ISSN 2304-0122 Ufa Mathematical Journal. Vol. 15. No 2 (2023). P. 85-99.

doi:10.13108/2023-15-2-85

ON STABILITY OF EQUILIBRIA OF NONLINEAR CONTINUOUS-DISCRETE DYNAMICAL SYSTEMS

M.G. YUMAGULOV, S.V. AKMANOVA

Abstract. In this paper the main attention is paid to discussing the issues on sufficient conditions for Lyapunov stability of nonlinear hybrid (continuous-discrete) systems, that is, the systems, the processes in which have several levels of different descriptions and the states involve both continuous and discrete components. It is well-known that by switchings between unstable regimes in a continuous dynamical system one can achieve a stability and vice versa, even when all regimes of the continuous system are stable, under the switching there can appear unstable regimes in the system. This is why it is important to make a detailed analysis on the stability issues while passing from continuous to the hybrid system.

In the present paper we propose new tests for Lyapunov stability of stationary regimes of nonlinear hybrid system with a constant discretization step $h > 0$. These tests are based on the methods for studying the stability by the linear approximation and on the formulae from the perturbation theory, which allow us to analyse the equilibria and cycles of the dynamical systems depending on a small parameter. The proposed approaches are based on a passage from the original hybrid system to equivalent in a natural sense dynamical system with a discrete time. We discuss relations between dynamical characteristics of hybrid and discrete systems. While studying the main problem on Lyapunov stability of an equilibrium of the hybrid system, we consider two formulations: the stability for small $h > 0$ and stability for arbitrary fixed $h = h_0 > 0$. Moreover, we discuss some questions on scenarios of bifurcation behavior of the hybrid system under the stability loss of the equilibrium. We adduce an example illustrating the efficiency of the obtained results in the problem on studying the stability of the equilibria of the hybrid systems.

Keywords: continuous-discrete system, hybrid system, equilibrium, periodic solutions, stability, bifurcation.

Mathematics Subject Classification: 37N35, 34D20, 34C23.

1. Introduction and formulation of problem

Many theoretical and practical problems lead to the need to study hybrid (continuous-discrete) systems, i.e. ones, the processes in which have several levels of heterogeneous description and the states contain both continuous and discrete components. Systems of this type are widely used in applied control problems of mechanical and technological processes, traffic in computer networks and in many other fields, see, for example, [1]–[5].

One of the main problems leading to hybrid systems is one on stabilizing the main regimes of a system. There is a large class of systems that cannot be stabilized only by a continuous control law with state feedback and, on the other hand, can be stabilized by an appropriate switching at certain time moments. In other words, by switching unstable regimes, it is possible to achieve their stability and vice versa, even when all modes of a continuous system are stable, their switchings can produces unstable regimes [6]–[8]. This explains the ever-increasing interest of

M.G. Yumagulov, S.V. Akmanova, On stability of equilibria of nonlinear continuous-discrete dynamical systems.

Submitted September 5, 2022.

specialists from various fields in studying the stability of hybrid systems, see, for instance, [9]–[16]. We note that most works are aimed at developing the method of Lyapunov functions and relevant applications in the problem of stability analysis linear and nonlinear hybrid systems, for a detailed review of works on this subject see [5].

This article proposes new Lyapunov stability tests in of stationary regimes of nonlinear hybrid systems. These tests are based on the methods for studying stability by the first approximation and formulas of the perturbations theory obtained in [17]. They allow us to analyze the stability of equilibria and cycles of dynamical systems depending on a small parameter. The results obtained in this work are a significant development of the results of the authors announced in [18].

We consider a nonlinear hybrid system described by the equations [19]:

$$\begin{cases} x'(t) = f(x(t), y(t_k)), & t_k \leqslant t < t_{k+1}, \\ y(t_{k+1}) = g(x(t_{k+1}), y(t_k)), & k = 0, 1, 2, \dots, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ are vectors characterizing the behavior of respectively continuous and discrete parts of the hybrid system; the time moments t_k define on $\mathbb{R}$ a uniform grid with a step $h > 0$:

$$0 = t_0 < t_1 = t_0 + h < t_2 = t_1 + h < \ldots < t_{k+1} = t_k + h < \ldots \tag{1.2}$$

The functions $f(x, y)$ and $g(x, y)$ in system (1.1) are continuously differentiable and generate the operators $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$.

The evolution of system (1.1) follows a standard scheme:

1) Initial condition $u_0 = (x_0, y_0)$ are prescribed;
2) By the solution $x = \varphi_0(t)$ to the Cauchy problem $x' = f(x, y_0)$, $x(t_0) = x_0$, we find the vectors $x_1 = \varphi_0(t_1)$ and $y_1 = g(x_1, y_0)$;
3) By the solution $x = \varphi_1(t)$ to the Cauchy problem $x' = f(x, y_1)$, $x(t_1) = x_1$, we find the vectors $x_2 = \varphi_1(t_2)$ and $y_2 = g(x_2, y_1)$;

and so forth.

Thus, a solution $u(t) = (x(t), y(t))$ of hybrid system (1.1) starting from the point $u_0 = (x_0, y_0)$ is a function

$$u(t) = (x(t), y(t)) = \begin{cases} (\varphi_0(t), y_0), & t_0 \leqslant t < t_1, \\ (\varphi_1(t), y_1), & t_1 \leqslant t < t_2, \\ (\varphi_2(t), y_2), & t_2 \leqslant t < t_3, \\ \quad \cdots\cdots \end{cases} \tag{1.3}$$

The first component $x(t)$ of the solution $u(t) = (x(t), y(t))$ is continuous for all $t \geqslant 0$, continuously differentiable on each interval $t_j < t < t_{j+1}$ but it is not necessarily differentiable at the switching moments $t = t_j$. Concerning the second component, the function $y(t)$, it is piecewise constant changing its values at the moments $t = t_j$.

