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ON RATE OF DECREASING OF EXTREMAL FUNCTION IN CARLEMAN CLASS

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Abstract. We study the issues related with Levinson-Sjöberg-Wolf type theorems in the complex analysis and, in particular, we discuss a famous question posed in 70s by E.M. Dyn'kin on an effective bound for majorant of the growth of an analytic function in the vicinity of the set of singular points and another close problem on the rate of decaying of an extremal function in a non-quasianalytic Carleman class in the vicinity of the point, at which all the derivatives of the functions from this class vanish. Exact asymptotic estimates for the best majorant for the growth in the vicinity of the singularities were found by V. Matsaev and M. Sodin in 2002.

Some bounds, both from above and below, for an extremal function in the Carleman class were obtained by A.M. Gaisin in 2018 but they turned out to be not very close to exact values of this function. In the present paper we obtain sharp two-sided estimates for the extremal function.

Keywords: non-quasianalytic Carleman class, Levinson-Sjöberg type theorem, extremal function, regular sequence, associated weight.

Mathematical Subject Classification: 26E10, 28A10.

1. Introduction

In 1938 N. Levinson proved a theorem, see [1], which provided a deep generalization of the maximum modulus principle, see [2]. This result played an important role in the theory of quasianalytic functions, see [3]–[7].

Theorem 1.1 (N. Levinson). *Let $M(y)$ be a positive monotonically decreasing in the semi-interval $(0,b]$ function, $M(y)\uparrow\infty$ as $y\downarrow 0$, $M(b)=e$. Let F_M be a family of functions analytic in the rectangle*

$$P=\{z=x+iy: |x|<a,\ |y|<b\}$$

and obeying the estimate $|F(z)|\leqslant M(|y|)$ in P. If

$$\int\limits_0^b \ln\ln M(y)dy<\infty, \tag{1.1}$$

then for each $\delta>0$ there exists a constant C depending only on δ and $M(y)$ such that all functions $f\in F_M$ satisfy the estimate $|F(z)|\leqslant C$ in the rectangle

$$P_\delta=\{z=x+iy: |x|<a-\delta,\ |y|<b\}.$$

We note that independently of N. Levinson, and it seems to be simultaneously with him, this theorem in a slightly different form was proved by N. Sjöberg, see [8]. However, the next theorem was established by T. Carleman much earlier, see [9].

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Theorem 1.2 (T. Carleman). *Let $M(\varphi)$ be a positive on the interval $(0, 2\pi)$ function such that $\ln M(\varphi) > 1$ and the integral*

$$\int_0^{2\pi} \ln \ln M(\varphi) d\varphi$$

converges. Then each entire function $f(z)$ satisfying the inequality

$$|f(z)| \leqslant M(\varphi), \quad \varphi = \arg z, \quad 0 < \varphi < 2\pi,$$

is constant: $f(z) \equiv \mathrm{const}$.

Exactly this result by T. Carleman was developed later by N. Levinson and N. Sjöberg, who extended it to the most general case. We however mention that Carleman's theorem is true with no additional restrictions for the majorant $M(\varphi)$. Later F. Wolf extended Levinson-Sjöberg theorem to a wider class of functions, see [10]. In [2] another simpler proof of Theorem 1.1 was given.

We provide one of the versions of this theorem, see [11], [12].

Theorem 1.3 (Y. Domar). *Let $D = \{z = x + iy : -a < x < a,\ 0 < y < b\}$ and $M(y)$ be a Lebesgue measurable function and $M(y) \geqslant e$ as $0 < y < b$. If integral (1.1) converges, then there exists a decreasing function $m(\delta)$ depending only on $M(y)$ and finite for $\delta > 0$ such that if $f(z)$ is analytic in D and*

$$|f(z)| \leqslant M(\operatorname{Im} z), \tag{1.2}$$

then

$$|f(z)| \leqslant m(\operatorname{dist}(z, \partial D)), \quad z \in D.$$

Corollary 1.1. *Let $J = \{f\}$ be a family of analytic in D functions satisfying condition (1.2). If integral (1.1) converges, then the family of the functions J is normal, that is, relatively compact.*

