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NONLOCAL PROBLEMS WITH DISPLACEMENT FOR MATCHING TWO SECOND ORDER HYPERBOLIC EQUATIONS

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Abstract. In this work we study two nonlocal problems with a displacement for two second order hyperbolic equations being a wave equation in one part of the domain and a degenerate hyperbolic equation of the first kind in the other. As a nonlocal boundary condition, in the considered problems we use a linear combination with variable coefficients of the first derivative and fractional derivative (in the Riemann-Liouville sense) of the unknown function on one of the characteristics and one the line of the type changing. By using the methods of integral equations, the solvability issue of the first problem is equivalently reduced to the solvability of a Volterra equation of the second kind with a weak singularity, while the solvability of the second problem is reduced to the solvability of aFredholm equation of the second kind with a weak singularity. For the first problem we prove a uniform convergence of the resolvent for the kernel of the obtained Volterra equation of the second kind and that it solution belongs to a needed class. For the second problem we find sufficient conditions for the given functions, which ensure the existence of the unique solution of the Fredholm equation of the second kind with a weak singularity in the needed class. In some particular cases the solutions of the problems are written explicitly.

Keywords: wave equation, degenerate hyperbolic equation of the first kind, Volterra integral equation, Fredholm integral equation, Trikomi method, method of integral equations, methods of fractional calculus.

Mathematics Subject Classification: 35L53, 35L80

1. Introduction. Formulation of problem

On the Euclidean plane of points $(x,\, y)$ we consider an equation

$$0 = \begin{cases} (-y)^m u_{xx} - u_{yy} + \lambda(-y)^{\frac{m-2}{2}} u_x, & y < 0, \\ u_{xx} - u_{yy} + f, & y > 0, \end{cases} \tag{1.1}$$

where m, λ are given numbers and $m > 0$, $|\lambda| \leqslant \frac{m}{2}$; $f = f\,(x,y)$ is a given function and $u = u\,(x,y)$ is a sought function.

As $y < 0$ equation (1.1) coincides with the equation

$$(-y)^m u_{xx} - u_{yy} + \lambda(-y)^{\frac{m-2}{2}} u_x = 0, \tag{1.2}$$

while for $y > 0$ equation (1.1) is an inhomogeneous wave equation

$$u_{xx} - u_{yy} + f(x,y) = 0. \tag{1.3}$$

Equation (1.2) belongs to an important class of degenerate hyperbolic equations of first kind [1]. An important property of equation (1.2) is the fact that as $|\lambda| \leqslant \frac{m}{2}$, the Cauchy problem

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for it is well-defined in the usual formulation with the data on the curve of the parabolic degeneration $y=0$ despite the Protter condition is violated [2]. As $m=2$, equation (1.2) becomes the Bitsadze-Lykov equation [3], [4], [5], while for $\lambda=0$ equation (1.2) becomes Gellerstedt equation, which, as it was shown in monograph [6], arises in the problem on determining the shape of a dam cut. A particular case of equation (1.2) is also the Tricomi equation, which has applications in the theory of transonic gas dynamics and aerodynamics [7], [8], [9].

Equation (1.1) is considered in the domain $\Omega=\Omega_1\cup\Omega_2\cup I$, where Ω_1 is the domain enveloped by the characteristics

$$\sigma_1=AC: x-\frac{2}{m+2}(-y)^{(m+2)/2}=0 \quad \text{and} \quad \sigma_2=CB: x+\frac{2}{m+2}(-y)^{(m+2)/2}=r$$

of equation (1.2) leaving the point

$$C=(r/2, y_C), \qquad y_C=-\left[\frac{r(m+2)}{4}\right]^{\frac{2}{m+2}},$$

passing respectively through the points $A=(0,0)$ and $B=(r,0)$, and by the segment $I=AB$ on the straight line $y=0$. The symbol Ω_2 denotes the domain enveloped by the characteristics $\sigma_3=AD: x-y=0$, $\sigma_4=BD: x+y=r$ of equation (1.3) leaving the points A and B and intersecting at the point $D=\left(\frac{r}{2},\frac{r}{2}\right)$ and by the segment $I=AB$.