We assume that system (1.1) possesses an equilibrium at the point $x = 0$, $y = 0$, that is,

$$f(0, 0) = 0, \quad g(0, 0) = 0. \tag{1.4}$$

The main problem of the present paper is to study the issue on sufficient conditions for the Lyapunov stability of the equilibrium $x = 0$, $y = 0$ of system (1.1). We also discuss the issue on the bifurcations in system (1.1) under the loss of stability of the equilibrium $x = 0$, $y = 0$.

The stability of the equilibria of system (1.1) is treated in the classical sense. Namely, the equilibrium $x = 0$, $y = 0$ of hybrid system (1.1) is called *Lyapunov stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that as $\|u_0\| < \delta$ the solution $u(t) = (x(t), y(t))$ of system (1.1) starting from the point $u_0 = (x_0, y_0)$ satisfies the inequality $\|u(t)\| < \varepsilon$ for all $t \geqslant 0$. We call the equilibrium $x = 0$, $y = 0$ of system (1.1) *asymptotically stable* if it is Lyapunov stable and there exists $\delta_0 > 0$ such that as $\|u_0\| < \delta_0$, a solution $u(t) = (x(t), y(t))$ of system (1.1) starting from the point $u_0 = (x_0, y_0)$ satisfies the convergence $\|u(t)\| \to 0$ as $t \to \infty$. Finally, the equilibrium $x = 0$, $y = 0$ of system (1.1) is called *unstable* if it is not Lyapunov stable. Hereinafter by $\|\cdot\|$ we denote Euclidean norms of the vectors in the spaces $\mathbb{R}^N$.

2. Passage from hybrid to discrete system

The main constructions of the present work are based on passing from hybrid system (1.1) to an auxiliary discrete system. The construction of this discrete system and the study of its properties is of an independent interest. In this section we provide the main points of this passage.

2.1. Shift operator and discrete system. We denote by $U(h)$ the operator of shift along the trajectories of system $x' = f(x, y)$ in time from $t = 0$ till $t = h$, see, for instance, [20]. The operator $U(h)$ maps the vector $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ into the vector $x_1 = x(h) \in \mathbb{R}^n$; here $x(t)$ is the solution of the Cauchy problem $x' = f(x, y_0)$, $x(0) = x_0$. Thus, $U(h)(x_0, y_0) = x_1$.

We consider a discrete system

$$\begin{cases} x_{k+1} = U(h)(x_k, y_k), \\ y_{k+1} = g(U(h)(x_k, y_k), y_k), \quad k = 0, 1, 2, \dots \end{cases} \tag{2.1}$$

in which $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^m$.

Discrete system (2.1) is equivalent to original hybrid system (1.1) in the following sense. System (2.1) fixes the values of solutions of hybrid system (1.1) at time moments $t = 0$, $t = h$, $t = 2h$ and so forth. In other words, the following lemma is true by the construction.

Lemma 2.1. *Each solution (1.3) of hybrid system (1.1) generates the solution*

$$(x_0, y_0), \quad (x_1, y_1), \quad (x_2, y_2), \quad \dots, \quad (x_k, y_k), \quad \dots, \tag{2.2}$$

of discrete system (2.1), in which $x_0 = \varphi_0(0)$, $x_1 = \varphi_1(h)$, $x_2 = \varphi_2(2h)$, ... At the same time, discrete system (2.1) possesses no other solutions.

Of course, while passing to discrete system (2.1) one loses the information on behavior of hybrid system (1.1) at intervals $0 < t < h$, $h < t < 2h$ and so forth, but the most part of important questions on qualitative characteristics of the equilibria or cycles of the original hybrid system (1.1) can be analyzed.

Let us show a relation between the equilibria and periodic solutions of systems (1.1) and (2.1).

Theorem 2.1. *Each equilibrium (x^*, y^*) of system (1.1) is an equilibrium of system (2.1). To each equilibrium (x^*, y^*) of system (2.1), there corresponds an h-periodic solution $(\varphi_0(t), y^*)$ of system (1.1) such that $\varphi_0(0) = \varphi_0(h) = x^*$. At the same time, the stability nature of the mentioned solutions to systems (1.1) and (2.1) are similar.*

The proof of this theorem and other main statements is provided in Section 7.

We note that the periodic function $x = \varphi_0(t)$ mentioned in Theorem 2.1 is continuously differentiable for all t. We also note that if the first equation of system (1.1) is scalar, then the function $x = \varphi_0(t)$ is constant. In other words, the following lemma holds.

Lemma 2.2. *Let $n = 1$. Then each equilibrium (x^*, y^*) of system (2.1) is an equilibrium of system (1.1) and vice versa.*

The issue on relation between the cycles of systems (1.1) and (2.1) is clarified by the following statement.

Lemma 2.3. *Each qh-periodic solution* $(x(t), y(t))$ *of system (1.1) forms a q-cycle*

$$(x_0^*, y_0^*),\ (x_1^*, y_1^*),\ \ldots,\ (x_{q-1}^*, y_{q-1}^*),\quad (x_0^*, y_0^*),\ (x_1^*, y_1^*),\ \ldots,\ (x_{q-1}^*, y_{q-1}^*),\ \ldots. \tag{2.3}$$

of system (2.1) so that

$$\begin{gathered} x(0) = x_0^*, \quad y(0) = y_0^*, \qquad x(h) = x_1^*, \quad y(h) = y_1^*, \quad \ldots, \\ x((q-1)h) = x_{q-1}^*, \quad y((q-1)h) = y_{q-1}^*, \qquad x(qh) = x_0^*, \quad y(qh) = y_0^*. \end{gathered} \tag{2.4}$$

To each q-cycle (2.3) of system (2.1), there corresponds a qh-periodic solution $(x(t), y(t))$ *of system (1.1) so that identities (2.4) hold.*

2.2. Transformation of discrete system. A passage to discrete system (2.1) will be more effective if we provide formulas allowing to find constructively the shift operator $U(h)$.

