) v_{\bar{t}}^{\frac{s}{p}} = \widetilde{\Lambda}_k v_{(k)} + \overset{*}{\Psi}, \qquad x_k \in \omega_{h_k}, \tag{11.2}$$

$$\frac{0.5h_k\varepsilon}{p} v_{\bar{t}}^{j+\frac{k}{p}} + \frac{0.5h_k}{p}\sum_{s=1}^{pj+k} \left( e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}} \right) v_{\bar{t}}^{\frac{s}{p}} = \Lambda_k^- v_{(k)} + \widetilde{\Psi}_{-k}, \qquad x_k = 0, \tag{11.3}$$

$$\frac{0.5h_k\varepsilon}{p} v_{\bar{t}}^{j+\frac{k}{p}} + \frac{0.5h_k}{p}\sum_{s=1}^{pj+k} \left( e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{p}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{p}}} \right) v_{\bar{t}}^{\frac{s}{p}} = \Lambda_k^+ v_{(k)} + \widetilde{\Psi}_{+k}, \qquad x_k = l_k, \tag{11.4}$$

$$v(x,0) = 0, \tag{11.5}$$

where

$$\widetilde{\Psi}_{\pm k}=\Lambda_k^{\pm}\eta_{(k)}+\overset{*}{\Psi}_{\pm k},\qquad \widetilde{\Psi}_k=\overset{*}{\Psi}+\widetilde{\Lambda}_k\eta_{(k)},\qquad \overset{*}{\Psi}=O\left(h_k^2+\tau\right),\qquad \overset{*}{\Psi}_{\pm k}=O\left(h_k^2+\tau\right).$$

To estimate the solution of problem (11.2)–(11.5) we employ Theorem 10.1 taking into consideration that $\eta^{j+\frac{k}{p}}=O\left(\frac{\tau}{\varepsilon+\gamma\tau}\right)$, $k=1,2,\ldots,p$, $j=0,1,\ldots,j_0-1$:

$$\|v^{j+1}\|_C\leqslant M\max_{0<j'+\frac{k}{p}\leqslant j+1}\|\eta^{j'+\frac{k}{p}}\|_{C_\gamma}+\sum_{j'=0}^{j}\frac{\tau}{\varepsilon+\gamma\tau}\sum_{k=1}^{p}\max_{0\leqslant s\leqslant k}\|\widetilde{\Psi}^{j'+\frac{s}{p}}\|_C. \tag{11.6}$$

In there exist continuous in the closed domain $\bar{Q}_T$ derivatives

$$\frac{\partial^2 u}{\partial t^2},\qquad \frac{\partial^4 u}{\partial x_k^2\partial x_\nu^2},\qquad \frac{\partial^3 u}{\partial x_k^2\partial t},\qquad \frac{\partial^2 f}{\partial x_k^2},\qquad 1\leqslant k,\qquad \nu\leqslant p,\qquad k\neq\nu,$$

then

$$\widetilde{\Lambda}_k\eta_{(k)}=-\frac{\tau}{\varepsilon+\gamma\tau}a_k\widetilde{\Lambda}_k\left(\overset{\circ}{\Psi}_{k+1}+\ldots+\overset{\circ}{\Psi}_p\right)=O\left(\frac{\tau}{\varepsilon+\gamma\tau}\right),\qquad \Lambda_k^{\pm}\eta_{(k)}=O\left(\frac{\tau}{\varepsilon+\gamma\tau}\right),$$

where $a_k$ are known constants. Then by (11.6) we obtain

$$\|v^{j+1}\|_C\leqslant M\left(\frac{h}{\tau+\varepsilon}+\frac{\tau}{(\tau+\varepsilon)^2}\right),\qquad h=\max_{1\leqslant k\leqslant p}h_k.$$

Therefore,

$$\|z^{j+1}\|_C\leqslant\|\eta^{j+1}\|_C+\|v^{j+1}\|_C=O\left(\frac{h}{\tau+\varepsilon}+\frac{\tau}{(\tau+\varepsilon)^2}\right).$$

Thus, the following theorem holds true.

**Theorem 11.1.** *Let problem (2.6), (2.8), (7.4) possesses a unique continuous solution $u(x,t)$ in $\overline{Q}_T$ for all values $\varepsilon$ and there exists continuous in $\overline{Q}_T$ derivatives*

$$\frac{\partial^2 u}{\partial t^2},\qquad \frac{\partial^4 u}{\partial x_k^2\partial x_\nu^2},\qquad \frac{\partial^3 u}{\partial x_k^2\partial t},\qquad \frac{\partial^2 f}{\partial x_k^2},\qquad 1\leqslant k,\qquad \nu\leqslant p,\qquad k\neq\nu,\qquad \alpha>0,$$

*and conditions (2.4), (7.2) holds. Then the solution of scheme (8.10), (8.11) converges uniformly to the solution of problem (2.6), (2.8), (7.4) at the rate*

$$O\left(\frac{h}{\tau+\varepsilon}+\frac{\tau}{(\tau+\varepsilon)^2}\right),\qquad h=o(\tau+\varepsilon),\qquad \tau=o((\tau+\varepsilon)^2),$$

*where $\varepsilon$ is a small parameter, $\alpha>0$.*

**Corollary 11.1.** *If $\varepsilon=\tau^\delta$, then the solution of schemes (4.1),(4.2) and (9.4), (9.5) for all $\alpha>0$ in view of (2.24), (7.6) converges uniformly to the solution of the original problem at the rate $O\left(\frac{h^2}{\tau^\delta}+\tau^{1-2\delta}+\tau^\delta\right)$, where $0<\delta<\frac{1}{2}$.*

**Remark 11.1.** *The obtained results hold also for the case when $G$ is a domain of a complicated shaped obeying two conditions in* [20, Ch.IX, Sect. 3, Item 5]. *Then estimates (5.14) and (10.8) respectively become*

$$\begin{aligned}\|y^{j+1}\|_C\leqslant&\|u_0\|_C+\max_{0<t'\leqslant t_j}\|\mu(x,t')\|_{C_\gamma}\\&+\max_{0<t'\leqslant t_j}\frac{\tau\|\overset{*}{\varphi}\|_{\overset{*}{C}}}{\varepsilon+\gamma\tau}+\sum_{j'=0}^{j}\frac{\tau}{\varepsilon+\gamma\tau}\sum_{k=1}^{p}\max_{0\leqslant s\leqslant k}\|\overset{\circ}{\varphi}{}_k^{j'+\frac{s}{p}}\|_{\overset{\circ}{C}},\end{aligned}\tag{11.7}$$

$$\|y^{j+1}\|_C \leqslant \|u_0\|_C + M \max_{0<t'\leqslant j\tau} \left(\|\bar{\mu}_{-k}(x,t')\|_{C_\gamma} + \|\bar{\mu}_{+k}(x,t')\|_{C_\gamma}\right) + \max_{0<t'\leqslant t_j} \frac{\tau\|\overset{*}{\varphi}\|_{\overset{*}{C}}}{\varepsilon+\gamma\tau} + \sum_{j'=0}^{j} \frac{\tau}{\varepsilon+\gamma\tau} \sum_{k=1}^{p} \max_{0\leqslant s\leqslant k} \|\overset{\circ}{\varphi}{}_k^{j'+\frac{s}{p}}\|_{\overset{\circ}{C}}, \tag{11.8}$$

*where* $\overset{\circ}{\varphi}{}_k^{j+\frac{k}{p}}$, $\overset{*}{\varphi}{}_k^{j+\frac{k}{p}}$ *are determined by the conditions*

$$\overset{\circ}{\varphi}{}_k^{j+\frac{k}{p}} = \begin{cases} \varphi_k, & x \in \overset{\circ}{\omega}_h, \\ 0, & x \in \overset{*}{\omega}_h, \end{cases} \qquad \overset{*}{\varphi}{}_k^{j+\frac{k}{p}} = \begin{cases} \varphi_k, & x \in \overset{*}{\omega}_h, \\ 0, & x \in \overset{\circ}{\omega}_h, \end{cases}$$

*so that* $\overset{\circ}{\varphi}_k + \overset{*}{\varphi}_k = \varphi_k$ *as* $x \in \omega_h$. *That is,* $\overset{*}{\varphi}_k$ *is non-zero only at near-boundary nodes,* $\overset{\circ}{\omega}$ *is some connected subset of the set* $\omega$, *and* $\overset{*}{\omega}$ *is a completement of* $\overset{\circ}{\omega}$ *to* $\omega$, *where*

$$h = \max_{1\leqslant k\leqslant p} h_k, \qquad \|y\|_C = \max_{x\in\omega_h} |y|, \qquad \|y\|_{C_\gamma} = \max_{x\in\gamma_h} |y|,$$
$$\|\varphi\|_{\overset{*}{C}} = \max_{x\in\overset{*}{\omega}_h} |\varphi|, \qquad \|\varphi\|_{\overset{\circ}{C}} = \max_{x\in\overset{\circ}{\omega}_h} |\varphi|.$$

**Remark 11.2.** *The operator*

$$\mathcal{B}u = \frac{1}{\alpha} \int_0^t e^{-\frac{1}{\alpha}(t-\xi)} \frac{\partial u}{\partial \xi}\, d\xi$$

*as* $\alpha = \frac{1-\gamma}{\gamma}$ *becomes a Caputo-Fabrizion fractional derivative* [30]*:*

$${}^{CF}\Delta_{0t}^{\gamma} u = \frac{\gamma}{1-\gamma} \int_0^t e^{-\frac{\gamma}{1-\gamma}(t-\xi)} \frac{\partial u}{\partial \xi}\, d\xi, \qquad 0<\gamma<1.$$

## 12. Algorithm of numerical solution

For a numerical solving of differential problem (2.1), (7.1), (2.3) we write calculation formulas ($0 \leqslant x_k \leqslant l_k$, $k = 1,2$, $p = 2$) and in order to do this, we rewrite problem (2.1), (7.1), (2.3) as $p$=2:

$$\begin{aligned} \frac{\partial u}{\partial t} =& \frac{\partial}{\partial x_1}\Big(\Theta_1(x_1,x_2,t)\frac{\partial u}{\partial x_1}\Big) + \frac{\partial}{\partial x_2}\Big(\Theta_2(x_1,x_2,t)\frac{\partial u}{\partial x_2}\Big) \\ &+ \alpha\frac{\partial^2}{\partial t\partial x_1}\Big(\Theta_1(x_1,x_2,t)\frac{\partial u}{\partial x_1}\Big) + \alpha\frac{\partial^2}{\partial t\partial x_2}\Big(\Theta_2(x_1,x_2,t)\frac{\partial u}{\partial x_2}\Big) + r_1(x_1,x_2,t)\frac{\partial u}{\partial x_1} \\ &+ r_2(x_1,x_2,t)\frac{\partial u}{\partial x_2} - q_1(x_1,x_2,t)u(x_1,x_2,t) - q_2(x_1,x_2,t)u(x_1,x_2,t) + f(x_1,x_2,t), \end{aligned} \tag{12.1}$$

$$\begin{cases} \Theta_1\dfrac{\partial u}{\partial x_1} + \alpha\dfrac{\partial\left(\Theta_1 u_{x_1}\right)}{\partial t} = \beta_{-1}(x,t)\dfrac{\partial u}{\partial t} - \mu_{-1}(x,t), & x_1 = 0, \quad 0\leqslant t\leqslant T, \\ -\left(\Theta_1\dfrac{\partial u}{\partial x_1} + \alpha\dfrac{\partial\left(\Theta_1 u_{x_1}\right)}{\partial t}\right) = \beta_{+1}(x,t)\dfrac{\partial u}{\partial t} - \mu_{+1}(x,t), & x_1 = l_1, \quad 0\leqslant t\leqslant T, \\ \Theta_2\dfrac{\partial u}{\partial x_2} + \alpha\dfrac{\partial\left(\Theta_2 u_{x_2}\right)}{\partial t} = \beta_{-2}(x,t)\dfrac{\partial u}{\partial t} - \mu_{-2}(x,t), & x_2 = 0, \quad 0\leqslant t\leqslant T, \\ -\left(\Theta_2\dfrac{\partial u}{\partial x_2} + \alpha\dfrac{\partial\left(\Theta_2 u_{x_2}\right)}{\partial t}\right) = \beta_{+2}(x,t)\dfrac{\partial u}{\partial t} - \mu_{+2}(x,t), & x_2 = l_2, \quad 0\leqslant t\leqslant T, \end{cases} \tag{12.2}$$

$$u(x_1,x_2,0) = u_0(x_1,x_2). \tag{12.3}$$

We consider a grid

$$x_k^{(i_k)} = i_k h_k, \qquad k = 1,2, \qquad t_j = j\tau, \qquad i_k = 0,1,\ldots,N_k,$$

$$h_k = \frac{l_k}{N_k}, \qquad j = 0, 1, \ldots, m, \qquad \tau = \frac{T}{m}.$$

We introduce one fractional step $t_{j+\frac{1}{2}} = t_j + 0.5\tau$. By $y_{i_1,i_2}^{j+\frac{k}{p}}$ we denote a grid function:

$$y_{i_1,i_2}^{j+\frac{k}{p}} = y^{j+\frac{k}{p}} = y(i_1h_1, i_2h_2, (j+0.5k)\tau), \quad k = 1, 2.$$

We write a locally one-dimensional scheme:

$$\begin{cases} \varepsilon\dfrac{y^{j+\frac{1}{2}} - y^j}{\tau} + \dfrac{1}{2}\sum\limits_{s=1}^{2j+1}\left(e^{-\frac{1}{\alpha}t_{j+\frac{1-s}{2}}} - e^{-\frac{1}{\alpha}t_{j+\frac{2-s}{2}}}\right) y_{\bar t}^{\frac{s}{2}} = \tilde\Lambda_1 y^{j+\frac{1}{2}} + \varphi_1, \\ \varepsilon\dfrac{y^{j+1} - y^{j+\frac{1}{2}}}{\tau} + \dfrac{1}{2}\sum\limits_{s=1}^{2j+2}\left(e^{-\frac{1}{\alpha}t_{j+\frac{2-s}{2}}} - e^{-\frac{1}{\alpha}t_{j+\frac{3-s}{2}}}\right) y_{\bar t}^{\frac{s}{2}} = \tilde\Lambda_2 y^{j+1} + \varphi_2, \end{cases} \tag{12.4}$$

$$\begin{cases} y_{0,i_2}^{j+\frac{1}{2}} = \varkappa_{11}(i_2h_2, t_{j+\frac{1}{2}}) y_{1,i_2}^{j+\frac{1}{2}} + \mu_{11}(i_2h_2, t_{j+\frac{1}{2}}), \\ y_{N_1,i_2}^{j+\frac{1}{2}} = \varkappa_{12}(i_2h_2, t_{j+\frac{1}{2}}) y_{N_1-1,i_2}^{j+\frac{1}{2}} + \mu_{12}(i_2h_2, t_{j+\frac{1}{2}}), \\ y_{i_1,0}^{j+1} = \varkappa_{21}(i_1h_1, t_{j+1}) y_{i_1,1}^{j+1} + \mu_{21}(i_1h_1, t_{j+1}), \\ y_{i_1,N_2}^{j+1} = \varkappa_{22}(i_1h_1, t_{j+1}) y_{i_1,N_2-1}^{j+1} + \mu_{22}(i_1h_1, t_{j+1}), \end{cases} \tag{12.5}$$

$$y_{i_1,i_2}^0 = u_0(i_1h_1, i_2, h_2), \tag{12.6}$$

where

$$\tilde\Lambda_k y^{j+\frac{k}{p}} = \varkappa_k\Big(a_k y_{\bar x_k}^{j+\frac{k}{p}}\Big)_{x_k} + b_k^+ a_k^{(+1)} y_{x_k}^{j+\frac{k}{p}} + b_k^- a_k y_{\bar x_k}^{j+\frac{k}{p}} - d_k y^{j+\frac{k}{p}}, \quad k = 1, 2,$$

$$\varphi_k = \frac{1}{2}\sum_{s=1}^{2j+k-1}\left(e^{-\frac{1}{\alpha}t_{j+\frac{k-s}{2}}} - e^{-\frac{1}{\alpha}t_{j+\frac{k-s+1}{2}}}\right) f^{\frac{s}{2}} - e^{-\frac{1}{\alpha}t_{j+\frac{k}{2}}}\left(y_{\bar x_1 x_1}^{j+\frac{k-1}{2}} + y_{\bar x_2 x_2}^{j+\frac{k-1}{2}}\right).$$

Let us provide calculation formulas for the solution of problem (12.4)–(12.6).

At the first step we find the solution $y_{i_1,i_2}^{j+\frac{1}{2}}$. In order to do this, for each value $i_2 = \overline{1, N_2-1}$ we solve the problem

$$\begin{aligned} &A_{1(i_1,i_2)} y_{i_1-1,i_2}^{j+\frac{1}{2}} - C_{1(i_1,i_2)} y_{i_1,i_2}^{j+\frac{1}{2}} + B_{1(i_1,i_2)} y_{i_1+1,i_2}^{j+\frac{1}{2}} = -F_{1(i_1,i_2)}^{j+\frac{1}{2}}, \quad 0 < i_1 < N_1, \\ &\begin{cases} y_{0,i_2}^{j+\frac{1}{2}} = \varkappa_{11}(i_2h_2, t_{j+\frac{1}{2}}) y_{1,i_2}^{j+\frac{1}{2}} + \mu_{11}(i_2h_2, t_{j+\frac{1}{2}}), \\ y_{N_1,i_2}^{j+\frac{1}{2}} = \varkappa_{12}(i_2h_2, t_{j+\frac{1}{2}}) y_{N_1-1,i_2}^{j+\frac{1}{2}} + \mu_{12}(i_2h_2, t_{j+\frac{1}{2}}), \end{cases} \end{aligned} \tag{12.7}$$

where

$$A_{1(i_1,i_2)} = \frac{(\varkappa_1)_{i_1,i_2}(a_1)_{i_1,i_2}}{h_1^2} - \frac{(b_1^-)_{i_1,i_2}(a_1)_{i_1,i_2}}{h_1},$$

$$B_{1(i_1,i_2)} = \frac{(\varkappa_1)_{i_1,i_2}(a_1)_{i_1+1,i_2}}{h_1^2} + \frac{(b_1^+)_{i_1,i_2}(a_1)_{i_1+1,i_2}}{h_1},$$

$$C_{1(i_1,i_2)} = A_{1(i_1,i_2)} + B_{1(i_1,i_2)} + \frac{\varepsilon}{\tau} + \frac{1}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right) + \frac{1}{2}d_{1(i_1,i_2)},$$

$$F_{1(i_1,i_2)}^{j+\frac{1}{2}} = \frac{\varepsilon}{\tau}y_{i_1,i_2}^j + \frac{1}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right) y_{i_1,i_2}^j - \frac{1}{2}\sum_{s=1}^{2j}\left(e^{-\frac{1}{\alpha}t_{j-\frac{s}{2}}} - e^{-\frac{1}{\alpha}t_{j-\frac{1-s}{2}}}\right) y_{\bar t}^{\frac{s}{2}} + \varphi_{1(i_1,i_2)},$$

$$\varkappa_{11}(i_2h_2, t_{j+\frac{1}{2}}) = \frac{\frac{(a_1)_{1,i_2}}{h_1}}{\frac{(a_1)_{1,i_2}}{h_1} + \frac{0.5h_1 + \beta_{-1,i_2}}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right) + 0.5h_1 d_{-1,i_2}^{j+\frac{1}{2}} + \frac{0.5h_1\varepsilon}{\tau}},$$

$$\varkappa_{12}(i_2h_2,t_{j+\frac{1}{2}})=\frac{\frac{(a_1)_{N_1,i_2}}{h_1}}{\frac{(a_1)_{N_1,i_2}}{h_1}+\frac{0.5h_1+\beta_{+1,i_2}}{\tau}\left(1-e^{-\frac{\tau}{2\alpha}}\right)+0.5h_1d^{j+\frac{1}{2}}_{+1,i_2}+\frac{0.5h_1\varepsilon}{\tau}},$$

$$\mu_{11}(i_2h_2,t_{j+\frac{1}{2}})=\frac{\widetilde{\mu}_{-1}(i_2h_2,t_{j+\frac{1}{2}})+\frac{\beta_{-1,i_2}}{\tau}\left(1-e^{-\frac{\tau}{2\alpha}}\right)y^j_{0,i_2}}{\frac{(a_1)_{1,i_2}}{h_1}+\frac{0.5h_1+\beta_{-1,i_2}}{\tau}\left(1-e^{-\frac{\tau}{2\alpha}}\right)+0.5h_1d^{j+\frac{1}{2}}_{-1,i_2}+\frac{0.5h_1\varepsilon}{\tau}}$$
$$-\frac{\frac{\beta_{-1,i_2}}{2}\sum\limits_{s=1}^{2j}\left(e^{-\frac{1}{\alpha}t_{j-\frac{s}{2}}}-e^{-\frac{1}{\alpha}t_{j-\frac{1-s}{2}}}\right)y^{\frac{s}{2}}_{\bar{t},0}}{\frac{(a_1)_{1,i_2}}{h_1}+\frac{0.5h_1+\beta_{-1,i_2}}{\tau}\left(1-e^{-\frac{\tau}{2\alpha}}\right)+0.5h_1d^{j+\frac{1}{2}}_{-1,i_2}+\frac{0.5h_1\varepsilon}{\tau}},$$

$$\mu_{12}(i_2h_2,t_{j+\frac{1}{2}})=\frac{\widetilde{\mu}_{+1}(i_2h_2,t_{j+\frac{1}{2}})+\frac{0.5h_1\varepsilon}{\tau}y^j_{N_1}+\frac{\beta_{+1,i_2}}{\tau}\left(1-e^{-\frac{\tau}{2\alpha}}\right)y^j_{N_1,i_2}}{\frac{(a_1)_{N_1,i_2}}{h_1}+\frac{0.5h_1+\beta_{+1,i_2}}{\tau}\left(1-e^{-\frac{\tau}{2\alpha}}\right)+0.5h_1d^{j+\frac{1}{2}}_{+1,i_2}+\frac{0.5h_1\varepsilon}{\tau}}$$
$$-\frac{\frac{\beta_{+1,i_2}}{2}\sum\limits_{s=1}^{2j}\left(e^{-\frac{1}{\alpha}t_{j-\frac{s}{2}}}-e^{-\frac{1}{\alpha}t_{j-\frac{1-s}{2}}}\right)y^{\frac{s}{2}}_{\bar{t},N_1}}{\frac{(a_1)_{N_1,i_2}}{h_1}+\frac{0.5h_1+\beta_{+1,i_2}}{\tau}\left(1-e^{-\frac{\tau}{2\alpha}}\right)+0.5h_1d^{j+\frac{1}{2}}_{+1,i_2}+\frac{0.5h_1\varepsilon}{\tau}}.$$

To calculate the right hand sides $F^{j+\frac{1}{2}}_{1(i_1,i_2)}$, $\mu^{j+\frac{1}{2}}_{11(i_2h_2)}$, $\mu^{j+\frac{1}{2}}_{12(i_2h_2)}$ on $(j+\frac{1}{2})$th layer, we need to take into consideration the values of the sought functions $y^j_{i_1,i_2}$ on all previous (lower) layers because of the term

$$\frac{1}{2}\sum_{s=1}^{2j}\left(e^{-\frac{1}{\alpha}t_{j-\frac{s}{2}}}-e^{-\frac{1}{\alpha}t_{j-\frac{1-s}{2}}}\right)y^{\frac{s}{2}}_{\bar{t}},$$

and this increases significantly the amount of calculations even for small partitions of the grid. To avoid this, we propose a recurrent formula for fast calculations, which allows us to keep at the previous layer the value of the mentioned sum and by the number of the operations this does not worse than a two-layer scheme.

Thus, as $p=2$ on $j+\frac{1}{2}$th layer the recurrent formula for the fast calculation, for instance, for $F^{j+\frac{1}{2}}_{1(i_1,i_2)}$, reads as

$$S^{j+\frac{1}{2}}=\frac{1}{\tau}\left(1-e^{-\frac{\tau}{2\alpha}}\right)\left(y^{j+\frac{1}{2}}-y^j\right)+e^{-\frac{\tau}{2\alpha}}S^j,$$

where

$$S^0=0,\qquad S^{j+\frac{1}{2}}=\frac{1}{2}\mathcal{B}_\tau y^{j+\frac{1}{2}}=\frac{1}{2}\sum_{s=0}^{2j}\left(e^{-\frac{1}{\alpha}t_{j-\frac{s}{2}}}-e^{-\frac{1}{\alpha}t_{j+\frac{1-s}{2}}}\right)y^{\frac{s}{p}}_{\bar{t}}.$$

At the second step we find the solution $y^{j+1}_{i_1,i_2}$. As in the first case, in order to do this, for each value $i_1=\overline{1,N_1-1}$ we solve the problem

$$\begin{aligned}&A_{2(i_1,i_2)}y^{j+1}_{i_1,i_2-1}-C_{2(i_1,i_2)}y^{j+1}_{i_1,i_2}+B_{2(i_1,i_2)}y^{j+1}_{i_1,i_2+1}=-F^{j+1}_{2(i_1,i_2)},\qquad 0<i_2<N_2,\\&\begin{cases}y^{j+1}_{i_1,0}=\varkappa_{21}(i_1h_1,t_{j+1})y^{j+1}_{i_1,1}+\mu_{21}(i_1h_1,t_{j+1}),\\ y^{j+1}_{i_1,N_2}=\varkappa_{22}(i_1h_1,t_{j+1})y^{j+1}_{i_1,N_2-1}+\mu_{22}(i_1h_1,t_{j+1}),\end{cases}\end{aligned}\tag{12.8}$$

where

$$A_{2(i_1,i_2)}=\frac{(\varkappa_2)_{i_1,i_2}(a_2)_{i_1,i_2}}{h_2^2}-\frac{(b_2^-)_{i_1,i_2}(a_2)_{i_1,i_2}}{h_2},$$

$$B_{2(i_1,i_2)}=\frac{(\varkappa_2)_{i_1,i_2}(a_2)_{i_1,i_2+1}}{h_2^2}+\frac{(b_2^+)_{i_1,i_2}(a_2)_{i_1,i_2+1}}{h_2},$$

$$C_{2(i_1,i_2)} = A_{2(i_1,i_2)} + B_{2(i_1,i_2)} + \frac{\varepsilon}{\tau} + \frac{1}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right) + \frac{1}{2}d_{2(i_1,i_2)},$$

$$F^{j+1}_{2(i_1,i_2)} = \frac{\varepsilon}{\tau}y^{j+\frac{1}{2}}_{i_1,i_2} + \frac{1}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right)y^{j+\frac{1}{2}}_{i_1,i_2} + \frac{1}{2}\sum_{s=1}^{2j+1}\left(e^{-\frac{1}{\alpha}t_{j+\frac{1-s}{2}}} - e^{-\frac{1}{\alpha}t_{j+\frac{2-s}{2}}}\right)y^{\frac{s}{2}}_{\bar{t}} + \varphi_{2(i_1,i_2)},$$

$$\varkappa_{21}(i_1h_1, t_{j+1}) = \frac{\frac{(a_2)_{i_1,1}}{h_2}}{\frac{(a_2)_{i_1,1}}{h_2} + \frac{0.5h_2+\beta_{-2,i_1}}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right) + 0.5h_2d^{j+1}_{-2,i_1} + \frac{0.5h_2\varepsilon}{\tau}},$$

$$\varkappa_{22}(i_1h_1, t_{j+1}) = \frac{\frac{(a_2)_{i_1,N_2}}{h_2}}{\frac{(a_2)_{i_1,N_2}}{h_2} + \frac{0.5h_2+\beta_{+2,i_1}}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right) + 0.5h_2d^{j+1}_{+2,i_1} + \frac{0.5h_2\varepsilon}{\tau}},$$

$$\mu_{21}(i_1h_1, t_{j+1}) = \frac{\widetilde{\mu}_{-2}(i_1h_1, t_{j+1}) + \frac{0.5h_2\varepsilon}{\tau}y^j_{i_1,0} + \frac{\beta_{-2,i_1}}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right)y^j_{i_1,0}}{\frac{(a_2)_{i_1,1}}{h_2} + \frac{0.5h_2+\beta_{-2,i_1}}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right) + 0.5h_2d^{j+1}_{-2,i_1} + \frac{0.5h_2\varepsilon}{\tau}}$$
$$- \frac{\frac{\beta_{-2,i_1}}{2}\sum_{s=1}^{2j+1}\left(e^{-\frac{1}{\alpha}t_{j-\frac{s}{2}}} - e^{-\frac{1}{\alpha}t_{j-\frac{1-s}{2}}}\right)y^{\frac{s}{2}}_{\bar{t},i_1,0}}{\frac{(a_2)_{i_1,1}}{h_2} + \frac{0.5h_2+\beta_{-2,i_1}}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right) + 0.5h_2d^{j+1}_{-2,i_1} + \frac{0.5h_2\varepsilon}{\tau}},$$

$$\mu_{22}(i_1h_1, t_{j+1}) = \frac{\widetilde{\mu}_{+2}(i_1h_1, t_{j+1}) + \frac{0.5h_2\varepsilon}{\tau}y^j_{i_1,N_2} + \frac{\beta_{+2,i_1}}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right)y^j_{i_1,N_2}}{\frac{(a_2)_{i_1,N_2}}{h_2} + \frac{0.5h_2+\beta_{+2,i_1}}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right) + 0.5h_2d^{j+1}_{+2,i_1} + \frac{0.5h_2\varepsilon}{\tau}}$$
$$- \frac{\frac{\beta_{+2,i_1}}{2}\sum_{s=1}^{2j+1}\left(e^{-\frac{1}{\alpha}t_{j-\frac{s}{2}}} - e^{-\frac{1}{\alpha}t_{j-\frac{1-s}{2}}}\right)y^{\frac{s}{2}}_{\bar{t},i_1,N_2}}{\frac{(a_2)_{i_1,N_2}}{h_2} + \frac{0.5h_2+\beta_{+2,i_1}}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right) + 0.5h_2d^{j+1}_{+2,i_1} + \frac{0.5h_2\varepsilon}{\tau}}.$$

On the $(j+1)$th layer the reccurent formula for the fast calculation reads as

$$S^{j+1} = \frac{1}{\tau}\left(1 - e^{-\frac{\tau}{2\alpha}}\right)\left(y^{j+1} - y^{j+\frac{1}{2}}\right) + e^{-\frac{\tau}{2\alpha}}S^{j+\frac{1}{2}},$$

where

$$S^{j+1} = \frac{1}{2}\mathcal{B}_\tau y^{j+1} = \frac{1}{2}\sum_{s=0}^{2j+1}\left(e^{-\frac{1}{\alpha}t_{j-\frac{1-s}{2}}} - e^{-\frac{1}{\alpha}t_{j+\frac{2-s}{2}}}\right)y^{\frac{s}{p}}_{\bar{t}}.$$

Each of problems (12.7), (12.8) is solved by a sweep method [20, Ch. I, Sect. 2, Item 5].

## 13. Conclusion

This work is devoted to the study of initial boundary value problems for a multi-dimensional pseudoparabolic equation with Dirichlet boundary conditions of a special form. For an approximate solution of the considered problems, the multi-dimensional pseudoparabolic equation is reduced to an integro-differential equation with a small parameter. We show that as the small parameter tends to zero, the solution of the corresponding modified problem converges to the solution of the original problem. For each problem we construct a Samarsky locally one-dimensional scheme, the main idea of which is to reduce the transition from layer to layer to a sequential solving of a number of one-dimensional problems in each of the coordinate directions. Using the maximum principle for each problem, we obtain apriori estimate for the solution of a locally one-dimensional scheme in the uniform metric, and the stability and convergence of the solution are proved. An algorithm for the numerical solution of the modified problem with boundary conditions of a special form is constructed. In view of the fact that in order to determine the solution on any time layer one needs to take into account the values of the desired function on all previous (lower) layers (in this case, the amount of calculations increases significantly), we propose a recursive formula for fast calculation in the multi-dimensional case.

## BIBLIOGRAPHY


1. G.I. Barenblatt, Yu.P. Zheltov, I.N. Kochina. *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks (strata)* // Prikl. Matem. Mekh. **24**:5, 852–864 (1960). [J. Appl. Math. Mech. **24**, 1286–1303 (1961).]
2. N.N. Kochina. *Regulation of the level of ground waters during irrigation* // DAN SSSR. **213**:1, 51–54 (1973). [Sov. Phys. Dokl. **18**, 689–691 (1973).]
3. E.S. Dzektser. *Equations of motion of underground water with a free surface in stratified media* // DAN SSSR. **220**:3, 540–543 (1975). (in Russian).
4. M. Hallaire. *Le potentiel efficace de l'eau dans le sol en regime de dessechement* // L'Eau et la Production Vegetale. Paris: Institut National de la Recherche Agronomique. **9**, 27–62 (1964).
5. R.E. Showalter, T.W. Ting. *Pseudoparabolic partial differential equations* // SIAM J. Math. Anal. **1**:1, 1–26 (1970).
6. A.F. Chudnovsky. *Soil thermophysics.* Nauka, Moscow (1976).
7. P.J. Chen, M.E. Curtin. *On a theory of heat conduction involving two temperatures* // J. Angew. Math. Phys. **19**:4, 614–627 (1968).
8. S.V. Nerpin, A.F. Chudnovsky. *Energy- and mass-transfer in a system plant-soil-air.* Gidrometeoizdat, Leningrad (1975). (in Russian).
9. V.Z. Kanchukoev. *Boundary value problem for a third order equation of a mixed hyperbolic-pseudoparabolic type* // Differ. Uravn. **16**:1, 177–178 (1980). (in Russian).
10. V.Z. Kanchukoev, M.Kh. Shkhanukov. *Boundary value problems for heat and mass transfer equations and their approximation by stable difference schemes* // in Coll. Sci. Works "Boundary value problems for mixed type equations and related problems in functional analysis and applied mathematics". **2**, 143–150 (1979). (in Russian).
11. D.L. Colton. *Pseudoparabolic equations in one space variable* // J. Diff. Equat. **12**:3, 559–565 (1972).
12. B.D. Coleman, R.J. Duffin, V.J. Mizel. *Instability, uniqueness, and nonexistence theorems for the equation* $u_t = u_{xx} - u_{xxt}$ *on a strip* // Arch. Rat. Mech. Anal. **19**:2, 100–116 (1965).
13. M.Kh. Shkhanukov. *Boundary-value problems for a third-order equation occurring in the modeling of water filtration in porous media* // Diff. Uravn. **18**:4, 689–699 (1982). [Diff. Equat. **18**:4, 509–517 (1982).]
14. T.W. Ting. *Certain non-steady flows of second-order fluids* // Arch. Ration Mech. Anal. **14**:1, 1–26 (1963).
15. V.A. Vodakhova. *A boundary-value problem with A. M. Nakhushev's nonlocal condition for a pseudoparabolic moisture-transfer equation* // Diff. Uravn. **18**:2, 280–285 (1982). [Diff. Equats. **18**:2, 226–229 (1982).]
16. M.H. Beshtokov. *Riemann method for solving non-local boundary value problems for the third order pseudoparabolic equations* // Vestnik SamGTU. Ser. Fiz.-Mat. Nauki. **4**:33, 1–10 (2013). (in Russian).
17. A.G. Sveshnikov, A.B. Al'shin, M.O. Korpusov, Yu.D. Pletner. *Linear and nonlinear equations of Sobolev type.* Fizmatlit, Moscow (2007). (in Russian).
18. A.P. Soldatov, M.Kh. Shkhanukov. *Boundary value problems with general nonlocal Samarskij condition for pseudoparabolic equations of higher order* // DAN SSSR. **297**:3, 547–552 (1987). [Sov. Math. Dokl. **36**:3, 507–511 (1988).]
19. A.A. Samarskii. *On an economical difference method for the solution of a multidimensional parabolic equation in an arbitrary region* // Zhurn. Vycisl. Matem. Matem. Fiz. **2**:5, 787–811 (1962). [U.S.S.R. Comput. Math. Math. Phys. **2**:5, 894–926 (1963).]
20. A.A. Samarskii. *The theory of difference schemes.* Nauka, Moscow (1977). [Marcel Dekker, New York (2001).]
21. A.A. Samarskii, A.V. Gulin. *Stability of difference schemes.* Nauka, Moscow (1973). (in Russian).

22. M.Kh. Beshtokov. *Finite difference method for a nonlocal boundary value problem for a third-order pseudoparabolic equation* // Diff. Uravn. **49**:9, 1170–1177 (2013). [Diff. Equat. **49**:9, 1134–1141 (2013).]
23. M.Kh. Beshtokov. *On the numerical solution of a nonlocal boundary value problem for a degenerating pseudoparabolic equation* // Diff. Uravn. **52**:10, 1393–1406 (2016). [Diff. Equat. **52**:10, 1341–1354 (2016).]
24. M.Kh. Beshtokov. *The third boundary value problem for loaded differential Sobolev type equation and grid methods of their numerical implementation* // IOP Conf. Ser.: Mater. Sci. Eng. **158**:1, 12–19 (2016).
25. M.Kh. Beshtokov. *Boundary value problems for degenerating and nondegenerating Sobolev-type equations with a nonlocal source in differential and difference forms* // Diff. Uravn. **54**:2, 249–266 (2018). [Diff. Equat. **54**:2, 250–267 (2018).]
26. M.Kh. Beshtokov. *Numerical analysis of initial-boundary value problem for a Sobolev-type equation with a fractional-order time derivative* // Zhurn. Vychisl. Matem. Matem. Fiz. **59**:2, 185–202 (2019). [Comp. Math. Math. Phys. **59**:2, 175–192 (2019).]
27. M.I. Vishik, L.A. Lyusternik. *Regular degeneration and boundary layer for linear differential equations with small parameter* // Uspekhi Matem. Nauk. **12**:5, 3–122 (1957).
28. S.K. Godunov, V.S. Ryabenk'ij. *Difference schemes.* Nauka, Moscow (1973). (in Russian).
29. O.A. Ladyzhenskaya. *Boundary value problems of mathematical physics.* Nauka, Moscow (1973). (in Russian).
30. M. Caputo, M. Fabrizio. *A New Definition of Fractional Derivative without Singular Kernel* // Progr. Fract. Differ. Appl. **1**:2, 1–13 (2015).

Murat Khamidbievich Beshtokov,
Institute of Applied Mathematics and Automation
of Kabardin-Balkar Scientific Center of RAS,
Shortanova str. 89a,
360000, Nalchik, Russia
E-mail: beshtokov-murat@yandex.ru