In what follows we make use of the following notations:

$$\varepsilon_1=\frac{m-2\lambda}{2(m+2)}, \qquad \varepsilon_2=\frac{m+2\lambda}{2(m+2)}, \qquad \varepsilon=\varepsilon_1+\varepsilon_2=\frac{m}{m+2}, \qquad \gamma_1=\frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon_2)},$$

$$\gamma_2=\frac{\Gamma(1-\varepsilon)(2-2\varepsilon)^{\varepsilon-1}}{\Gamma(1-\varepsilon_1)}, \qquad a(x)=\frac{\beta(x)+\gamma_1\alpha(x)}{\gamma(x)-\gamma_2\alpha(x)}, \qquad b(x)=\frac{1}{a(x)}=\frac{\gamma(x)-\gamma_2\alpha(x)}{\beta(x)+\gamma_1\alpha(x)},$$

$$\theta_{00}(x)=\left(\frac{x}{2},-(2-2\varepsilon)^{\varepsilon-1}x^{1-\varepsilon}\right), \qquad \theta_{01}(x)=\left(\frac{x}{2},\frac{x}{2}\right), \qquad \theta_{r1}(x)=\left(\frac{r+x}{2},\frac{r-x}{2}\right)$$

are the affixes of the intersection points of the characteristics leaving the points $(x,0)$ with the characteristics AC of equation (1.2) and the characteristics AD and BD of equation (1.3), respectively. The functions

$$B(p,q)=\int\limits_0^1 t^{p-1}(1-t)^{q-1}dt, \quad \Gamma(x)=\int\limits_0^\infty \exp(-t)t^{x-1}dt, \quad B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

are the Euler integrals of the first and second kind and the relation between them;

$$E_\rho(z,\mu)=\sum_{n=0}^\infty \frac{z^n}{\Gamma\left(\mu+n\rho^{-1}\right)}$$

is a function of Mittag-Leffler type [10], while as $\mu=1$ it coincides with the Mittag-Leffler function $E_\rho(z,1)=E_{1/\rho}(z)$. The formula

$$D_{cx}^{\alpha}\varphi(t)=\begin{cases} \frac{\operatorname{sgn}(x-c)}{\Gamma(-\alpha)}\int\limits_c^x \frac{\varphi(t)\,dt}{|x-t|^{1+\alpha}}, & \alpha<0,\\ \operatorname{sgn}^{[\alpha]+1}(x-c)\frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}}D_{cx}^{\alpha-[\alpha]-1}\varphi(t), & \alpha>0,\end{cases}$$

defines an operator of Riemann-Liouville fractional integro-differentiation of order $|\alpha|$, where $[\alpha]$ is the integer part of a number α [5], [11].

A *regular in the domain* Ω *solution* of equation (1.1) is a function $u=u\,(x,\,y)$ from the class $C\left(\bar\Omega\right)\cap C^1\left(\Omega\right)\cap C^2\left(\Omega_1\cup\Omega_2\right)$, the substitution of which into equation (1.1) transforms the latter into an identity.

Problem 1. *Find a regular in the domain Ω solution to equation (1.1) obeying the conditions*

$$u\left[\theta_{01}\left(x\right)\right]=\psi_1(x),\qquad 0\leqslant x\leqslant r, \tag{1.4}$$

$$\alpha(x)x^{\varepsilon_1}D_{0x}^{1-\varepsilon_2}u\left[\theta_{00}(t)\right]+\beta(x)D_{0x}^{1-\varepsilon}u(t,0)+\gamma(x)u_y(x,0)=\psi_2(x),\qquad 0<x<r, \tag{1.5}$$

where $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\psi_1(x)$, $\psi_2(x)$ are given function defined on the segment $[0,r]$ and $\alpha^2(x)+\beta^2(x)+\gamma^2(x)\neq 0$ as $x\in[0,r]$.

Problem 2. *Find a regular in the domain Ω solution of equation (1.1) in the class $u_x(x,0)$, $D_{0x}^{1-\varepsilon}u(t,0)\in L_1\left(0,r\right)$ satisfying nonlocal condition (1.5) and the boundary condition*

$$u\left[\theta_{r1}\left(x\right)\right]=\psi_1(x),\quad 0\leqslant x\leqslant r, \tag{1.6}$$

where $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\psi_1(x)$, $\psi_2(x)$ are given functions defined on the segment $[0,r]$ and, as in Problem 1, $\alpha^2(x)+\beta^2(x)+\gamma^2(x)\neq 0$ as $x\in[0,r]$.

Goursat problem for a degenerating inside a domain hyperbolic equation was studied earlier in works [12], [13]. In work [12] a uniqueness criterion was studied for the solution of the Goursat problem for equation of form (1.2), while in [13] there was explicitly written a solution for the Goursat problem for a degenerating inside a domain model equation. In work [14] the Dirichlet problem for a degenerating inside a domain hyperbolic equation was considered. In works [15]–[17], there were studied boundary value problems for degenerating hyperbolic equations in a characteristic quadrilateral with the data on opposite characteristics. Problems with shift for a degenerating inside a domain hyperbolic equations were studied in works [18]–[21]. Problems with shift for a degenerating hyperbolic equation of first kind of form (1.2) as a generalization of the Dirichlet and Neumann Darboux problems were studied in (1.1).