In order to do this, we observe that by identities (1.4) the functions $f(x,y)$ and $g(x,y)$ can be represented as

$$f(x,y) = A_1x + B_1y + a(x,y), \qquad g(x,y) = A_2x + B_2y + b(x,y), \qquad (x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m), \tag{2.5}$$

where

$$A_1 = f'_x(0,0), \qquad B_1 = f'_y(0,0), \qquad A_2 = g'_x(0,0), \qquad B_2 = g'_y(0,0),$$

and the smooth nonlinearities $a(x,y)$ and $b(x,y)$ satisfy the relations

$$a(x,y) = o(\|x\| + \|y\|), \quad b(x,y) = o(\|x\| + \|y\|) \quad \text{as} \quad \|x\| + \|y\| \to 0. \tag{2.6}$$

The matrices A_1, B_1, A_2, B_2 generate linear operators

$$A_1 : \mathbb{R}^n \to \mathbb{R}^n, \quad B_1 : \mathbb{R}^m \to \mathbb{R}^n, \quad A_2 : \mathbb{R}^n \to \mathbb{R}^m, \quad B_2 : \mathbb{R}^m \to \mathbb{R}^m.$$

The following lemma holds true.

Lemma 2.4. *Let* $\det A_1 \neq 0$. *Then the shift operator* $U(h)$ *can be represented as*

$$U(h)(x_0, y_0) = e^{A_1h}x_0 + A_1^{-1}(e^{A_1h} - I)B_1y_0 + \varepsilon(x_0, y_0; h), \tag{2.7}$$

where

$$\varepsilon(x_0, y_0; h) = e^{hA_1} \int_0^h e^{-sA_1} a(x(s, x_0, y_0), y_0)\, ds. \tag{2.8}$$

Here $x = x(t, x_0, y_0)$ *is the solution of the Cauchy problem*

$$x' = A_1x + B_1y_0 + a(x, y_0), \quad x(0) = x_0. \tag{2.9}$$

This lemma implies that discrete system (2.1) can be represented as

$$\begin{cases} x_{k+1} = e^{A_1h}x_k + A_1^{-1}(e^{A_1h} - I)B_1y_k + \varepsilon(x_k, y_k; h), \\ y_{k+1} = A_2e^{A_1h}x_k + (A_2A_1^{-1}(e^{A_1h} - I)B_1 + B_2)y_k + c(x_k, y_k; h); \end{cases} \tag{2.10}$$

here the nonlinearity $c(x_k, y_k; h)$ is given by the identity

$$c(x_k, y_k; h) = A_2\varepsilon(x_k, y_k; h) + b(U(h)(x_k, y_k), y_k).$$

We represent system (2.10) in a more compact form:

$$u_{k+1} = A(h)u_k + \xi(u_k, h), \quad k = 0, 1, 2, \ldots, \tag{2.11}$$

where

$$u_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \qquad \xi(u_k, h) = \begin{bmatrix} \varepsilon(x_k, y_k; h) \\ c(x_k, y_k; h) \end{bmatrix},$$

and $A(h)$ is a square matrix of order $n+m$:

$$A(h) = \begin{bmatrix} e^{A_1h} & A_1^{-1}(e^{A_1h} - I)B_1 \\ A_2e^{A_1h} & A_2A_1^{-1}(e^{A_1h} - I)B_1 + B_2 \end{bmatrix}. \tag{2.12}$$

Lemma 2.5. *The function $\xi(u,h)$ satisfies the relation*

$$\xi(u,h) = o(\|u\|) \quad as \quad \|u\| \to 0. \tag{2.13}$$

3. Stability of zero equilibrium of system (1.1)

We return back to the main problem, the question on conditions ensuring the Lyapunov stability of zero equilibrium $x = 0$, $y = 0$ of hybrid system (1.1). In what follows, to shorten formulations, we shall employ the following notation:

- $A < 0 \Leftrightarrow$ all eigenvalues of a square matrix A have negative real parts;
- $|A| < 1 \Leftrightarrow$ all eigenvalues λ of the square matrix A satisfies the inequality $|\lambda| < 1$.

The notations $A \leqslant 0$ and $|A| \leqslant 1$ have the similar meanings.

3.1. Stability for a fixed $h > 0$. We first discuss the stability of the equilibrium $x = 0$, $y = 0$ of hybrid system (1.1) for a fixed $h = h_0 > 0$.

We begin with the following statement, in which A_1, B_1, A_2, B_2 are the matrices from identities (2.5); this statement was announced in [18].

Theorem 3.1. *Let $A_1 < 0$ and $|B_2| < 1$. Then for $h = h_0 > 0$ there exists $\delta = \delta(h_0) > 0$ such that if $\|A_2\| < \delta$ and $\|B_1\| < \delta$, then the equilibrium $x = 0$, $y = 0$ of hybrid system (1.1) for $h = h_0$ is asymptotically stable.*

In other words, if $A_1 < 0$ and $|B_2| < 1$, then the equilibrium $x = 0$, $y = 0$ of hybrid system (1.1) is asymptotically stable once the quantities $\|A_2\|$ and $\|B_1\|$ are small enough.

We observe that Theorem 3.1 states the following simple fact. As $A_2 = 0$ and $B_1 = 0$, the linearized in the vicinity of equilibrium $x = 0$, $y = 0$ system associated with (1.1) splits into two uncoupled linear equations:

$$x'(t) = A_1 x(t) \tag{3.1}$$

and

$$y(t_{k+1}) = B_2 y(t_k), \quad k = 0, 1, 2, \dots. \tag{3.2}$$

Conditions $A_1 < 0$ and $|B_2| < 1$ mean that each of these linear equations is asymptotically stable. This is why it is natural to expect that the zero solution of system (1.1) for a given $h = h_0 > 0$ is asymptotically stable if the norms $\|A_2\|$ and $\|B_1\|$ are small enough.

Now we provide another stability condition for the equilibrium $x = 0$, $y = 0$ of system (1.1), which also was announced in [18].

Theorem 3.2. *Let the matrix A_1 be invertible, that is,*

$$\det A_1 \neq 0. \tag{3.3}$$

Let for $h = h_0 > 0$ matrix (2.12) satisfy condition $|A(h_0)| < 1$. Then the equilibrium $x = 0$, $y = 0$ of hybrid system (1.1) as $h = h_0$ is asymptotically stable. If the absolute value of at least one eigenvalue of the matrix $A(h_0)$ exceeds 1, then the equilibrium $x = 0$, $y = 0$ of system (1.1) as $h = h_0$ is unstable.

In what follows condition (3.3) is supposed to be satisfied.

From the practical point of view, the following question is important. Let the matrix A_1 possess at least one eigenvalue with a real positive part, then linear system (3.1) is unstable. Whether it is possible, for a given $h = h_0 > 0$, to choose matrices A_2, B_1 and B_2 so that the matrix (2.12) satisfies the condition $|A(h_0)| < 1$? The answer is positive and this is demonstrated by appropriate formulas in Section 4.

3.2. Stability for small $h > 0$. We shall say that the equilibrium $x = 0$, $y = 0$ of hybrid system (1.1) is *Lyapunov stable (asymptotically stable, unstable) for small* $h > 0$ if there exists $h_0 > 0$ such that for each $h \in (0, h_0)$ the equilibrium $x = 0$, $y = 0$ of system (1.1) is Lyapunov stable (asymptotically stable, unstable).

To discuss the issue on stability of the equilibrium $x = 0$ and $y = 0$ of system (1.1) for small $h > 0$, by matrix (2.12) we define new matrices:

$$A_0 = A(0) = \begin{bmatrix} I & 0 \\ A_2 & B_2 \end{bmatrix}, \tag{3.4}$$

and

$$A' = A'(0) = \begin{bmatrix} A_1 & B_1 \\ A_2A_1 & A_2B_1 \end{bmatrix}. \tag{3.5}$$

The above matrix is the derivative of the matrix (2.12) for $h = 0$.

The next lemma is obvious.

Lemma 3.1. *The matrix A_0 possesses an eigenvalue $\lambda_0 = 1$, the multiplicity of which is at least n. This eigenvalue is semi-simple of multiplicity n if the matrix B_2 has no eigenvalue 1. Other eigenvalues of the matrix A_0 coincides with ones of the matrix B_2.*

In particular, if $|B_2| < 1$, then the matrix A_0 and the corresponding transposed matrix A_0^* possesses a semi-simple eigenvalue $\lambda_0 = 1$ of multiplicity n.

The next statement gives a necessary stability condition for zero solution of hybrid system (1.1) for small $h > 0$.

Theorem 3.3. *Let the zero condition of system (1.1) be stable for small $h > 0$. Then $|B_2| \leqslant 1$.*

We observe that the condition $|B_2| \leqslant 1$ is also necessary for the stability of linear discrete system (3.2). We also note that condition $A_1 \leqslant 0$, which is necessary for the stability of linear continuous system (3.1) is not necessary for stability of system (1.1) for small $h > 0$. We also discuss this issue below.

In what follows we suppose that $|B_2| < 1$. Since the matrix A_0 has a semi-simple eigenvalue $\lambda_0 = 1$ of multiplicity n, then there exists a linearly independent system of eigenvectors e_i: $A_0 e_i = e_i$, $i = \overline{1, n}$. The transposed matrix A_0^* also has a semi-simple eigenvalue 1 of multiplicity n with associated eigenvectors e_i^*: $A_0^* e_i^* = e_i^*$, $i = \overline{1, n}$. The vectors e_i and e_j^* can be chosen by the relations

$$(e_i, e_i^*) = 1, \qquad (e_i, e_j^*) = 0 \quad \text{as} \quad i \neq j, \qquad i = \overline{1, n}, \quad j = \overline{1, n}. \tag{3.6}$$

We define a matrix

$$D = \begin{bmatrix} (A'e_1, e_1^*) & (A'e_2, e_1^*) & \cdots & (A'e_n, e_1^*) \\ (A'e_1, e_2^*) & (A'e_2, e_2^*) & \cdots & (A'e_n, e_2^*) \\ \vdots & \vdots & \ddots & \vdots \\ (A'e_1, e_n^*) & (A'e_2, e_n^*) & \cdots & (A'e_n, e_n^*) \end{bmatrix}. \tag{3.7}$$

Theorem 3.4. *Let $|B_2| < 1$ and $D < 0$. Then the zero solution of hybrid system (1.1) is asymptotically stable for all $h > 0$.*

4. Stability of zero equilibrium of system (1.1) in case $n = m = 1$

Here we consider an important case of system (1.1) when $n = m = 1$. In this case identities (2.5) and (2.6) become

$$f(x, y) = a_1 x + b_1 y + a(x, y), \quad g(x, y) = a_2 x + b_2 y + b(x, y), \quad x \in \mathbb{R}^1, \quad y \in \mathbb{R}^1, \tag{4.1}$$

where the numbers a_1, b_1, a_2, b_2 are determined by the identities $a_1 = f'_x(0,0)$, $b_1 = f'_y(0,0)$, $a_2 = g'_x(0,0)$, $b_2 = g'_y(0,0)$, while the nonlinearities $a(x,y)$ and $b(x,y)$ satisfy the relations

$$a(x,y) = o(|x|+|y|), \quad b(x,y) = o(|x|+|y|) \quad \text{as} \quad |x|+|y| \to 0. \tag{4.2}$$

Then system (1.1) casts into the form

$$\begin{cases} x'(t) = a_1 x(t) + b_1 y(t_k) + a(x(t), y(t_k)), & t_k \leqslant t < t_{k+1}, \\ y(t_{k+1}) = a_2 x(t_{k+1}) + b_2 y(t_k) + b(x(t_{k+1}), y(t_k)). \end{cases} \tag{4.3}$$

We first provide an analogue of Theorem 3.2.

Theorem 4.1. *Let $a_1 \neq 0$ and for $h = h_0 > 0$ the inequalities hold:*

$$|b_2 e^{a_1 h_0}| < 1, \qquad \left| \frac{a_2 b_1}{a_1}(e^{a_1 h_0} - 1) + b_2 + e^{a_1 h_0} \right| < b_2 e^{a_1 h_0} + 1. \tag{4.4}$$

Then the zero solution of hybrid system (4.3) for $h = h_0$ is asymptotically stable. If at least one of inequalities (4.4) holds with an opposite sign, then the zero solution of hybrid system (4.3) as $h = h_0$ is unstable.

A simple analysis of inequalities (4.4) gives a positive answer to the above formulated question: if $a_1 > 0$, whether it is possible, for a given $h = h_0 > 0$, to choose the numbers a_2, b_1 and b_2 to ensure inequalities (4.4)?

Now we provide an analogue of Theorem 3.4. We let

$$\gamma = a_1(1 - b_2) + a_2 b_1. \tag{4.5}$$

Theorem 4.2. *Let $|b_2| < 1$ and $\gamma < 0$ ($\gamma > 0$). Then the zero solution of hybrid system (4.3) is asymptotically stable (unstable) for all small $h > 0$.*

This statement implies a series of important corollaries.

1. If $a_1 < 0$ and $|b_2| < 1$, and the quantity $|a_2 b_1|$ is small enough, then the zero solution of hybrid system (4.3) is asymptotically stable for all small $h > 0$.
2. At the same time, conditions $a_1 < 0$ and $|b_2| < 1$ (that is, as linear systems $x' = a_1 x$ and $y_{k+1} = b_2 y_k$ are stable) do not ensure the stability of zero solution of hybrid system (4.3) for small $h > 0$.
3. And vice versa, if $a_1 > 0$ (that is, as the linear system $x' = a_1 x$ is unstable), nevertheless, the zero solution of hybrid system (4.3) can turn out to be stable for small $h > 0$.

Similar corollaries are true for system (1.1) in the general setting.

5. Bifurcation of equilibria

The evolution of hybrid system (1.1) depends on the quantity h, which can be regarded as a parameter of the system. In particular, as h varies, the zero solution of this system can change its stability character and this leads to various bifurcations scenarios. According to Theorem 3.2, the stability character of this solution can change as h passes h_0 such that matrix (2.12) satisfies condition $|A(h_0)| = 1$.

Following the classical theory of bifurcations, see, for instance, [21], a value $h = h_0$ is called a *bifurcation point* of system (1.1) if the matrix $A(h_0)$ possesses at least one eigenvalue μ_0 such that $|\mu_0| = 1$.

One more interesting question is on possible bifurcation scenarios in the vicinity of the zero solution of hybrid system (1.1) as the parameter h passes the bifurcation point h_0. Let us discuss this issues in the case when the matrix $A(h_0)$ possesses a simple eigenvalue 1 or -1.

We denote by e and g the eigenvectors of the matrix $A_0 = A(h_0)$ and the adjoint matrix $A_0^* = A^*(h_0)$ associated with a simple eigenvalue 1 (eigenvalue -1).

Theorem 5.1. *Let the matrix $A_0 = A(h_0)$ possesses a simple eigenvalue* 1 *(simple eigenvalue* -1*), while absolute values of its eigenvalues are less than one. Let $(A'(h_0)e, g) \neq 0$; here $A'(h)$ is the matrix of the derivatives of the entries of the matrix $A(h)$. Then as the parameter h passes h_0, the matter of a qualitative restructuring of the behavior of system (1.1) in the vicinity of the equilibrium $x = 0$, $y = 0$ is the appearance of non-zero periodic solution: there exist $h_k \to h_0$ such that, as $h = h_k$, system (1.1) has a non-zero h_k-periodic ($2h_k$-periodic) solution $(x_k(t), y_k)$ so that $(x_k(t), y_k) \to (0, 0)$ as $k \to \infty$.*

We note that the h_k-periodic function $x_k(t)$ involved in this theorem for the case of the eigenvalue 1 is continuously differentiable for all t. Therefore, as $n = 1$, this function is constant. In other words, in the mentioned case in system (1.1) there is a bifurcation of a multiple equilibrium: as the parameter h passes h_0, in the vicinity of the zero equilibrium $x = 0$, $y = 0$ nonzero equilibria appear.

The discussion of some directions of developing the results of Theorem 5.1 is provided below in its proof.

6. Examples

6.1. Problems on stability of unstable equilibrium of dynamical system. As a first illustration of the obtained results we consider the problem on stability of the zero equilibrium $\varphi = 0$, $y = 0$ of a hybrid system described by the equations

$$\begin{cases} \varphi''(t) = \sin\varphi(t) + y(t_k), & t_k \leqslant t < t_{k+1}, \\ y(t_{k+1}) = \alpha\varphi(t_{k+1}) + \beta\varphi'(t_{k+1}), & k = 0, 1, 2, \dots, \end{cases} \tag{6.1}$$

where α and β are real parameters, while the time moments t_k form on $\mathbb{R}$ uniform grid (1.2) with the step $h > 0$. We restrict ourselves by discussing the stability of the equilibrium $\varphi = 0$, $y = 0$ for small $h > 0$.

The above issue can be interpreted as a question of stabilization of the zero solution $\varphi = 0$ to a continuous dynamical system $\varphi'' = \sin\varphi$ by means of fast switching at the time moments $t = t_k$, $k = 1, 2, 3, \dots$ The questions of such kind arise, for instance, in the problem on stabilization of the upper unstable pendulum position, which was widely studied in literature, see, for instance, [22].

Letting $x_1 = \varphi$, $x_2 = \varphi'$, we pass from (6.1) to system

$$\begin{cases} x_1' = x_2, \\ x_2' = \sin x_1 + y(t_k), & t_k \leqslant t < t_{k+1}, \\ y(t_{k+1}) = \alpha x_1(t_{k+1}) + \beta x_2(t_{k+1}), & k = 0, 1, 2, \dots \end{cases} \tag{6.2}$$

This is a system of form (1.1) with $n = 2$, $m = 1$ and

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} \alpha & \beta \end{bmatrix}, \qquad B_2 = 0.$$

Here matrices (3.4) and (3.5) are respectively of the form

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & 0 \end{bmatrix}, \qquad A' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ \beta & \alpha & \beta \end{bmatrix}.$$

The vectors involved in formulas (3.6) can be chosen as

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \alpha \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ \beta \end{bmatrix}, \qquad e_1^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad e_2^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then matrix (3.7) is of the form

$$D = \begin{bmatrix} 0 & 1 \\ 1+\alpha & \beta \end{bmatrix}.$$

This matrix satisfies the condition $D < 0$ if $\alpha < -1$ and $\beta < 0$.

By Theorem 3.4 this implies that the zero solution of hybrid system (6.1) is asymptotically stable for small $h > 0$ if $\alpha < -1$ and $\beta < 0$. If it least one of these inequalities holds with an opposite sign, then the mentioned solution is unstable for all small $h > 0$.

6.2. Doubling period bifurcation. As a second illustration we consider a problem on bifurcations in hybrid systems under the change of the stability nature of the equilibrium. Namely, we consider hybrid system (4.3) of form

$$\begin{cases} x'(t) = -x(t) + y(t_k) + a(x(t), y(t_k)), & t_k \leqslant t < t_{k+1}, \\ y(t_{k+1}) = 2x(t_{k+1}) - 2y(t_k) + b(x(t_{k+1}), y(t_k)), \end{cases} \tag{6.3}$$

in which the switching moments t_k form on $\mathbb{R}$ a uniform grid (1.2) with step $h > 0$.

Matrix (2.12) for system (6.3) reads as

$$A(h) = \begin{bmatrix} e^{-h} & 1 - e^{-h} \\ 2e^{-h} & -2e^{-h} \end{bmatrix}.$$

As $h = h_0 = \ln 3$ this matrix is

$$A_0 = A(h_0) = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}.$$

The matrix A_0 possesses eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2/3$. As the eigenvectors e and g of the matrices A_0 and A_0^* associated with the eigenvalue $\lambda_1 = -1$ we can take the vectors

$$e = g = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

We then have $(A'(h_0)e, g) = 3 \neq 0$.

Thus, all assumptions of Theorem 5.1 for system (6.3) are satisfied and therefore, as the parameter h passes the value $h_0 = \ln 3$, in the vicinity of the equilibrium $x = 0$, $y = 0$ of this system there arise non-zero periodic solutions with the period T so that $T \approx 2\ln 3$. The bifurcation directions, that is, for what values of h (smaller or greater than h_0) there arise periodic solutions as well as the stability nature of periodic solutions depend on the properties of nonlinearities $a(x, y)$ and $b(x, y)$ involved in the right hand sides of system (6.3). The corresponding study can be done on the base of works [23] and [24].

7. Proof of main statements

7.1. Proof of Theorem 2.1. We consider an autonomous system

$$\frac{dx}{dt} = f(x, y), \quad x \in \mathbb{R}^n, \tag{7.1}$$

depending on the parameter $y \in \mathbb{R}^m$, in which $f(x, y)$ is a smooth functions such that $f(0, 0) = 0$, that is, as $y = 0$, system (7.1) possesses a zero equilibrium $x = 0$. We suppose that for all x_0 and y_0 the solution $x = x(t)$ of the Cauchy problem $x' = f(x, y_0)$, $x(0) = x_0$, is well-defined for all $t \geqslant 0$.

In order to prove Theorem 2.1, we shall need the following auxiliary statement.

Lemma 7.1. *For all $\rho > 0$ and $h > 0$ there exists r, $0 < r < \rho$, such that if $\|u_0\| < r$, $u_0 = (x_0, y_0)$, then the solution $x(t)$ of the Cauchy problem*

$$\begin{cases} x' = f(x, y_0), \\ x(0) = x_0, \end{cases} \tag{7.2}$$

satisfies the inequality $\|x(t)\| \leqslant \rho$ for all $t \in [0, h]$.

Proof. We argue by contradiction assuming that there exist numbers $\rho_0 > 0$ and $h_0 > 0$ as well as a sequence of vectors $u_n = (x_n, y_n) \to (0, 0)$ such that the solutions $x_n(t)$ of the Cauchy problem

$$\begin{cases} x' = f(x, y_n), \\ x(0) = x_n, \end{cases} \tag{7.3}$$

for some $t = t_n \in [0, h_0]$ satisfy the inequality $\|x_n(t_n)\| > \rho_0$. Since $(x_n, y_n) \to (0, 0)$, by theorem on the continuous dependence of the solution of the Cauchy problem on parameters and initial data, the solution $x_n(t)$ of the Cauchy problem (7.3) uniformly in $t \in [0, h_0]$ tends to the solution Cauchy problems $x' = f(x, 0)$, $x(0) = 0$, that is, to the function $x(t) \equiv 0$. In other words, $\max\limits_{0 \leqslant t \leqslant h_0} \|x_n(t)\| \to 0$ as $n \to \infty$. This contradicts the inequality $\|x_n(t_n)\| > \rho_0 > 0$. The proof is complete. □

We proceed to proving Theorem 2.1.

Proof. The first assertion of this theorem follows from Lemma 2.1. Let us show the validity of the second assertion. Let the system (2.1) possesses an equilibrium point (x^*, y^*), that is,

$$\begin{cases} x^* = U(h)(x^*, y^*), \\ y^* = g(U(h)(x^*, y^*), y^*). \end{cases} \tag{7.4}$$

The first of these identities means that the solution $x = \varphi_0(t)$ of the Cauchy problem $x' = f(x, y^*)$, $x(0) = x^*$, possesses the property $\varphi_0(0) = \varphi_0(h) = x^*$, that is, it is h-periodic. The second of identities (7.4) is of the form $y^* = g(x^*, y^*)$, which means that (x^*, y^*) is a constant solution of the second equation in original hybrid system (1.1). Finally we get that hybrid system (1.1) has an h-periodic solution $(\varphi_0(t), y^*)$.

It remains to establish the validity of the third assertion. We restrict ourselves by proving the fact that if (x^*, y^*) is the equilibrium of each of systems (1.1) and (2.1) and, moreover, it is Lyapunov stable for one of these systems, then it will be stable for the other system. We can suppose that the equilibrium (x^*, y^*) is zero, that is, $x^* = 0, y^* = 0$. We let $u = (x, y)$ and $u^* = (x^*, y^*) = (0, 0)$. By $\|u\|$ we denote the norm of the vector u.

First let u^* be the zero equilibrium of each of system (1.1) and (2.1) and let it be Lyapunov stable for hybrid system (1.1). We are going to show that it is also Lyapunov stable for discrete system (2.1).

Indeed, since the solution u^* of hybrid system (1.1) is stable, then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|u_0\| < \delta$, then the solution $u(t) = (x(t), y(t))$ of hybrid system (1.1) (see formula (1.3)) starting from the point $u_0 = (x_0, y_0)$ satisfies the inequality $\|u(t)\| < \varepsilon$ for all $t \geqslant 0$. Then by Lemma 2.1 the mentioned property of the solution u^* to hybrid system (1.1) also holds for the solution u^* to discrete system (2.1), that is, it is also stable.

Let hybrid system (1.1) possesses a zero equilibrium u^*; then discrete system (2.1) also possesses the same equilibrium. Let the equilibrium u^* of discrete system (2.1) is Lyapunov stable. Let us show that the equilibrium u^* of hybrid system (1.1) is also Lyapunov stable.

We argue by contradiction assuming that there exist a number $\varepsilon_0 > 0$ and a sequence $u_n \to 0$ such that the solution $u_n(t)$ of hybrid system (1.1) starting from the point u_n satisfies the inequality $\|u_n(\tau_n)\| \geqslant \varepsilon_0$ for some $t = \tau_n > 0$. We let $\rho_0 = \varepsilon_0/2$. By Lemma 7.1 for given

$\rho_0 > 0$ and $h > 0$ there exist r, $0 < r < \rho_0$ such that if $\|u_0\| < r$, $u_0 = (x_0, y_0)$, then the solution $x(t)$ of the Cauchy problem $x' = f(x, y_0)$, $x(0) = x_0$, satisfies the inequality $\|x(t)\| \leqslant \rho_0$ for all $t \in [0, h]$.

By Lemma 2.1, each solution $u_n(t)$ of hybrid system (1.1) generates a corresponding solution $u_{k,n} = (x_{k,n}, y_{k,n})$ to discrete system (2.1) so that at the time moments $t = 0$, $t = h$, $t = 2h$, etc. these solutions coincide. Each point τ_n is located in some interval $t_{k(n)} < t < t_{k(n)+1}$, see grid (1.2). Hence, the first component $x_n(t)$ in the solution $u_n(t) = (x_n(t), y_n(t))$ of hybrid system (1.1) as $t_{k(n)} \leqslant t < t_{k(n)+1}$ solves the Cauchy problem

$$\begin{cases} x' = f(x, y_{k(n),n}), \\ x(t_{k(n)}) = x_{k(n),n}. \end{cases} \tag{7.5}$$

Since the zero solution u^* of discrete system (2.1) is stable, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|u_0\| < \delta$, then solution $u_k = (x_k, y_k)$ of hybrid system (2.1) starting from the point $u_0 = (x_0, y_0)$ satisfies the inequality $\|u_k\| < \varepsilon$ for all $k = 0, 1, 2, \ldots$ In what follows as $\varepsilon > 0$ we choose $\varepsilon = r$.

Since $u_n \to 0$, we can suppose that $\|u_n\| < \delta$ for all n. Then the solution $u_{k,n} = (x_{k,n}, y_{k,n})$ of discrete system (2.1) satisfies the inequality $\|u_{k,n}\| < r$ for all $k = 0, 1, 2, \ldots$ and all n. By Lemma 7.1 this implies that the solution $x_n(t)$ of the Cauchy problem (7.5) satisfies the inequality $\|x_n(t)\| \leqslant \rho_0$ for all $t \in [t_{k(n)}, t_{k(n)+1}]$. This contradicts the above inequality $\|u_n(\tau_n)\| \geqslant \varepsilon_0$ since it implies the inequality $\|x_n(\tau_n)\| > \rho_0$. The proof is complete. □

7.2. Proof of Lemma 2.4. By the first identity in (2.5) the operator $U(h)$ maps the vector (x_0, y_0) into the vector $x_1 = x(h)$, where $x(t)$ is the solution of Cauchy problem (2.9), or, equivalently, the solution of the integral equation

$$x(t) = e^{tA_1}x_0 + e^{tA_1}\int_0^t e^{-sA_1}[B_1 y_0 + a(x(s), y_0)]\, ds.$$

Since $\det A_1 \neq 0$, the inverse matrix A_1^{-1} is well-defined. This is why

$$\int_0^t e^{-sA_1}\, ds = A_1^{-1}(I - e^{-A_1 t}).$$

This implies identity (2.7).

7.3. Proof of Lemma 2.5. To simplify the presentation, we restrict ourselves by considering the case $n = m = 1$. We also restrict ourselves by proving relation (2.13) only for the first component of the vector $\xi(u, h)$, that is, we are going to show that

$$\varepsilon(x, y; h) = o(\|u\|), \qquad \|u\| \to 0\,; \tag{7.6}$$

here $u = (x, y)$ and $\|u\| = |x| + |y|$. The proof for the second component of the vector $\xi(u, h)$ is similar.

So, we consider system (4.3). For this system function (2.8) reads as

$$\varepsilon(x_0, y_0; h) = e^{ha_1}\int_0^h e^{-sa_1}a(x(s), y_0)\, ds\,; \tag{7.7}$$

here $x = x(t)$ is the solution of the Cauchy problem

$$x' = a_1 x + b_1 y_0 + a(x, y_0), \quad x(0) = x_0. \tag{7.8}$$

To prove relation (7.6), it is sufficient to show that for each $\delta > 0$ there exists $r > 0$ such that if $|x_0| + |y_0| < r$, then

$$|\varepsilon(x_0, y_0; h)| < \delta(|x_0| + |y_0|). \tag{7.9}$$

We take an arbitrary $\delta > 0$ and by this number we determine another number δ_1 by two conditions, the first being the inequality $0 < \delta_1 < \delta$, while the second will be provided a bit later.

By the first of relations (2.6), for δ_1 there exists $r_1 > 0$ such that if $|x_0| + |y_0| < r_1$, then

$$|a(x_0, y_0)| < \delta_1 \cdot (|x_0| + |y_0|). \tag{7.10}$$

By the theorem on the continuous dependence of the solution of the Cauchy problem on initial data and parameters there exists $\rho_1 > 0$ such that

$$|x_0| + |y_0| < \rho_1, \tag{7.11}$$

then the solution $x = x(t)$ of the Cauchy problem (7.8) satisfies the inequality $|x(t)| < \dfrac{r_1}{2}$ for all $t \in [0, h]$. We can suppose that $\rho_1 < \dfrac{r_1}{2}$. Then if inequality (7.11) holds, then $|x(t)| + |y_0| < r_1$ and therefore, by (7.10) we obtain

$$|a(x(t), y_0)| < \delta_1(|x(t)| + |y_0|) \tag{7.12}$$

for all $t \in [0, h]$.

The solution $x = x(t)$ of Cauchy problem (7.8) satisfies the integral equation

$$x(t) = e^{ta_1}x_0 + e^{ta_1}\int\limits_0^t e^{-sa_1}[b_1y_0 + a(x(s), y_0)]\,ds.$$

We let $M_1 = \max\limits_{0\leqslant t\leqslant h} e^{ta_1}$. Then

$$|x(t)| \leqslant M_1\left[|x_0| + |b_1y_0|h + \int\limits_0^t |a(x(s), y_0)|ds\right].$$

Under inequality (7.11) and therefore, under inequality (7.12) we have

$$|x(t)| \leqslant M_1\left[|x_0| + |b_1y_0|h + \delta_1|y_0|h + \delta_1\int_0^t |x(s)|ds\right].$$

By Grönwall inequality, see, for instance, [25], we get

$$|x(t)| \leqslant k\exp\left(\int_0^t |M_1\delta_1|ds\right) \leqslant ke^{M_1\delta_1 h},$$

where $k = M_1\left[|x_0| + |b_1y_0|h + \delta_1|y_0|h\right]$. We let $M_0 = \max\{M_1,\ M_1h(|b_1| + \delta_1)\}$. Then $k \leqslant M_0(|x_0| + |y_0|)$. This is why under inequality (7.11)

$$|x(t)| \leqslant M_0e^{M_1\delta_1 h}(|x_0| + |y_0|) \tag{7.13}$$

for all $t \in [0, h]$.

We return back to function (7.7). Under inequalities (7.11) by (7.12) and (7.13) we obtain

$$|\varepsilon(x_0, y_0; h)| \leqslant M_1\int\limits_0^h |a(x(s), y_0)|ds \leqslant M_1h(M_0e^{M_1\delta_1 h} + 1)(|x_0| + |y_0|)\delta_1.$$

We recall that for the number δ_1 we should provide the second condition; the first is the inequality $0 < \delta_1 < \delta$. This second condition is the inequality

$$\delta_1M_1h(M_0e^{M_1\delta_1 h} + 1) < \delta.$$

Then under inequality (7.11) we arrive at desired estimate (7.9). The proof of Lemma 2.5 is complete.

We proceed to proving Theorems 3.1–4.2. By Theorem 2.1, these theorems for hybrid systems (1.1) and (4.3) are implied by their analogues for corresponding discrete system (2.11).

7.4. