ISSN 2304-0122 Ufa Mathematical Journal. Vol. 15. No 3 (2023). P. 3-12.

doi:10.13108/2023-15-3-3

BEHAVIOR OF ENTIRE DIRICHLET SERIES OF CLASS $\underline{D}(\Phi)$ ON CURVES OF BOUNDED K-SLOPE

N.N. AITKUZHINA, A.M. GAISIN, R.A. GAISIN

Abstract. We study an asymptotic behavior of the sum of an entire Dirichlet series $F(s) = \sum\limits_n a_n e^{\lambda_n s}$, $0 < \lambda_n \uparrow \infty$, on curves of a bounded K-slope naturally going to infinity. For entire transcendental functions of finite order having the form $f(z) = \sum\limits_n a_n z^{p_n}$, $p_n \in \mathbb{N}$, Pólya showed that if the density of the sequence $\{p_n\}$ is zero, then for each curve γ going to infinity there exists an unbounded sequence $\{\xi_n\} \subset \gamma$ such that, as $\xi_n \to \infty$, the relation holds:

$$\ln M_f(|\xi_n|) \sim \ln |f(\xi_n)|\,;$$

here $M_f(r)$ is the maximum of the absolute value of the function f. Later these results were completely extended by I.D. Latypov to entire Dirichlet series of finite order and finite lower order according in the Ritt sense. A further generalization was obtained in works by N.N. Yusupova–Aitkuzhina to more general classes $D(\Phi)$ and $\underline{D}(\Phi)$ defined by the convex majorant Φ. In this paper we obtain necessary and sufficient conditions for the exponents λ_n ensuring that the logarithm of the absolute value of the sum of any Dirichlet series from the class $\underline{D}(\Phi)$ on the curve γ of a bounded K-slope is equivalent to the logarithm of the maximum term as $\sigma = \operatorname{Re} s \to +\infty$ over some asymptotic set, the upper density of which is one. We note that for entire Dirichlet series of an arbitrarily fast growth the corresponding result for the case of $\gamma = \mathbb{R}_+$ was obtained by A.M. Gaisin in 1998.

Keywords: Dirichlet series, maximal term, curve of a bounded slope, asymptotic set.

Mathematics Subject Classification: 30D10

1. Introduction

We briefly dwell on the history of a question. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{p_n} \tag{1.1}$$

be an entire transcendental function, $P = \{p_n\}$ be a sequence of natural numbers having a density

$$\Delta = \lim_{n\to\infty} \frac{n}{p_n}.$$

Pólya [1] showed that if $\Delta = 0$, then in each angle $\{z : |\arg(z-\alpha)| \leqslant \delta\}$, $\delta > 0$, the function f possesses the same order as in the entire plane. A corresponding result for the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad 0 < \lambda_n \uparrow \infty, \tag{1.2}$$

N.N. Aitkuzhina, A.M. Gaisin, R.A. Gaisin, Behavior of entire Dirichlet series of class $\underline{D}(\Phi)$ on curves of bounded K-slope.

Submitted January 31, 2023.

The work of the second author (Introduction) is supported by the Russian Science Foundation (grant no., https://rscf.ru/project/21-11-00168 . The work of the third author (Section 3) is made in the framework of the State Task of the Ministry of Science and Higher Education of Russian Federation (theme FMRS-2022-0124).

absolutely converging in the entire plane was proved in [2]: if a sequence $\Lambda = \{\lambda_n\}$ satisfies the conditions $\Delta = 0$ and $\lambda_{n+1} - \lambda_n \geqslant h > 0$, $n \geqslant 1$, then the R-order of the function F on a positive ray $\mathbb{R}_+ = [0, \infty)$ is equal to the R-order ρ_R of the function F in the entire plane. A more general result was proved in [3], where, in particular, it was shown that if $\Delta = 0$ and the condensation index δ of the sequence Λ is equal to zero, then $\rho_R = \rho_\gamma$, where

$$\rho_\gamma = \varlimsup_{s\in\gamma,\, s\to\infty} \frac{\ln\ln|F(s)|}{\sigma}, \qquad \sigma = \operatorname{Re}\ s,$$

is Ritt order on the curve γ going to infinity so that if $s \in \gamma$ and $s \to \infty$, then $\operatorname{Re} s \to +\infty$.

A more general result of a bit different nature was established in paper [4]. In order to formulate it, we introduce appropriate notation and definitions.

Let $\Gamma = \{\gamma\}$ be a family of all curves going to infinity so that if $s \in \gamma$ and $s \to \infty$, then $\operatorname{Re} s \to +\infty$.

By $D(\Lambda)$ we denote the class of entire functions F represented by Dirichlet series (1.2) in the entire plane, while by $D(\Lambda, R)$ we denote a subclass $D(\Lambda)$ consisting of functions F possessing a finite Ritt order $\rho_R(F)$:

$$\rho_R(F) = \varlimsup_{\sigma\to+\infty} \frac{\ln\ln M_F(\sigma)}{\sigma}, \qquad M_F(\sigma) = \sup_{|t|<\infty} |F(\sigma + it)|.$$

For $F \in D(\Lambda)$, $\gamma \in \Gamma$ we let

$$d(F;\gamma) \overset{def}{=} \varlimsup_{s\in\gamma,\, s\to\infty} \frac{\ln|F(s)|}{\ln M_F(\operatorname{Re} s)}, \qquad d(F) = \inf_{\gamma\in\Gamma} d(F;\gamma).$$

By L we denote the class of all continuous and unboundedly increasing on $[0, \infty)$ positive functions.

A sequence $\{b_n\}$ ($b_n \neq 0$ as $n \geqslant N$) is called $\overline{W}$-normal[1] if there exists a function $\theta \in L$ such that [4]

$$\lim_{x\to\infty} \frac{1}{\ln x} \int_1^x \frac{\theta(t)}{t^2}\,dt = 0, \qquad -\ln|b_n| \leqslant \theta(\lambda_n), \qquad n \geqslant N.$$

We consider a Weierstrass product

$$Q(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right), \qquad 0 < \lambda_n \uparrow \infty.$$

It is known that Q is an entire function of exponential type if and only if the sequence Λ possesses a finite upper density.

In [4] the following theorem was proved.

Theorem 1.1. *Let the sequence Λ possesses a finite upper density. Assume that the sequence $\{Q'(\lambda_n)\}$ is $\overline{W}$-normal. Then for each function $F \in D(\Lambda, R)$ the identity $d(F) = 1$ holds if and only if*

$$\varliminf_{x\to\infty} \frac{1}{\ln x} \sum_{\lambda_n\leqslant x} \frac{1}{\lambda_n} = 0. \tag{1.3}$$

Let an entire function f of a finite order be of the form (1.1). If the sequence P has the density $\Delta = 0$, then $d(f) = 1$ ($d(f)$ is an analogue of quantity $d(F)$, which is defined by all curves arbitrarily going to infinity). This fact was first established by Pólya in [1]. We note that the identity $d(f) = 1$ follows from a more general Theorem 1.1. Indeed, since $\Delta = 0$, then obviously

$$\lim_{x\to\infty} \frac{1}{\ln x} \sum_{p_n\leqslant x} \frac{1}{p_n} = 0.$$

[1]In this paper we use the term "$W(\ln)$-normal sequence".

Since $\Delta = 0$ and $p_n \in \mathbb{N}$, then, as it is known, see, for instance [5],

$$\delta = \lim_{n\to\infty} \frac{1}{p_n} \ln \left| \frac{1}{Q'(p_n)} \right| = 0.$$

This means that there exists a function $\theta \in L$, $\theta(x) = o(x)$ as $x \to \infty$, such that

$$-\ln |Q'(p_n)| \leqslant \theta(p_n), \quad n \geqslant 1.$$

Hence, the sequence $\{Q'(p_n)\}$ is $\overline{W}$-normal ($W(\ln)$-normal).

Finally, if f is an entire function of finite order, then letting $z = e^s$, we note that

$$F(s) = f(e^s) = \sum_{n=1}^{\infty} a_n e^{p_n s}$$

is an entire function of a finite R-order. Therefore, $d(f) = d(F)$ and all facts are implied by Theorem 1.1.

However, the identity $d(F) = 1$ generally does not imply the identity $\rho_R(F) = \rho_\gamma$ for the Ritt orders of the function F in the entire plane and on the curve $\gamma \in \Gamma$. It turns out that if, in Theorem 1.1, we replace condition (1.3) by a stronger one

$$\lim_{x\to\infty} \frac{1}{\ln x} \sum_{\lambda_n \leqslant x} \frac{1}{\lambda_n} = 0,$$

then $\rho_R(F) = \rho_\gamma$ for each function $F \in D(\Lambda, R)$, see [6].

As in work [6], here we consider a more general situation, namely, we study the class of Dirichlet series (1.2) determined by some convex growth majorant. For the curves $\gamma \in \Gamma$ having a bounded slope, we prove a stronger asymptotic estimate than the identity $d(F) = 1$ obtained in [6] for the functions in the same class.

By definition, the curve $\gamma \in \Gamma$ defined by the equation $y = g(x)$, $x \in \mathbb{R}_+ = [0, +\infty)$, possesses a bounded slope if

$$\sup_{\substack{x_1, x_2 \in \mathbb{R} \\ x_1 \neq x_2}} \left| \frac{g(x_2) - g(x_1)}{x_2 - x_1} \right| = K < \infty. \tag{1.4}$$

Condition (1.4) means that the absolute values of the tangents of all chords of the curve γ does not exceed K. In this case γ is called a curve of a bounded K-slope.

In a series of papers, there was found a close relation between the regularity of the growth of the sum of the Dirichlet series (1.2) on $\gamma \in \Gamma$ with the incompleteness of the system of exponentials $\{e^{\lambda_n z}\}$ on the arcs $\gamma' \subset \gamma$ and especially with a strong incompleteness of this exponential system in a vertical strip, see [7]–[9]. It should be noted that the results of works [8], [9] on the incompleteness of the system $\{e^{\lambda_n z}\}$ on the arcs can be applied to studying the uniqueness theorems and asymptotic properties of entire Dirichlet series (1.2) with no restrictions for the growth $M_F(\sigma)$, that is, in the most general case.

The aim of the present paper is to show, under the same assumptions for Λ as in [6], that if

$$\varliminf_{\sigma\to+\infty} \frac{\ln M_F(\sigma)}{\Phi(\sigma)} < \infty$$

(Φ is some convex on $\mathbb{R}_+$ function), then for each curve $\gamma \in \Gamma$ of a bounded K-slope, as $s \in \gamma$, $\sigma = \operatorname{Re} s \to +\infty$ over some asymptotic set $A \subset \mathbb{R}_+$ with the upper density $DA = 1$, a Pólya asymptotic identity

$$\ln |F(s)| \sim \ln M_F(\sigma), \quad s \in \gamma,$$

holds. It is clear that this relation is essentially better than the identity $d(F) = 1$.

2. Auxiliary statements. Main results

Let $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ be a sequence having a finite upper density D. Then $Q(z)$ is an entire function of exponential type at most πD^*, where D^* is an averaged upper density of the sequence Λ:

$$D^* = \overline{\lim_{t\to\infty}} \frac{N(t)}{t}, \quad N(t) = \int_0^t \frac{n(x)}{x}\, dx, \quad n(t) = \sum_{\lambda_j \leqslant t} 1.$$

It always holds $D^* \leqslant D \leqslant eD^*$, see, for instance, [5], [10].

Let L be the class of all continuous and unboundedly increasing on $\mathbb{R}_+$ positive function, Φ be a convex function in L,

$$D_m(\Phi) = \big\{F \in D(\Lambda) : \ln M_F(\sigma) \leqslant \Phi(m\sigma)\big\}, \qquad m \geqslant 1,$$

where $M_F(\sigma) = \sup\limits_{|t|<\infty} |F(\sigma + it)|$. We let

$$D(\Phi) = \bigcup_{m=1}^{\infty} D_m(\Phi).$$

We suppose that the above introduced function Φ is such that

$$\overline{\lim_{x\to\infty}} \frac{\varphi(x^2)}{\varphi(x)} < \infty, \tag{2.1}$$

where φ is a function inverse to Φ. For our purposes we shall need the following class of monotone functions:

$$W(\varphi) = \left\{ w \in L : \sqrt{x} \leqslant w(x), \lim_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{w(t)}{t^2} dt = 0 \right\}.$$

We note that the condition $\sqrt{x} \leqslant w(x)$ in this definition does not restrict the generality; it is introduced just for a convenience. Let $\Gamma = \{\gamma\}$ be the family of curves γ introduced above and let for $F \in D(\Lambda)$

$$d(F;\gamma) \overset{def}{=} \overline{\lim_{s\in\gamma,\, s\to\infty}} \frac{\ln|F(s)|}{\ln M_F(\operatorname{Re} s)}, \quad d(F) = \inf_{\gamma\in\Gamma} d(F;\gamma). \tag{2.2}$$

By $\mu(\sigma)$ we denote a maximal term in series (1.2).

In work [11], there was proved a criterion of validity of the identity $d(F) = 1$ for each function F in the class $D(\Phi)$, while in [6] the same was done for the class $\underline{D}(\Phi)$, where

$$\underline{D}(\Phi) = \bigcup_{m=1}^{\infty} \underline{D}_m(\Phi),$$

$$\underline{D}_m(\Phi) = \{F \in D(\Lambda) : \exists\{\sigma_n\} : 0 < \{\sigma_n\} \uparrow \infty,\ \ln M_F(\sigma_n) \leqslant \Phi(m\sigma_n)\}, \qquad m \geqslant 1.$$

We shall say that the sequence $\big\{Q'(\lambda_n)\big\}$ is $W(\varphi)$-normal if there exists $\theta \in L$ such that

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{\theta(t)}{t^2} dt = 0, \qquad -\ln\big|Q'(\lambda_n)\big| \leqslant \theta(\lambda_n), \quad n \geqslant 1. \tag{2.3}$$

The following theorem was proved in [6].

Theorem 2.1. *Let the sequence Λ possesses a finite upper density. Suppose that the sequence $\{Q'(\lambda_n)\}$ is $W(\varphi)$-normal.*

The identity $d(F) = 1$ holds for each function $F \in \underline{D}(\Phi)$ if and only if the condition

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \sum_{\lambda_n \leqslant x} \frac{1}{\lambda_n} = 0 \tag{2.4}$$

is satisfied.

We note that in the definition of the class $\underline{D}(\Phi)$ we can consider, for example, the function

$$\Phi(\sigma) = \underbrace{\exp\exp\ldots\exp}_{k}(\sigma), \quad k \geqslant 1.$$

Therefore, Theorem 2.1 implies a corresponding result in [4] proven for the case $k = 1$.

Now we are in position to formulate our main result.

Let Φ be the above introduction function and φ be its inverse. The following theorem is true.

Theorem 2.2. *Let the upper density of the sequence Λ be finite and the sequence $\{Q'(\lambda_n)\}$ be $W(\varphi)$-normal. If condition (2.4) is satisfied, then for each function $F \in \underline{D}(\Phi)$, for each curve $\gamma \in \Gamma$ of a bounded K-slope, as $s \in \gamma$, $\sigma = \mathrm{Re}s \to +\infty$ over some asymptotic set $A \subset \mathbb{R}_+$ with the upper density $DA = 1$, the asymptotic identity*

$$\ln|F(s)| = (1 + o(1))\ln M_F(\sigma), \quad s \in \gamma, \tag{2.5}$$

holds true.

Now we formulate lemmas, which will be employed for the proof of Theorem 2.2.

Lemma 2.1. *Let $\Phi \in L$ and its inverse function φ satisfies condition (2.1). Let $u(\sigma)$ be a non-decreasing positive continuous on $[0,\infty)$ function and $\lim\limits_{\sigma\to\infty} u(\sigma) = \infty$, and for some sequence $\{\tau_n\}$ and $m \in \mathbb{N}$ the estimate holds:*[1]

$$u(\tau_n) \leqslant \ln\Phi(m\tau_n).$$

Suppose that the function w belongs to the class $W(\varphi)$. If $v = v(\sigma)$ is a solution of the equation

$$w(v) = e^{u(\sigma)},$$

then as $\sigma \to \infty$ outside some set $E \subset [0,\infty)$,

$$\mathrm{mes}(E \cap [0,\tau_n]) = o(\varphi(v(\tau_n))), \quad \tau_n \to \infty,$$

the estimate holds:

$$u\left(\sigma + \frac{w(v(\sigma))}{v(\sigma)}\right) < u(\sigma) + o(1).$$

This lemma was proved in [12].

Lemma 2.2. *Let a function $g(z)$ be analytic and bounded in the circle*

$$D(0,R) = \{z : \ |z| < R\}, \qquad |g(0)| \geqslant 1.$$

If $0 < r < 1 - N^{-1}$, $N > 1$, then there exist at most countably many circles

$$V_n = \{z : |z - z_n| \leqslant \rho_n\}, \qquad \sum_n \rho_n \leqslant Rr^N(1 - r) \tag{2.6}$$

such that for all z in the circle $\{z : |z| \leqslant rR\}$ but outside $\bigcup\limits_n V_n$ the estimate

$$\ln|g(z)| \geqslant \frac{R - |z|}{R + |z|}\ln|g(0)| - 5NL \tag{2.7}$$

[1] In [12] Lemma 2.1 was proved under the estimate $u(\tau_n) \leqslant C\Phi(\tau_n)$. It is obviously true as $u(\tau_n) \leqslant \Phi(m\tau_n)$.

holds, where

$$L = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \left| g(Re^{i\theta}) \right| d\theta - \ln |g(0)|.$$

This lemma was proved in [13].

3. Proof of Theorem 2.2

The sequence $\{Q'(\lambda_n)\}$ is $W(\varphi)$-normal and $\Lambda = \{\lambda_n\}$ possesses a finite upper density. Therefore,

$$\overline{\lim_{x\to\infty}} \frac{N(x)}{x} < \infty, \qquad -\ln |Q'(\lambda_n)| \leqslant \theta(\lambda_n), \qquad n \geqslant 1, \qquad \theta \in W(\varphi).$$

Since, see [6],

$$\sup_{x>0} \left| \sum_{\lambda_n \leqslant x} \frac{1}{\lambda_n} - \int_0^x \frac{N(t)}{t^2} \right| = a < \infty,$$

then in view of (2.3), (2.4) we obtain

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \int_0^x \frac{N(t)}{t^2} dt = 0.$$

We let $w(t) = \max(\sqrt{t}, N(et) + \theta(t))$, where θ is the function from condition (2.3). It is clear that $w \in W(\varphi)$. Then it is obviously exists a function $w^* \in W(\varphi)$ such that $w^*(x) = \beta(x)w(x)$, $\beta \in L$.

Let $v = v(\sigma)$ be a solution of equation

$$w^*(v) = 3 \ln \mu(\sigma). \tag{3.1}$$

We let

$$h = \frac{w(v(\sigma))}{v(\sigma)}, \quad h^{(1)} = \frac{w_1(v)}{v}, \quad h^* = \frac{w^*(v(\sigma))}{v(\sigma)},$$

where $w^*(v) = \sqrt{\beta(x)}w(x)$. Let

$$R_v = \sum_{\lambda_j > v} |a_j| \, e^{\lambda_j \sigma}, \quad v = v(\sigma).$$

Since the sequence Λ possesses a finite upper density, then $C = \sum\limits_{n=1}^{\infty} \lambda_n^{-2} < \infty$. Therefore, the estimate holds, see, for instance, [7]:

$$R_v \leqslant C\mu(\sigma + h^*) \exp\left[-(1 + o(1))w^*(v)\right]. \tag{3.2}$$

We consider a function $u(\sigma) = \ln 3 + \ln \ln \mu(\sigma)$. Since $F \in \underline{D}(\Phi)$, then there exists a sequence $\{\tau_j\}$, $0 < \tau_j \uparrow \infty$, such that

$$u(\sigma) \leqslant \ln \Phi(m\sigma), \qquad \sigma = \tau_j, \qquad m \geqslant 1.$$

Therefore, in view of (3.1), as $\sigma = \tau_j$, $j \geqslant 1$, we have:

$$\ln w^*(v(\sigma)) = u(\sigma) \leqslant \ln \Phi(m\sigma), \qquad m \geqslant 1.$$

Hence,

$$\frac{1}{\sigma} \leqslant \frac{m}{\varphi(w^*(v(\sigma)))}, \qquad \sigma = \tau_j, \qquad m \geqslant 1. \tag{3.3}$$

Taking into consideration condition (2.1) and the fact that $\sqrt{x} \leqslant w^*(x)$, we get:

$$\varphi(x) \leqslant C_1\varphi(w^*(x)), \qquad x \geqslant x_0, \qquad 0 < C_1 < \infty. \tag{3.4}$$

Thus, by (3.3) and (3.4) we obtain the estimates:

$$\frac{1}{\sigma} \leqslant \frac{C_2}{\varphi(v(\sigma))}, \qquad \sigma = \tau_j, \qquad j \geqslant 1, \qquad 0 < C_2 < \infty. \tag{3.5}$$

Since $w^* \in W(\varphi)$ and the function φ is concave, then

$$\lim_{x\to\infty} \frac{w^*(x)}{x\varphi(x)} = 0, \tag{3.6}$$

which is implies by the identity

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{w^*(t)}{t^2}\, dt = 0. \tag{3.7}$$

Applying Lemma 2.1 for the functions u and w^* and taking into consideration (3.5), as $\sigma \to \infty$ outside some set $E_1 \subset [0,\infty)$,

$$\operatorname{mes}(E_1 \cap [0,\tau_j]) \leqslant o(\varphi(v(\tau_j))) = o(\tau_j), \qquad \tau_j \to \infty, \tag{3.8}$$

we obtain that

$$\mu(\sigma + 3h^*(\sigma)) = \mu(\sigma)^{1+o(1)}. \tag{3.9}$$

Therefore, by (3.2), (3.9) we obtain that as $\sigma \to \infty$ outside the set E_1 with the lower density $dE_1 = 0$,

$$R_v \leqslant C\mu(\sigma)^{1+o(1)} \exp\left[-w^*(v)(1+o(1))\right] = \mu(\sigma)^{-2(1+o(1))}. \tag{3.10}$$

This implies that $\lambda_{\nu(\sigma)} \leqslant v(\sigma)$ as $\sigma \geqslant \sigma_1$, $\sigma \notin E_1$, where $\lambda_{\nu(\sigma)}$ is the central indicator ($\nu(\sigma)$ is the central index) of series (1.2).

In the same way as (3.10) we show that as $\sigma \to \infty$, outside the same set E_1, see [7],

$$\sum_{\lambda_j > v(\sigma)} |a_j| e^{\lambda_j(\sigma + h^{(1)})} \leqslant \mu^{-2(1+o(1))}(\sigma). \tag{3.11}$$

Borel-Nevanlinna relation (3.9) allows us to do this since $h^{(1)}(\sigma) = o(h^*(\sigma))$ as $\sigma \to \infty$; properties (3.6), (3.7) are needed for the proof of Lemma 2.1.

Let

$$F_a(s) = \sum_{\lambda_n \leqslant a} a_n e^{\lambda_n s}, \qquad s = \sigma + it.$$

Then for $\lambda_n \leqslant a$ we have, see [5]:

$$a_n = e^{-\alpha\lambda_n} \frac{1}{2\pi i} \int_C \varphi_n(t) F_a(t+\alpha)\, dt, \tag{3.12}$$

where α is an arbitrary parameter,

$$\varphi_n(t) = \frac{1}{Q_a'(\lambda_n)} \int_0^\infty \frac{Q_a(\lambda)}{\lambda - \lambda_n} e^{-\lambda t}\, d\lambda, \qquad Q_a(\lambda) = \prod_{\lambda_n \leqslant a} \left(1 - \frac{\lambda^2}{\lambda_n^2}\right), \tag{3.13}$$

and C is an arbitrary closed contour enveloping $\overline{D}$, which the conjugate diagram $Q_a(\lambda)$. But $Q_a(\lambda)$ is a polynomial and therefore, $\overline{D} = \{0\}$.

We let $a = v(\sigma)$, $\alpha = \sigma + it$, where t is such that $\alpha \in \gamma$. As C we take the contour $\{t : |t| = h^{(1)}\}$, where $h^{(1)} = h^{(1)}(\sigma) = \frac{h^*(\sigma)}{\sqrt{\beta(v(\sigma))}}$. Then by assumption

$$-\ln |Q'(\lambda_n)| \leqslant \theta(\lambda_n) \leqslant w(\lambda_n), \qquad n \geqslant 1.$$

Therefore, in view of identity (3.1) we obtain that for each $\lambda_n \leqslant v(\sigma)$ as $\sigma \to \infty$ we get:

$$\frac{1}{|Q_v'(\lambda_n)|} \leqslant \frac{1}{|Q'(\lambda_n)|} \leqslant e^{\theta(\lambda_n)} \leqslant e^{w(v)} = e^{o(w^*(v))} = \mu(\sigma)^{o(1)}.$$

But then by (3.12), (3.13) we get that for all $\lambda_n \leqslant v(\sigma)$ as $\sigma \to \infty$ outside the set E_1

$$|a_n|e^{\lambda_n \sigma} \leqslant \mu(\sigma)^{o(1)} h^{(1)} \left[\max_{|\xi-\alpha|\leqslant h^{(1)}} |F(\xi)| + \sum_{\lambda_j > v} |a_j| \; e^{\lambda_j(\sigma+h^{(1)})} \right] \int\limits_0^\infty \left| \frac{Q_v(\lambda)}{\lambda - \lambda_n} \right| |e^{-\lambda t}||d\lambda|, \quad (3.14)$$

where $\alpha = \sigma + it \in \gamma$.

It is easy to show that [14]

$$\max_{|\lambda|=r} \left| \frac{Q_v(\lambda)}{\lambda - \lambda_n} \right| \leqslant M(1) M_v(r), \quad (3.15)$$

where $M(1) = \max\limits_{|z|=1} |Q(z)|$, $M_v(r) = \max\limits_{|z|=r} |Q_v(z)|$.

Since $\lambda_\nu(\sigma) \leqslant v(\sigma)$ outside E_1 as $\sigma \geqslant \sigma'$, taking into consideration (3.11), (3.15), by (3.14) as $\sigma \to \infty$ outside E_1 we obtain:

$$\mu(\sigma)^{1+o(1)} \leqslant h^{(1)} \left[\max_{|\xi-\alpha|\leqslant h^{(1)}} |F(\xi)| + \mu(\sigma)^{-2(1+o(1))} \right] \int\limits_0^\infty M_v(r) e^{-rh^{(1)}} dr. \quad (3.16)$$

Then, taking into consideration the definition of the quantities $v = v(\sigma)$, $h^{(1)} = h^{(1)}(\sigma)$, as well as the inequalities $n(x) \leqslant N(ex)$, $\ln(1 + x^2) < x$, $x > 0$, we have:

$$\ln M(r) = n(v) \ln \left(1 + \frac{r^2}{v^2}\right) + 2r^2 \int\limits_0^v \frac{n(t)}{t(t^2 + r^2)} dt \leqslant \frac{n(v)}{v} r + 2N(v) = o(1) h^{(1)} r + o(1) \ln \mu(\sigma).$$

Therefore, by (3.16) we obtain that as $\sigma \to \infty$ outside E_1

$$\mu(\sigma)^{1+o(1)} \leqslant \max_{|\xi-\alpha|\leqslant h^{(1)}} |F(\xi)| = |F(\xi^*)|, \quad (3.17)$$

where $|\xi^* - \alpha| = h^{(1)}$, $\alpha = \sigma + it \in \gamma$. In view of estimate (3.15), as $\sigma \to \infty$ outside E_1 we also have

$$\begin{aligned} \mu(\sigma) \leqslant M_F(\sigma) \leqslant M_F(\sigma + 2h^*) \leqslant \sum_{n=1}^\infty |a_n| e^{\lambda_n(\sigma+2h^*)} \\ \leqslant \mu(\sigma + 3h^*) \left[n(v) + \sum_{\lambda_j > v(\sigma)} e^{-h^* \lambda_j} \right] < \mu(\sigma)^{1+o(1)}. \end{aligned} \quad (3.18)$$

Let $B = \mathbb{R}_+ \setminus E_1$, $h = \frac{w(v(\sigma))}{v(\sigma)}$. Then there exists a sequence $\{\sigma_j\}$, $\sigma_j \in B$, $\sigma_j \uparrow 0$, $\sigma_j + h_j \leqslant \sigma_{j+1}$, $j \geqslant 1$, such that, see [13],

$$B \subset \bigcup_{j=1}^\infty [\sigma_j - h_j, \sigma_j + h_j],$$

where $h_j = \frac{w(v_j)}{v_j}$, $v_j = v(\sigma_j)$,, $j \geqslant 1$.

We let $g(z) = F(z + \xi^*)$. By (3.17) we see that $|g(0)| \geqslant 1$ as $\sigma \geqslant \sigma'' > \sigma'$ outside E_1. We apply Lemma 2.1 to the function $g(z)$, letting $\alpha_j = \sigma_j + it_j$, $h^{(1)} = h_j^{(1)} = \frac{w(v_j)}{v_j}\sqrt{\beta(v_j)}$ in (3.17) and $N = 4$, $r = \frac{1}{\sqrt{\beta(v_j)}}$, $R = h_j^*$ in estimates (2.6), (2.7), where $h_j^* = \frac{w^*(v_j)}{v_j}$, $j \geqslant j_1$. Then in the circle $\{z : |z| \leqslant h_j^{(1)}\}$ but outside exceptional circles V_{nj} with the total sum of the radii

$$\sum_n \rho_n \leqslant \frac{h_j}{\beta_j}, \qquad \beta_j = \beta(v(\sigma_j)), \qquad j \geqslant j_1, \quad (3.19)$$

estimate (2.7) holds true.

Let γ_j be a part of γ connecting vertical straight lines passing through the end-points of the segment $\Delta_j = [\sigma_j - h_j,\, \sigma_j + h_j]$. Since the curve γ possesses a K-slope, then γ_j is located in some rectangle $P_j = \Delta_j \times [c_j,\, d_j]$, $d_j - c_j \leqslant 2Kh_j$, with the center at the point $\alpha_j = \sigma_j + it_j$ and connects its vertical sides.

Since the rectangle P_j is located in the circle $\{z : |z| \leqslant h_j^{(1)}\}$, then for all $z \in P_j$ but outside the circles V_{nj} with the total sum of radii obeying estimate (3.19), as $j \to \infty$ we obtain that

$$\ln|g(z)| \geqslant \left[1 + o(1) - \frac{20L}{\ln|g(0)|}\right] \ln|g(0)|. \tag{3.20}$$

Taking into consideration (3.17), (3.18), as well as that $|g(0)| \geqslant 1$, we confirm that as $j \to \infty$ the asymptotic identity

$$\frac{L}{\ln|g(0)|} = o(1)$$

holds, where

$$L = \frac{1}{2\pi} \int\limits_0^{2\pi} \ln^+ |g(Re^{i\theta})| d\theta - \ln|g(0)|,$$

$$g(0) = F(\xi^*), \qquad |\mathrm{Re}\ \xi^* - \sigma_j| \leqslant h^{(1)}, \qquad \alpha_j = \sigma_j + it_j \in \gamma.$$

Therefore, by (3.20), for all z in the rectangle P_j but outside the circles V_{nj} as $j \to \infty$ we have

$$\ln|g(z)| \geqslant (1 + o(1)) \ln|g(0)|. \tag{3.21}$$

But then, taking into consideration that $g(z) = F(z + \xi^*)$ and using estimates (3.17)–(3.21), we obtain that for all z in P_j with the center at the point $\alpha_j = \sigma_j + it_j$ but outside exceptional circles V_{nj} with the total sum of radii not exceeding $\frac{h_j}{\beta_j}$ we have

$$\ln|F(z)| > (1 + o(1)) \ln \mu(\sigma_j), \qquad j \to \infty. \tag{3.22}$$

Let E_2 be the projection of all exceptional circles of the set $\bigcup\limits_j P_j$ on B, where $\alpha_j = \sigma_j + it_j$ is the center of P_j, $B \subset \bigcup\limits_{j=1}^{\infty} [\sigma_j - h_j, \sigma_j + h_j]$, $\sigma_j \in B$, $\sigma_j + h_j \leqslant \sigma_{j+1}$, $j \geqslant 1$. Let us show that $DE_2 = 0$. Indeed, let $\sigma_j \leqslant \sigma < \sigma_{j+1}$. According to (3.6),

$$h_j \leqslant h_j^{(1)} < h_j^* = o(\sigma_j), \quad j \to \infty.$$

And since $\beta_j \uparrow \infty$ as $j \to \infty$, then it is obvious that

$$\lim_{\sigma \to \infty} \frac{\mathrm{mes}(E_2 \cap [0, \sigma])}{\sigma} = 0.$$

Thus, $DE_2 = 0$, and therefore, $dE = 0$, where $E = E_1 \cup E_2$.

Estimate (3.22) holds in each P_j with the center $\alpha_j = \sigma_j + it_j \in \gamma$ but outside exceptional circles V_{nj}, the total sum of radii of which obeys estimate (3.19).

The projection p_j of the arc γ_j on $\mathbb{R}_+$ is a segment $[\sigma_j - h_j, \sigma_j + h_j]$. We let $A = P \setminus E$, where $P = \bigcup\limits_{j=1}^{\infty} p_j$. On this set asymptotic estimates (3.18), (3.22); A is called asymptotic set. This implies that as $s \in \gamma$, $\mathrm{Re}\, s = \sigma \to \infty$ over the set A

$$\ln|F(s)| = (1 + o(1)) \ln \mu(\sigma) = (1 + o(1)) \ln M_F(\sigma).$$

It remains to estimate DA. Taking into consideration that $B \subset P$ and $\mathrm{mes}(E \cap [0, \tau_j]) = o(\tau_j)$, $\tau \to \infty$, we get:

$$DA = \varlimsup_{\sigma \to \infty} \frac{\mathrm{mes}(A \cap [0, \sigma])}{\sigma} \geqslant \varlimsup_{\tau_j \to \infty} \frac{\mathrm{mes}(P \cap [0, \tau_j])}{\tau_j} - \varlimsup_{\tau_j \to \infty} \frac{\mathrm{mes}(E \cap [0, \tau_j])}{\tau_j} = 1.$$

Here $\{\tau_j\}$ is the above introduced sequence. Hence, $DA = 1$. The proof of Theorem 2.2 is complete.

As it was shown in [6], the assumptions of Theorem 2.2 are also necessary in order each function $F \in \underline{D}(\Phi)$ on some set $A \subset \mathbb{R}_+$ having a positive upper density DA asymptotic identity (2.5) to hold. Therefore, the statement of Theorem 2.2 is also sufficient.

BIBLIOGRAPHY

1. G. Pólya. *Untersuchungen über Lücken und Singularitäten von Potenzeihen* // Math. Z. **29**, 549–640 (1929).
2. M.N. Sheremeta. *Growth on the real axis of an entire function represented by a Dirichlet series* // Matem. Zamet. **33**:2, 235–245 (1983). [Math. Notes. **33**:2, 119–124 (1983).]
3. A.M. Gaisin. *Behavior of the sum of a Dirichlet series having a prescribed growth* // Matem. Zamet. **50**:4, 47–56 (1991). [Math. Notes. **50**:4, 1018–1024 (1991).]
4. A.M. Gaisin, I.D. Latypov. *Asymptotic behavior of the sum of the Dirichlet series of prescribed growth on curves* // Matem. Zamet. **78**:1, 37-51 (2005). [Math. Notes. **78**:1, 33–46 (2005).]
5. A.V. Leont'ev. *Sequences of exponential polynomials.* Nauka, Moscow (1980). (in Russian).
6. A.M. Gaisin, N.N. Yusupova. *Behaviour of the sum of Dirichlet series with a given majorant of a growth on curves* // Ufimskij Matem. Zhurn. **1**:2, 17–28 (2009). (in Russian).
7. A.M. Gaisin. *Estimates of the growth and decrease on curves of an entire function of infinite order* // Matem. Sb. **194**:8, 55–82 (2002). [Sb. Math. **194**:8, 1167–1194 (2003).]
8. A.M. Gaisin, R.A. Gaisin. *Incomplete system of exponentials on arcs and nonquasianalytic Carleman classes. II* // Alg. Anal. **27**:1, 49–73 (2015). [St.-Petersburg Math. J. **27**:1, 33–50 (2016).]
9. R.A. Gaisin. *Interpolation sequences and nonspanning systems of exponentials on curves* // Matem. Sborn. **212**:5, 58–79 (2021). [Sb. Math. **212**:5, 655–675 (2021).]
10. A.F. Leontiev. *Exponential series.* Nauka, Moscow (1976). (in Russian).
11. N.N. Yusupova. *Asymptotics of Dirichlet series of a prescribed growth.* PhD thesis. Ufa, (2009). (in Russian).
12. A.M. Gaisin, N.N. Aitkuzhina. *Stability preserving perturbation of the maximal terms of Dirichlet series* // Probl. Anal. Issues Anal. **11(29)**:3, 30–44 (2022).
13. A.M. Gaisin. *On a conjecture of Polya* // Izv. RAN. Ser. Matem. **58**:2, 73–92 (1994). [Russ. Acad. Sci. Izv. Math. **44**:2, 281–299 (1995).]
14. A.M. Gaisin. *Properties of series of exponentials whose exponents satisfy to a condition of Levinson type* // Matem. Sborn. **197**:6, 25–46 (2006). [Sb. Math. **197**:6, 813–833 (2006).]

Narkes Nurmukhametovna Aitkuzhina,
Ufa University of Science and Technologies,
Zaki Validi str. 32,
450076, Ufa, Russia
E-mail: YusupovaN@rambler.ru

Akhtyar Magazovich Gaisin,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: gaisinam@mail.ru

Rashit Akhtyarovich Gaisin,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: rashit.gajsin@mail.ru