In the present work we study two nonlocal Problems 1 and 2, which belong to the class of Zhegalov-Nakhushev boundary value problems with displacement [23]–[26] and are generalizations of the Goursat problem and the problems with data on opposite characteristics for equations of form (1.1). We find sufficient conditions for the given functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\psi_1(x)$, $\psi_2(x)$ and $f(x,y)$, under which there exists a unique regular in the considered domain solution to Problems 1 and 2. In a particular case, when $a(x)=\frac{\beta(x)+\gamma_1\alpha(x)}{\gamma(x)-\gamma_2\alpha(x)}=a=const$, the solutions of Problems 1 and 2 are written explicitly.

2. Study of Problem 1

The following theorem is true.

Theorem 2.1. *Let the given functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\psi_1(x)$, $\psi_2(x)$, $f(x,y)$ be such that*

$$\alpha(x),\beta(x),\gamma(x)\in C^1[0,r]\cap C^2(0,r), \tag{2.1}$$

$$\psi_1(x),\psi_2(x)\in C[0,r]\cap C^2(0,r), \tag{2.2}$$

$$f(x,y)\in C^1(\overline{\Omega_2}), \tag{2.3}$$

and one of the following conditions holds true: either

$$\gamma(x)-\gamma_2\alpha(x)\neq 0\qquad \textit{for all}\quad x\in[0,r]; \tag{2.4}$$

or

$$\gamma(x)-\gamma_2\alpha(x)\equiv 0,\quad \beta(x)+\gamma_1\alpha(x)\neq 0\quad \textit{for all}\quad x\in[0,r]. \tag{2.5}$$

Then there exists a unique regular in the domain Ω solution to Problem 1.

Proof. We prove by using the method of integral equations. We introduce the notations:

$$u\left(x,0\right)=\tau\left(x\right),\quad 0\leqslant x\leqslant r\qquad\text{and}\qquad u_y\left(x,\,0\right)=\nu\left(x\right),\quad 0<x<r. \tag{2.6}$$

Let us find fundamental relations between the sought functions $\tau\left(x\right)$ and $\nu\left(x\right)$ brought from the corresponding parts Ω_1 and Ω_2 of the domain Ω on the line $y=0$. First we consider the case $|\lambda| < \frac{m}{2}$. In this case a regular in the domain Ω_1 solution to Problem (2.6) for equation (1.2) is written by the formula [27]:

$$\begin{aligned} u(x,y) =& \frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon_1)\,\Gamma(\varepsilon_2)} \int\limits_0^1 \tau\left(x + (1-\varepsilon)(-y)^{\frac{1}{1-\varepsilon}}(2t-1)\right) t^{\varepsilon_2 - 1}(1-t)^{\varepsilon_1 - 1}\, dt \\ &+ \frac{\Gamma(2-\varepsilon)\, y}{\Gamma(1-\varepsilon_1)\,\Gamma(1-\varepsilon_2)} \int\limits_0^1 \nu\left(x + (1-\varepsilon)(-y)^{\frac{1}{1-\varepsilon}}(2t-1)\right) t^{-\varepsilon_1}\,(1-t)^{-\varepsilon_2}\, dt, \end{aligned} \tag{2.7}$$

where $\tau(x) \in C[0,r] \cap C^2(0,r)$, $\nu(x) \in C^1(0,r) \cap L_1(0,r)$.

By (2.7) we find:

$$\begin{aligned} u[\theta_{00}(x)] =& u\left(\frac{x}{2}, -(2-2\varepsilon)^{\varepsilon-1}x^{1-\varepsilon}\right) = \frac{1}{B(\varepsilon_1,\varepsilon_2)} \int\limits_0^1 \tau(xt)\, t^{\varepsilon_2-1}\,(1-t)^{\varepsilon_1-1}\, dt \\ &- \frac{(2-2\varepsilon)^{\varepsilon-1}x^{1-\varepsilon}}{B(1-\varepsilon_1, 1-\varepsilon_2)} \int\limits_0^1 \nu(xt)\, t^{-\varepsilon_1}\,(1-t)^{-\varepsilon_2}\, dt. \end{aligned}$$

Introducing a new variable of integration $z = xt$, we rewrite this identity as

$$\begin{aligned} u[\theta_{00}(x)] =& \frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon_1)\,\Gamma(\varepsilon_2)} x^{1-\varepsilon} \int\limits_0^x \frac{\tau(z)\, z^{\varepsilon_2-1}}{(x-z)^{1-\varepsilon_1}}\, dz \\ &- \frac{\Gamma(2-\varepsilon)}{\Gamma(1-\varepsilon_1)\,\Gamma(1-\varepsilon_2)} (2-2\varepsilon)^{\varepsilon-1} \int\limits_0^x \frac{z^{-\varepsilon_1}\nu(z)}{(x-z)^{\varepsilon_2}}\, dz. \end{aligned}$$

In terms of the operator $D^{\alpha}_{cx}\varphi(t)$ of the Riemann-Liouville fractional integro-differentiation, we rewrite the latter identity as

$$u[\theta_{00}(x)] = \frac{\Gamma(\varepsilon)\, x^{1-\varepsilon}}{\Gamma(\varepsilon_2)} D^{-\varepsilon_1}_{0x}\left[t^{\varepsilon_2-1}\tau(t)\right] - \frac{(2-2\varepsilon)^{\varepsilon-1}\,\Gamma(1-\varepsilon)}{\Gamma(1-\varepsilon_1)} D^{\varepsilon_2-1}_{0x}\left[t^{-\varepsilon_1}\nu(t)\right]. \tag{2.8}$$

Using the following composition law for the operators of fractional differentiation and integration [6], [11]

$$D^{\alpha}_{0x}\, t^{\alpha+\beta}\, D^{\beta}_{0t}\, g(s) = x^{\beta}\, D^{\alpha+\beta}_{0x}\, t^{\alpha} g(t), \quad 0 < \alpha \leqslant 1, \quad \beta < 0,$$

by (2.8) we find:

$$x^{\varepsilon_1} D^{1-\varepsilon_2}_{0x} u[\theta_{00}(t)] = \gamma_1 D^{1-\varepsilon}_{0x}\tau(t) - \gamma_2\nu(x). \tag{2.9}$$

In view (2.9), condition (1.5) is rewritten as

$$[\gamma(x) - \gamma_2\alpha(x)]\,\nu(x) + [\beta(x) + \gamma_1\alpha(x)]\, D^{1-\varepsilon}_{0x}\tau(t) = \psi_2(x). \tag{2.10}$$

The obtained relation is exactly the fundamental relation between the sought functions $\tau(x)$ and $\nu(x)$ brought from the domain Ω_1 to the straight line $y = 0$.

Let us find a fundamental relation between the functions $\tau(x)$ and $\nu(x)$ brought from the domain Ω_1 on the straight line $y = 0$. In order to do this, we employ a representation for a regular in the domain Ω_1 solution to problem (2.6) for equation (1.3), which is written by means of the D'Alambert formula [28]:

$$u(x,y) = \frac{\tau(x+y) + \tau(x-y)}{2} + \frac{1}{2}\int\limits_{x-y}^{x+y} \nu(t)\, dt + \frac{1}{2}\int\limits_0^y \int\limits_{x-y+t}^{x+y-t} f(s,t)\, ds dt, \tag{2.11}$$

where $\tau(x) \in C[0,r] \cap C^2(0,r)$, $\nu(x) \in C^1(0,r) \cap L_1(0,r)$. Substituting this representation into condition (1.4), we get

$$u\left[\theta_{01}(x)\right] = u\left(\frac{x}{2}, \frac{x}{2}\right) = \frac{\tau(x) + \tau(0)}{2} + \frac{1}{2}\int_0^x \nu(t)\,dt + \frac{1}{2}\int_0^{\frac{x}{2}}\int_t^{x-t} f(s,t)\,dsdt = \psi_1(x).$$

Differentiating this identity, we arrive at the relation

$$\nu(x) = 2\psi_1'(x) - \tau'(x) - \int_0^{\frac{x}{2}} f(x-t,t)\,dt. \tag{2.12}$$

Relation (2.12) is a fundamental relation between the functions $\tau(x)$ and $\nu(x)$ brought from the domain Ω_2 to the straight line $y = 0$.

Excluding the sought function $\nu(x)$ from (2.10) and (2.12), in view of the matching condition $\tau(0) = \psi_1(0)$ and condition (2.4) of Theorem 2.1, for the function $\tau(x)$ we obtain the following problem for the first order ordinary differential equation involving a fractional derivative in the lower order terms:

$$\tau'(x) - a(x)\,D_{0x}^{1-\varepsilon}\tau(t) = 2\psi_1'(x) - \frac{\psi_2(x)}{\gamma(x) - \gamma_2\alpha(x)} - \int_0^{\frac{x}{2}} f(x-t,t)\,dt, \quad 0 < x < r, \tag{2.13}$$

$$\tau(0) = \psi_1(0). \tag{2.14}$$

Integrating equation (2.13) in x from 0 to x, we arrive at an integral equation associated with problem (2.13), (2.14):

$$\tau(x) - \frac{1}{\Gamma(\varepsilon)}\int_0^x K(x,t)\tau(t)dt = F_1(x), \tag{2.15}$$

where

$$K(x,t) = \frac{a(x)}{(x-t)^{1-\varepsilon}} - \int_t^x \frac{a'(s)}{(s-t)^{1-\varepsilon}}\,ds,$$

$$F_1(x) = 2\psi_1(x) - \psi_1(0) - \int_0^x \frac{\psi_2(t)}{\gamma(t) - \gamma_2\alpha(t)}dt - \int_0^x\int_0^{\frac{t}{2}} f(t-s,s)ds\,dt.$$

It follows from (2.1), (2.2), (2.3) that equation (2.15) is an integral Volterra equation of the second kind with the kernel $K(x,t) \in L_1([0,r] \times [0,r])$ having a weak singularity as $x = t$ and with the right hand side $F_1(x) \in C[0,r] \cap C^2(0,r)$. In accordance with the general theory of Volterra integral equations, equation (2.15) is uniquely solvable and is written by the formula

$$\tau(x) = F_1(x) + \int_0^x R(x,t)F_1(t)dt, \tag{2.16}$$

where

$$R(x,t) = \sum_{n=0}^{\infty} \frac{K_n(x,t)}{\Gamma^{n+1}(\varepsilon)}$$

is the resolvent of the kernel $K(x,t)$;

$$K_0(x,t) = K(x,t), \quad K_{n+1}(x,t) = \int\limits_t^x K(x,s)K_n(s,t)\,ds.$$

Let us show that the resolvent $R(x,t)$, as well as the kernel $K(x,t)$ of equation (2.15) satisfies $R(x,t) \in L_1([0,r]\times[0,r])$ and has a weak singularity at $x=t$, while a solution $\tau(x)$ to this equation as well as its right hand side $F_1(x)$ satisfies $\tau(x) \in C[0,r] \cap C^2(0,r)$.

Indeed, taking into consideration that $a(x) \in C^1[0,r] \cap C^2(0,r)$, we find an estimate for the iterated kernels $\frac{K_n(x,t)}{\Gamma^{n+1}(\varepsilon)}$. Let $|a(x)| \leqslant M_1$ and $|a'(x)| \leqslant M_2$ for all $x \in [0,r]$. Then the first iterated kernel $\frac{K_0(x,t)}{\Gamma(\varepsilon)}$ obeys the estimate

$$\begin{aligned}\frac{1}{\Gamma(\varepsilon)}|K_0(x,t)| = \frac{1}{\Gamma(\varepsilon)}|K(x,t)| = \frac{1}{\Gamma(\varepsilon)}\left|\frac{a(x)}{(x-t)^{1-\varepsilon}} - \int\limits_t^x \frac{a'(s)}{(s-t)^{1-\varepsilon}}\,ds\right| \\ \leqslant \frac{M_1(x-t)^{\varepsilon-1}}{\Gamma(\varepsilon)} + \frac{M_2(x-t)^{\varepsilon}}{\Gamma(\varepsilon+1)}.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{\Gamma^2(\varepsilon)}|K_1(x,t)| =& \frac{1}{\Gamma^2(\varepsilon)}\left|\int\limits_t^x K(x,s)K_0(s,t)ds\right| \\ \leqslant& \int\limits_t^x \left(\frac{M_1}{\Gamma(\varepsilon)(x-s)^{1-\varepsilon}} + \frac{M_2(x-s)^{\varepsilon}}{\Gamma(\varepsilon+1)}\right)\left(\frac{M_1}{\Gamma(\varepsilon)(s-t)^{1-\varepsilon}} + \frac{M_2(s-t)^{\varepsilon}}{\Gamma(\varepsilon+1)}\right) ds \\ =& \frac{M_1^2}{\Gamma^2(\varepsilon)}(x-t)^{2\varepsilon-1}\int\limits_0^1 y^{\varepsilon-1}(1-y)^{\varepsilon-1}\,dy \\ &+ \frac{M_1M_2}{\varepsilon\Gamma^2(\varepsilon)}(x-t)^{2\varepsilon}\int\limits_0^1 y^{\varepsilon}(1-y)^{\varepsilon-1}\,dy + \frac{M_1M_2}{\varepsilon\Gamma^2(\varepsilon)}(x-t)^{2\varepsilon}\int\limits_0^1 y^{\varepsilon-1}(1-y)^{\varepsilon}\,dy \\ &+ \frac{M_2^2}{\Gamma^2(\varepsilon+1)}(x-t)^{2\varepsilon+1}\int\limits_0^1 y^{\varepsilon}(1-y)^{\varepsilon}\,dy \\ =& \frac{M_1^2(x-t)^{2\varepsilon-1}}{\Gamma(2\varepsilon)} \\ &+ \frac{2M_1M_2(x-t)^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + \frac{M_2^2(x-t)^{2\varepsilon+1}}{\Gamma(2\varepsilon+2)}.\end{aligned}$$

In the same way we obtain

$$\begin{aligned}\frac{1}{\Gamma^3(\varepsilon)}|K_2(x,t)| =\frac{1}{\Gamma^3(\varepsilon)}\left|\int\limits_t^x K(x,s)K_1(s,t)ds\right| \leqslant \frac{M_1^3(x-t)^{3\varepsilon-1}}{\Gamma(3\varepsilon)} + \frac{3M_1^2M_2(x-t)^{3\varepsilon}}{\Gamma(3\varepsilon+1)} \\ + \frac{3M_1M_2^2(x-t)^{3\varepsilon+1}}{\Gamma(3\varepsilon+2)} + \frac{M_2^3(x-t)^{3\varepsilon+2}}{\Gamma(3\varepsilon+3)}.\end{aligned}$$

It is clear that

$$\frac{1}{\Gamma^n(\varepsilon)}|K_{n-1}(x,t)| \leqslant \sum_{k=0}^{n} \frac{C_n^k\, M_1^{n-k}\, M_2^k\,(x-t)^{n\varepsilon+k-1}}{\Gamma(n\varepsilon+k)}, \tag{2.17}$$

where $C_n^k = \frac{n!}{k!\,(n-k)!}$ is the number of k-combinations of n elements. Observing that $\Gamma(n\varepsilon+k) > \Gamma(n\varepsilon)$, by (2.17) we get the estimate

$$\begin{aligned}\frac{1}{\Gamma^n(\varepsilon)}|K_{n-1}(x,t)| &< \frac{1}{\Gamma(n\varepsilon)} \sum_{k=0}^{n} C_n^k\, M_1^{n-k}\, M_2^k\,(x-t)^{n\varepsilon+k-1} \\ &= \frac{1}{\Gamma(n\varepsilon)}\,(M_1+M_2(x-t))^n\,(x-t)^{n\varepsilon-1}.\end{aligned} \tag{2.18}$$

For sufficiently large n, the exponent $n\varepsilon-1$ of $(x-t)$ in formula (2.18) is positive. At the same time, the difference $(x-t)$ can be replaced by a large scalar quantity r. Thus, for the resolvent $R(x,t)$ of the kernel $K(x,t)$ we have the estimate

$$|R(x,t)| = \left|\sum_{n=1}^{\infty} \frac{K_{n-1}(x,t)}{\Gamma^n(\varepsilon)}\right| < \sum_{n=1}^{\infty} \frac{(M_1+M_2r)^n\, r^{n\varepsilon-1}}{\Gamma(n\varepsilon)}. \tag{2.19}$$

Employing the Striling's formula for the Gamma function

$$\Gamma(n) = \frac{1}{\sqrt{2\pi n}} n^n e^{-n+\frac{\eta}{12n}}, \quad 0<\eta<1,$$

and the Cauchy test for the convergence of the scalar series, we easily confirm that the series in the right hand side of identity (2.19) converges. Thus, the series for the resolvent $R(x,t)$ of the kernel $K(x,t)$ converges absolutely and uniformly and this implies the continuity of the resolvent of the kernel for each $0<\varepsilon<1$ and each $x \neq t \in [0,r]$ as well as a weak singularity at $x=t$.

Representation (2.18) and estimate (2.19) for a continuous right hand side $F_1(x) \in C[0,r]$ implies the following estimate for the solution:

$$|\tau(x)| = \left|F_1(x) + \frac{1}{\Gamma(\varepsilon)} \int_0^x R(x,t) F_1(t)dt\right| < M_3\left[1 + \sum_{n=1}^{\infty} \frac{(M_1+M_2r)^n\, r^{n\varepsilon}}{\Gamma(n\varepsilon)}\right], \tag{2.20}$$

where $M_3 = \max\limits_{x\in[0,r]} |F_1(x)|$.

The convergence of the majorizing scalar series in the right hand side of inequality (2.20)) by the Weierstrass M-test implies an absolute and uniform convergence of the solution. This yields the continuity of the limiting function $\tau(x) \in C[0,r]$.

Let $F_1(x) \in C^2(0,r)$. In this case by a twice integration by parts in the integral in the right hand side of representation (2.16) we easily confirm that $\tau(x) \in C^2(0,r)$, that is, the solution $\tau(x)$ of integral equation (2.15) as well as its right hand side $F_1(x)$ belong to the class $\tau(x) \in C[0,r] \cap C^2(0,r)$.

As $a(x) = a = const$, the solution of equation (2.15) can be written explicitly:

$$\tau(x) = F_1(x) + a\int_0^x (x-t)^{\varepsilon-1} E_{1/\varepsilon}[a(x-t)^\varepsilon;\varepsilon] F_1(t)dt. \tag{2.21}$$

If condition (2.5) is satisfied, then by system (2.10) and (2.12) we immediately find

$$\tau(x) = D_{0x}^{\varepsilon-1}\left[\frac{\psi_2(t)}{[\beta(t)+\gamma_1\alpha(t)]}\right],$$

$$\nu(x) = -D_{0x}^{\varepsilon}\left[\frac{\psi_2(t)}{[\beta(t)+\gamma_1\alpha(t)]}\right] + 2\psi_1'(x) - \int\limits_0^{\frac{x}{2}} f\,(x-t,t)\,dt.$$

As $\lambda = \pm\frac{m}{2}$, the sought function $\tau(x)$ is again determined by one of the formulas (2.16) or (2.21), but

$$\varepsilon_2 = 0, \qquad \varepsilon = \varepsilon_1 = \frac{m}{m+2}, \qquad \gamma_1 = 0, \qquad \gamma_2 = \frac{(2-2\varepsilon)^{\varepsilon}}{2} \qquad \text{as} \qquad \lambda = -\frac{m}{2}$$

and

$$\varepsilon_1 = 0, \qquad \varepsilon = \varepsilon_2 = \frac{m}{m+2}, \qquad \gamma_1 = 1, \qquad \gamma_2 = (2-2\varepsilon)^{\varepsilon-1}\Gamma(2-\varepsilon) \qquad \text{as} \qquad \lambda = \frac{m}{2}.$$

Once the function $\tau(x)$ is found, the second sought function $\nu(x)$ is determined by one of formulas (2.10) and (2.12). Then the regular in the domain Ω_1 solution of the studied problem is written by formula (2.7) or by one of the following formulas [27]:

$$\begin{aligned} u\,(x,y) =& \frac{2y}{m+2}\int\limits_0^1 \nu\left(x - \frac{2}{m+2}\,(-y)^{\frac{m+2}{2}}\,(2t-1)\right)(1-t)^{-\frac{m}{m+2}}\,dt \\ &+ \tau\left(x - \frac{2}{m+2}\,(-y)^{\frac{m+2}{2}}\right), \qquad \lambda = -\frac{m}{2}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} u\,(x,y) =& \frac{2y}{m+2}\int\limits_0^1 \nu\left[x + \frac{2}{m+2}\,(-y)^{\frac{m+2}{2}}\,(2t-1)\right](1-t)^{-\frac{m}{m+2}}\,dt \\ &+ \tau\left(x + \frac{2}{m+2}\,(-y)^{\frac{m+2}{2}}\right), \qquad \lambda = \frac{m}{2}, \end{aligned} \tag{2.23}$$

while in the domain Ω_2 the solution to the Cauchy problem for equation (1.3) is found by formula (2.11). The proof is complete. □

3. Study of Problem 2

We proceed to studying Problem 2. Writing condition (1.6) for (2.11), we get

$$u\left[\theta_{r1}\,(x)\right] = u\left(\frac{r+x}{2},\frac{r-x}{2}\right) = \frac{\tau\,(x)+\tau\,(r)}{2} + \frac{1}{2}\int\limits_x^r \nu\,(t)\,dt + \frac{1}{2}\int\limits_0^{\frac{r-x}{2}}\int\limits_{x+t}^{r-t} f\,(s,t)\,dsdt = \psi_1(x).$$

Differentiating this identity, we arrive at the relation

$$\nu(x) = \tau'(x) - 2\psi_1'(x) - \int\limits_0^{\frac{r-x}{2}} f\,(x+t,t)\,dt. \tag{3.1}$$

This identity is a fundamental relation between the functions $\tau(x)$ and $\nu(x)$ brought from the domain Ω_2 on the straight line $y=0$ in the case of Problem 2.

Thus, for the sought functions $\tau(x)$ and $\nu(x)$ we obtain a system of equations expressed by relations (2.10) and (3.1). Excluding the function $\nu(x)$ from (2.10) and (3.1), in view of matching condition $\tau(r) = \psi_1(r)$, as in studying Problem 1, for $\tau(x)$ we obtain a boundary value problem for a first order ordinary differential equation involving a fractional derivative in the lower order terms:

$$\tau'(x) + a(x)\, D_{0x}^{1-\varepsilon}\tau(t) = 2\psi_1'(x) + \frac{\psi_2(x)}{\gamma(x) - \gamma_2\alpha(x)} + \int\limits_0^{\frac{r-x}{2}} f\,(x+t, t)\; dt, \quad 0 < x < r, \tag{3.2}$$

$$\tau(r) = \psi_1(r). \tag{3.3}$$

Integrating equation (3.2) in x from 0 to x, in view of condition (3.3), we arrive at an integral equation corresponding to problem (3.2), (3.3):

$$\tau(x) - \frac{1}{\Gamma(\varepsilon)} \int\limits_0^r L(x,t)\tau(t)dt = F_2(x), \tag{3.4}$$

where

$$L(x,t) = \begin{cases} K(r,t), & 0 \leqslant x < t, \\ K(r,t) - K(x,t), & t < x \leqslant r, \end{cases}$$

$$F_2(x) = 2\psi_1(x) - \psi_1(r) - \int\limits_x^r \frac{\psi_2(t)}{\gamma(t) - \gamma_2\alpha(t)} dt - \int\limits_x^r \int\limits_0^{\frac{r-t}{2}} f\,(t+s,\, s)\, ds\, dt.$$

If the given functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\psi_1(x)$, $\psi_2(x)$, $f(x,y)$ possess properties (2.1), (2.2), (2.3) formulated in Theorem 2.1, then equation (3.4) is a Fredholm integral equation of second kind with the kernel $L(x,t) \in L_1\left([0,r] \times [0,r]\right)$ and the right hand side $F_2(x) \in C[0,r] \cap C^2(0,r)$.

Let us find sufficient conditions for the given functions ensuring the unique solvability of equation (3.4). In order to do this, we consider a homogeneous problem corresponding to Problem by letting $\psi_1(x) = \psi_2(x) \equiv 0$ for all $x \in [0,r]$, $f(x,y) \equiv 0$ for all $(x,y) \in \overline{\Omega}_2$. Then problem (3.2), (3.3) becomes a corresponding homogeneous problem

$$b(x)\tau'(x) + D_{0x}^{1-\varepsilon}\tau(t) = 0, \quad 0 < x < r, \tag{3.5}$$

$$\tau(r) = 0. \tag{3.6}$$

We multiply equation (3.5) by the function $\tau(x)$ and integrate by parts the obtained identity in x from 0 to r taking into consideration condition (3.6). This gives

$$\int\limits_0^r b(x)\tau(x)\tau'(x)dx + \int\limits_0^r \tau(x) D_{0x}^{1-\varepsilon}\tau(t)\, dx$$
$$= -\frac{b(0)\tau^2(0)}{2} - \frac{1}{2}\int\limits_0^r b'(x)\tau^2(x)\, dx + \int\limits_0^r \tau(x) D_{0x}^{1-\varepsilon}\tau(t)\, dx = 0.$$

It is known [5] that

$$\int\limits_0^r \tau(x) D_{0x}^{1-\varepsilon}\tau(t)\, dx \geqslant 0,$$

and

$$\int\limits_0^r \tau(x) D_{0x}^{1-\varepsilon}\tau(t) dx = 0$$

if and only if $\tau(x) \equiv 0$ for all $x \in [0,r]$. Hence, if the function $b(x)$ is decaying and negative, the latter identity can hold if and only if $\tau(x) \equiv 0$ for all $x \in [0,r]$. Therefore, under the mentioned conditions for the given functions equation (3.4) can possess only a unique solution τ in the class $C[0,r]\cap C^2(0,r)$.

Thus, we have proved the following theorem.

Theorem 3.1. *Let the given functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\psi_1(x)$, $\psi_2(x)$, $f(x,y)$ possess properties (2.1), (2.2), (2.3) formulated in Theorem 2.1 and*

$$[\beta(x)+\gamma_1\alpha(x)]\,[\gamma(x)-\gamma_2\alpha(x)]\neq 0 \quad \text{for all} \quad x\in[0,r],$$
$$b'(x)\leqslant 0, \quad b(0)<0 \quad \text{for all} \quad x\in[0,r].$$

Then there exists a unique regular in the domain Ω solution to Problem 2.

In the case $a(x)=a=const$ the solution of problem (3.2), (3.3) can be written explicitly:

$$\tau(x)=\frac{E_\varepsilon\left[-ax^\varepsilon\right]}{E_\varepsilon\left[-ar^\varepsilon\right]}\left(\psi_1(r)+F_2(x)-F_2(r)+a\int_0^r(r-t)^{\varepsilon-1}E_{1/\varepsilon}\left[-a(r-t)^\varepsilon;\varepsilon\right]F_2(t)dt\right.$$
$$\left.-a\int_0^x(x-t)^{\varepsilon-1}E_{1/\varepsilon}\left[-a(x-t)^\varepsilon;\varepsilon\right]F_2(t)dt\right),$$

and

$$E_\varepsilon\left[-ar^\varepsilon\right]\neq 0. \tag{3.7}$$

It follows from Theorem 3.1 that inequality (3.7) holds true for instance for all $a<0$.

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