As P. Koosis showed, condition (1.1) is necessary and sufficient for the Levinson's theorem, see [11]: if integral (1.1) diverges, then there exists a sequence of polynomials $P_n(z)$ such that

1) $|P_n(z)| \leqslant KM(|y|)$, $K = \mathrm{const}$, $n \geqslant 1$, for all z in the rectangle

$$P = \{z = x + iy : |x| < a,\ |y| < b\};$$

2) as $n \to \infty$,

$$P_n(z) \to F(z) = \begin{cases} 1, & \text{if} \quad z \in P \cap \mathbb{C}_+, \\ -1, & \text{if} \quad z \in P \cap \mathbb{C}_-. \end{cases}$$

Here $\mathbb{C}_+ = \{z = x + iy : y > 0\}$, $\mathbb{C}_- = \{z = x + iy : y < 0\}$.

We note that under some additional restrictions for the behavior of the function $M(y)$ a similar statement was proved by N. Levinson in [1]. It was shown [12] that in Levinson's theorem the monotonicity condition for the function $M(y)$ can be replaced by the Lebesgue measurability of this function.

In [7] a generalization of the Levinson's theorem was obtained to the case, when the real segment $[-a, a]$ is replaced by some rectifiable arc γ, namely, an arc of a bounded slope.

Let E be a compact set in $\mathbb{R}$ and M be the majorant from the Levinson's theorem, for which bilogarithmic condition (1.1) holds. In [4] there was introduced a set $F_E^0(M)$ of functions f defined and analytic outside E such that

$$|f(z)| \leqslant M(|\operatorname{Im} z|), \quad z \in \mathbb{C} \setminus E.$$

In this estimate M is an arbitrary decreasing on $\mathbb{R}_+ = (0, +\infty)$ function coinciding with the majorant from Theorem 1.1 on $(0, b]$. In what follows we assume that $M(y) \downarrow 0$ as $y \to +\infty$.

According to Theorem 1.1, the set $F_E^0(M)$ is normal, that is, for each $\delta > 0$

$$M^*(\delta) = \sup\left\{|f(z)| : f \in F_E^0(M),\ \rho(z,E) \geqslant \delta\right\} < \infty.$$

Here $\rho(z,E) = \inf\limits_{\xi\in E} |z-\xi|$, $z \in \mathbb{C}$. Thus, M^* is a smallest function obeying

$$|f(z)| \leqslant M^*(\rho(z,E)), \qquad z \in \mathbb{C}\setminus E,$$

for all $f \in F_E^0(M)$. For the sake of definiteness, we can suppose that E is the segment $I = [0,1]$.

In [4] there was posed a problem (Problem 1) on an "effective estimate for the majorant M^*".

Suppose that the function M is logarithmically convex, that is, $\ln M(e^{-\sigma})$ is a convex function of σ. We let

$$M_n = \sup_{\delta>0} \frac{n!}{M(\delta)\delta^{n+1}}, \qquad n \geqslant 0.$$

Then, as it is known, the Carleman class on the segment I,

$$C_I(M_n) = \left\{f : \left\|f^{(n)}\right\| \leqslant cK_f^n M_n,\ n \geqslant 0\right\}, \qquad \|f\| = \max_I |f(x)|,$$

is quasianalytic if and only if integral (1.1) diverges, see [4]. In what follows by $C_I^N(M_n)$ we denote a normalized class, $C_I(M_n)$ with constant $c = 1$, $K_f = 1$. Following work [4], we also introduce the notation

$$P(\delta) = \sup\left\{|f(\delta)| : f \in C_I^N(M_n),\ \ f^{(n)}(0) = f^{(n)}(1) = 0,\ n \geqslant 0\right\}, \qquad 0 < \delta \leqslant 1.$$

As it was stated in work [6], see Remark to Theorem 3 in Sect. 2.4, the problem on effective estimate for the majorant "in form $M^* \simeq P^{-1}$ with unknown P was established in [4]". Hereinafter the writing $M^* \simeq P^{-1}$ means that

$$AP^{-1}(a\delta) \leqslant M^*(\delta) \leqslant BP^{-1}(b\delta), \tag{1.3}$$

where $0 < a < b$, $0 < A < B$ are some constants. We stress that estimates (1.3) in [4] were not written and only a lower bound was provided. The proof of the upper bound in the same inequalities in [4] was not given. It was shown in [13] that estimates of kind (1.3) in fact hold not for the function M^*, but for a so-called associated weight.

As in work [4], here we consider only regular sequences $\{M_n\}$, i.e. such that the numbers $m_n = \frac{M_n}{n!}$ satisfy the conditions:

$$1)\ m_n^{\frac{1}{n}} \to \infty,\quad n\to\infty; \qquad 2)\ \sup_{n\geqslant 0}\left(\frac{m_{n+1}}{m_n}\right)^{\frac{1}{n}} < \infty;$$
$$3)\ m_n^2 \leqslant m_{n-1}m_{n+1},\quad n \geqslant 1.$$

The Carleman class $C_I\left((n!)^{1+\alpha}\right)$ $(\alpha > 0)$ is called the a Gevrey class. This class is regular since the numbers $M_n = (n!)^{1+\alpha}$ satisfy all Conditions 1)–3).

An *associated weight* is a function $H^*(r) = [h^*(r)]^{-1}$, see [4],

$$h^*(r) = \inf_{n\geqslant 0}(m_n r^n).$$

It is clear that $h^*(r) \uparrow \infty$ as $r \to \infty$, $h^*(0+) = 0$. By Property 2) in the definition of regular sequences we see that $h^*(r) \leqslant rh^*(qr)$ for some $q > 1$. We also have

$$\frac{1}{h^*(r)} = \sup_{n\geqslant 0}\frac{1}{m_n r^n} = \sup_{n\geqslant 0}\frac{n!}{M_n r^n} \overset{def}{\equiv} H^*(r).$$

As it is known, see [4],

$$M_n = \sup_{r>0}\frac{n!}{H^*(r)r^n}, \quad n \geqslant 0.$$

The class $C_I(M_n)$ is quasianalytic if and only if one of the following equivalent conditions hold [4]:

$$1)\ \sum_{n=0}^{\infty}\frac{M_n}{M_{n+1}}=\infty; \qquad 2)\ \int_0^1 \ln^+\ln H^*(t)dt=\infty.$$

Using a duality, V. Matsaev showed that the Levinson-Sjöberg is equivalent to Denjoy-Carleman theorem on the quasianalyticity of the class $C_I(M_n)$, see [3]. Later this fact was rediscovered by E.M. Dynkin in [5], while in work [6] in terms of the quantity

$$J_M(s)=\sup\left\{|g(s)|:\sup_I|g^{(n)}(t)|\leqslant M_n, \qquad g^{(n)}(0)=0, \qquad n\geqslant 0\right\}$$

another problem was formulated, dual to Problem 1, namely, a relation $M^*\simeq J_M$. In the same work two-sided estimates for M^* have been established, but they turned to be not only non-sharp but also wrong, for more details see the survey and discussion in [14], [15]. Sharp estimates for the majorant M^* in another way were obtained in [14]. Let us formulate this result providing an answer for Problem 1 from [4].

Let

$$P_\varphi(s)=\sup_{y>0}\left[\frac{2y}{\pi}\int_0^\infty\frac{\varphi(t)dt}{t^2+y^2}-ys\right], \tag{1.4}$$

where the weight function (logarithmic weight) satisfies the conditions:

1) $\varphi(t)\geqslant 0$, $t\in\mathbb{R}_+$;

2) $\varphi(t)\uparrow\infty$ as $t\to+\infty$, $\lim\limits_{t\to\infty}\dfrac{\varphi(t)}{\ln t}=\infty$;

3) $\displaystyle\int_{\mathbb{R}}\frac{\varphi(t)}{t^2+1}dt<\infty$;

4) $\varphi(e^x)$ is convex in x on $\mathbb{R}_+$.

Sometimes an additional condition is imposed on the function φ:

5) the function $\varphi(t)$ is concave on $\mathbb{R}_+$ and

$$\lim_{t\to\infty}t\varphi'(t+0)=\infty.$$

For the logarithm of the majorant M in Problem 1 (it obeys condition (1.1)) we consider a *lower Legendre transformation*

$$\varphi(r)=\inf_{s>0}\left(\ln M(s)+rs\right).$$

Suppose that

$$\lim_{s\to 0}s^nM(s)=\infty \tag{1.5}$$

for each $n>0$. Then the weight function φ immediately satisfies Conditions 1)–3) as well as 5), see [14]. If the functions $\ln M(e^{-s})$ and $\ln M(t)$ are convex, then the function $\varphi(e^x)$ is convex in $x\in\mathbb{R}_+$, that is, Condition 4) holds, see [14].

In [14] the following theorem was proved.

Theorem 1.4. *Let the majorant M from Theorem 1.1 satisfies Conditions (1.1), (1.5), while the functions $\ln M(e^{-s})$ and $\ln M(t)$ are convex. Then*

$$\ln M^*(s)=(1+o(1))\ln P_\varphi(s),$$

as $s\to 0$, where P_φ is a function given by formula (1.4) and φ is a lower Legendre transform of the function $\ln M(t)$[1].

[1] In [14] this theorem was proved for the case $E=\{0\}$. But it is true also for the segment I.

The aim of the present paper is to find sharp two-sided estimates for $J_M(s)$ on I.

2. Second Dyn'kin problem on estimate of function $J_M(s)$

The history of the issue goes back to work by T. Bang [16].

Let $\{M_n\}_{n=0}^{\infty}$ be an arbitrary positive sequence, $M_n^{\frac{1}{n}} \to \infty$ and it is not necessarily regular. Then there exists a maximal logarithmically convex minorant $\{M_n^c\}_{n=0}^{\infty}$, i.e., such that

$$M_n^c \leqslant M_n, \quad n \geqslant 0; \qquad M_n^2 \leqslant M_{n-1}M_{n+1}, \quad n \geqslant 1.$$

The sequence $\{M_n^c\}$ is called a *convex regularization of* $\{M_n\}$ *by means of logarithms*, see [17].

Let $P = \{n_i\}$ be a sequence of main indices, that is, $M_{n_i} = M_{n_i}^c$, $i \geqslant 1$. In [16] for each function $f \in C^\infty(I)$ the quantity

$$B_f(x) = \inf_{p\in P}\left[\max\left(e^{-p}, \max_{0\leqslant n\leqslant p}\frac{|f^{(n)}(x)|}{e^n M_n^c}\right)\right] \tag{2.1}$$

is considered. It is continuous in x on I.

The main statement in [16] is the following Bang's theorem.

Theorem 2.1. *If $f \in C^\infty(I)$ and $\left\|f^{(n)}\right\| \leqslant M_n$, $n \geqslant 0$, then the estimate*

$$B_f(x) \geqslant e^{-q}$$

with some $q \in \mathbb{N}$ implies that

$$B_f(x+h) \leqslant B_f(x)\exp\left(e|h|\frac{M_q^c}{M_{q-1}^c}\right). \tag{2.2}$$

We note that in this theorem q is not necessarily belongs to the set of main indices P. The parameter h is chosen so that the shift $x+h$ belongs to I.

Remark 2.1. *Denoting $L(x) = \ln B_f(x)$, under the assumptions of Bang theorem the estimates hold* [16]*:*

1) for all x, $x+h \in I$,

$$|L(x+h) - L(x)| \leqslant e\frac{M_q^c}{M_{q-1}^c}|h|;$$

2) at points, where the derivative $L'(x)$ exists, the estimate

$$|L'(x)| \leqslant e\frac{M_q^c}{M_{q-1}^c}$$

holds.

As q, we can take an index $p \in P$, at which the infimum in (2.1) is attained.

Theorem 2.1 was employed by T. Bang for proving a criterion of a quasianalyticity of the class $C_I(M_n)$. We are interesting only in its sufficient part since its proof implies a simple estimate for each function f in the class $C_I^0(M_n) = \{f : f \in C_I^N(M_n),\ f^{(n)}(0) = f^{(n)}(1) = 0,\ n \geqslant 0\}$ in the vicinity of the point $x = 0$, which in a series is incorrectly extended also for the extremal function $J_M(M_n)$, see [6], [14].

Let us provide a short proof of Bang's statement: *if the class $C_I(M_n)$ is non-quasianalytic, then*

$$\sum_{n=0}^{\infty}\frac{M_n^c}{M_{n+1}^c} < \infty.$$

By assumption, there exist a function f in the class $C_I^0(M_n)$, $f(x) \not\equiv 0$. Hence, $B_f(x) \not\equiv 0$. Therefore, there exist $p_1 \in P$ and $x_1 \in I$ such that $B_f(x_1) = e^{-p_1}$. Then by induction we construct a sequence $\{x_n\}_{n=1}^{\infty}$: $x_n \downarrow 0$, $B_f(x_j) = e^{-p_j}$, $p_1 < p_2 < \ldots < p_n < \ldots$, $p_j \in P$. If we let $x = x_j$, $x + h = x_{j-1}$, then $h > 0$. By Theorem 2.1, in accordance with (2.2),

$$B_f(x_{j-1}) \leqslant B_f(x_j) \exp\left[e|x_j - x_{j-1}|\frac{M^c_{p_j}}{M^c_{p_j-1}}\right].$$

This yields

$$p_j - p_{j-1} \leqslant e|x_j - x_{j-1}|\frac{M^c_{p_j}}{M^c_{p_j-1}},$$

or

$$(p_j - p_{j-1})\,\frac{M^c_{p_j-1}}{M^c_{p_j}} \leqslant e|x_j - x_{j-1}|. \tag{2.3}$$

But the left hand side of the latter estimate is equal to the sum

$$\sum_{n=p_{j-1}}^{p_j-1} \frac{M^c_n}{M^c_{n+1}},$$

all terms of which are mutually equal and their total number is $p_j - p_{j-1}$. This can be easily seen by a geometric meaning of the regularization of the sequence $\{M_n\}$ by means of the logarithms, see [17]. Since

$$\sum_{j=2}^{\infty} |x_j - x_{j-1}| \leqslant x_1,$$

by (2.3) we obtain that

$$\sum_{n=p_1}^{\infty} \frac{M^c_n}{M^c_{n+1}} \leqslant ex_1 < \infty. \tag{2.4}$$

The statement was proved but we are interesting in inequality (2.4), since by this inequality T. Bang obtained an important estimate for the function f, namely, of $x \in I$ and

$$x < \frac{1}{e}\sum_{n=p_1}^{\infty} \frac{M^c_n}{M^c_{n+1}},$$

then

$$|f(x)| < M^c_0 e^{-p_1}. \tag{2.5}$$

It should be noted that the number p_1 depends on a particular function f: less is $\|f\|$, bigger is the number $p_1 = p_1(f)$.

3. Main result

Let

$$C^0_I(M_n) = \left\{f : f \in C^N_I(M_n),\ f^{(n)}(0) = f^{(n)}(1) = 0,\ n \geqslant 0\right\}.$$

Using the Taylor formula, for each function $f \in C^0_I(M_n)$ T. Bang obtained a simpler inequality [16], which implies the estimate

$$J_M(x) \leqslant \inf_{n\geqslant 0} \frac{M_n x^n}{n!}, \quad x \in I. \tag{3.1}$$

In order to understand how sharp this estimate is, let us consider an example.

We take a sequence of numbers M_n:

$$M_n = n![\ln(n+e)]^{(1+\beta)n}, \quad \beta > 0, \quad n \geqslant 0.$$

Let f be an arbitrary functions from the class $C_I^0(M_n)$, which is obviously non-analytic, and $f(x) \not\equiv 0$. Then by the Taylor series we obtain

$$|f(x)| \leqslant \frac{1}{\sup\limits_{n\geqslant 0} \frac{n!}{M_n x^n}} = \frac{1}{H_1(x)}, \tag{3.2}$$

where

$$H_1(x) \asymp \exp\exp\left[c_1\left(\frac{1}{x}\right)^{\frac{1}{1+\beta}}\right], \quad 0 < x \leqslant 1,$$

c_1 is a positive constant independent of f; we shall write $H_1 \asymp H_2$ if there exist $a_1 > 0$, $a_2 > 0$ such that $a_1 H_1(x) \leqslant H_2(x) \leqslant a_2 H_1(x)$.

Taking into consideration a rapid growth of the function $H_1(x)$ as $x \to 0$, we rewrite estimate (3.2) as

$$\ln\ln\frac{1}{|f(x)|} \geqslant c_2\left(\frac{1}{x}\right)^{\frac{1}{1+\beta}}, \tag{3.3}$$

where $0 < c_2 < c_1$ and c_2 is also independent of f; in fact, c_2 depends only on the sequence $\{M_n\}$.

The non-quasianalyticity of the class $C_I^N(M_n)$ is easily implied by the condition

$$\sum_{n=0}^{\infty}\frac{M_n}{M_{n+1}} < \infty,$$

and also by

$$\int_0^1 \ln^+ \ln H_1(x)dx < \infty. \tag{3.4}$$

But for $\beta = 0$ integral (3.4), as well as the series, converges and the class $C_I^N(M_n)$ becomes non-quasianalytic, what has to be expected. This hints that estimate (3.1) is rather sharp.

However, once we employ estimate (2.5) by T. Bang, we can obtain even sharper estimate but for a fixed function f, see [16]: there exists $x_0 = x_0(f)$ such that for all x, $0 < x < x_0(f)$ and some $c = c(f) > 0$

$$\ln\ln\frac{1}{|f(x)|} \geqslant c\left(\frac{1}{x}\right)^{\frac{1}{\beta}}. \tag{3.5}$$

There arises a natural question: which of estimates (3.3) and (3.5) indeed reflects the behavior of the extremal function $J_M(x)$? In [6] there was made an unsuccessful attempt to answer this question, see [13].

Let $\{M_n\}$ be a regular sequence and H_0 be a corrected associated weight, that is,

$$H_0(y) = \sup_{n\geqslant 0}\frac{n!}{M_n y^{n+1}}.$$

Then, as it is known,

$$M_n = \sup_{y>0}\frac{n!}{H_0(y)y^{n+1}}.$$

We also introduce a function

$$H(y) = \sum_{n=0}^{\infty}\frac{n!}{M_n y^{n+1}}. \tag{3.6}$$

Then a criterion of the quasianalyticity of the class $C_I^N(M_n)$ can be written as

$$\int_0^d \ln h(t)dt < \infty, \tag{3.7}$$

where $h(t) = \ln H(t)$ and $d > 0$ is such that $h(d) = 1$. This criterion is equivalent to the convergence of the Lebesgue-Stieltjes integral, see [13]:

$$-\int_0^d t\psi'(t)dt, \quad \psi(t) = \ln h(t).$$

As in work [13], by $\theta = \theta(y)$ we denote the inverse function for

$$y = -\int_0^\theta t\psi'(t)\,dt.$$

In [6] the following result was obtained.

Theorem 3.1. *Let $t|\psi'(t)| \to \infty$ as $t \to 0$. Then the following statements hold:*
1) if integral (3.7) diverges, then $J_M(x) \equiv 0$.
2) if integral (3.7) converges, then there exists a function $f \in C_I^0(M_n)$, for which

$$H_0\left(q_1\theta(x)\right) \leqslant f(x) \leqslant H_0\left(q_2\theta(x)\right),$$

where $0 < q_1 < q_2 < \infty$.

It is easy to see that the sequence $M_n = n![\ln(n+e)]^{(1+\beta)n}$, $\beta > 0$, $n \geqslant 0$, satisfies

$$h(y) \asymp y^{-\frac{1}{1+\beta}}, \qquad \theta(y) \asymp y^{\frac{1+\beta}{\beta}}.$$

Hence, by Theorem 3.1, there exists a function $f \in C_I^0(M_n)$ such that

$$c_f x^{-\frac{1}{\beta}} \leqslant \ln\ln\frac{1}{|f(x)|} \leqslant C_f x^{-\frac{1}{\beta}}, \qquad 0 < x \leqslant 1.$$

In [6] an appropriate estimate is given, where instead of $\frac{1}{|f(x)|}$ the quantity $\delta_{\{M_n\}}(s) = \sup\{|g(s)|,\ g \in C_I^0(M_n)\}$ is involved and this is wrong, see [13].

Thus, asymptotic estimate (3.5) by T. Bang for each fixed function f is better than estimate (3.3). However, as it was shown in [13], it does not reflect a real behavior of the quantity $J_M(x)$.

In [13] the following theorem was proved.

Theorem 3.2. *Let $\{M_n\}$ be a regular sequence. If the function H defined by formula (3.6) satisfies bilogarithmic condition (3.7), then the extremal function $J_M(x)$ satisfies the estimates*

$$\frac{1}{q_1 H\left(\frac{x}{2}\right)} \leqslant J_M(x) \leqslant \frac{1}{H\left(2q_2 x\right)}, \quad 0 < x \leqslant 1, \tag{3.8}$$

where q_1 is some positive constant depending only on the functions H, that is, on the numbers M_n, while

$$q_2 = \sup_{n\geqslant 1}\sqrt[n]{\frac{m_n}{m_{n-1}}} < \infty, \quad m_n = \frac{M_n}{n!}.$$

Now we are in position to formulate the main result, which essentially specifies estimates (3.2).

Theorem 3.3. *Let $\{M_n\}$ be a regular sequence, H_0 be an associated weight treated in the following sense:*

$$H_0(t) = \sup_{n\geqslant 0} \frac{n!}{M_n t^{n+1}}, \qquad t > 0.$$

If for this weight the bilogarithmic integral converges, which is equivalent to condition (3.7), then the extremal function $J_M(x)$ satisfies the estimates

$$\frac{1}{KH_0(x)} \leqslant J_M(x) \leqslant \frac{1}{xH_0(x)},$$

where $K = (1+L)C$ and C, $0 < C < \infty$, is a constant independent of x[1] *and*

$$L = \sup_{n\geqslant 1} \frac{nM_{n-1}}{M_n}.$$

Thus, as $x \to 0$,

$$\ln J_M(x) = -\ln H_0(x) + O\left(\ln \frac{1}{x}\right) = -(1 + O(1)) \ln H_0(x).$$

4. Proof of Theorem 3.3

Let $\{M_n\}$ be a regular sequence and H_0 be an associated weight introduced above.

If integral (3.7) converges, the integral

$$\int_0^{d_0} \ln \ln H_0(t) dt < \infty, \quad H_0(d_0) = e, \tag{4.1}$$

converges as well. Hence, there exists a function $f \in C_I^0(M_n)$, $f(x) \not\equiv 0$. Then by the Taylor formula we obtain

$$|f(x)| \leqslant \inf_{n\geqslant 0} \frac{M_n x^n}{n!} = \frac{1}{\sup\limits_{n\geqslant 0} \frac{n!}{M_n x^n}} = \frac{1}{xH_0(x)}, \quad x \in I.$$

This yields

$$J_M(x) \leqslant \frac{1}{xH_0(x)}, \quad x \in I,$$

and we obtain an upper bound for $J_M(x)$.

In order to estimate $J_M(x)$ from below, we consider a normed space $F_I(H_0)$ of the functions analytic outside the segment $I = [0,1]$ and satisfying the estimate

$$|f(z)| \leqslant C_f H_0(\operatorname{dist}(z, I)), \quad z \in \mathbb{C} \setminus I,$$

with the norm

$$\|f\|_0 = \sup_{\operatorname{Im} z \neq 0} \frac{|f(z)|}{H_0(|\operatorname{Im} z|)}.$$

By $F_I^0(H_0)$ we denote the unit ball in $F_I(H_0)$.

Instead of I we can consider an arbitrary closed set $E \subset \mathbb{R}$, see [4]. This is why, letting $E = \{0\}$, in the space $F_{\{0\}}(H_0)$ we consider a linear functional $\langle G, f\rangle = f(\delta)$, $\delta \in (0,1]$. Then we obviously have: $|\langle G, f\rangle| \leqslant C_f H_0(\delta)$. Since integral (4.1) converges, by N. Levinson theorem, the set of functions $F_{\{0\}}^0$ is normal. This means that if $C_f^0 = \inf C_f$, then $\sup\limits_{f\in F_{\{0\}}^0(H_0)} C_f^0 = C < \infty$. Therefore, $\|G\| \leqslant CH_0(\delta)$, where a positive constant C is independent of δ. Since $F_{\{0\}}(H_0) \subset F_I(H_0)$, by the Hahn-Banach theorem, the functional G admits a continuation on

[1] C is the extremal (best possible) constant uniquely determined by the family of the functions $F_{\{0\}}^0(H_0)$, see the proof of Theorem 3.3.

the entire space $F_I(H_0)$ with the preservation of the norm. Keeping the same notation for this continuation, we consider a function

$$\eta(t) = \left\langle G, \frac{1}{z-t} \right\rangle, \quad t \in I.$$

Then $\eta \in C^\infty(I)$ and

$$\left|\eta^{(n)}(t)\right| = \left|\left\langle G, \frac{n!}{(z-t)^{n+1}} \right\rangle\right| \leqslant CH_0(\delta) \left\|n!(z-t)^{-n-1}\right\| = CH_0(\delta)M_n, \qquad n \geqslant 0,$$

where

$$M_n = \sup_{y>0} \frac{n!}{H_0(y)y^{n+1}}.$$

We also observe that

$$\eta^{(n)}(0) = \left\langle G, \frac{n!}{z^{n+1}} \right\rangle = \frac{n!}{\delta^{n+1}}, \qquad n \geqslant 0.$$

Now we choose the function g letting $g(t) = 1 + \eta(t)(t-\delta)$. Since

$$g^{(n)}(t) = \eta^{(n)}(t)(t-\delta) + n\eta^{n-1}(t), \qquad n \geqslant 1,$$

we obtain

$$g^{(n)}(0) = 0, \quad n \geqslant 0, \qquad \left|g^{(n)}(t)\right| \leqslant CH_0(\delta)\left(M_n + nM_{n-1}\right), \quad n \geqslant 1.$$

But the sequence $\{M_n\}$ is logarithmically convex, that is, $M_n^2 \leqslant M_{n-1}M_{n+1}$, $n \geqslant 1$. Hence, the sequence $\left\{\frac{M_{n-1}}{M_n}\right\}$ is non-increasing. Then it follows from the convergence of the series

$$\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n}$$

that $nM_{n-1} = o(M_n)$ as $n \to \infty$. Hence,

$$\sup_{n\geqslant 1} \frac{nM_{n-1}}{M_n} = L < \infty,$$

and therefore

$$\sup_I \left|g^{(n)}(t)\right| \leqslant C(1+L)M_nH_0(\delta), \quad \delta \in (0,1], \quad n \geqslant 0.$$

Thus, we finally obtain that

1) $g^{(n)}(0) = 0$, $n \geqslant 0$;
2) $\left\|g^{(n)}\right\| \leqslant KH_0(\delta)M_n$, $n \geqslant 0$, $K = (1+L)C$;
3) $g(\delta) = 1$.

Therefore, the function

$$\psi(t) = \frac{g(t)}{KH_0(\delta)}$$

belongs to the class $C_I^0(M_n)$. It remains to observe that

$$J_M(\delta) \geqslant \frac{1}{KH_0(\delta)}, \quad \delta \in (0,1], \quad K = (1+L)C.$$

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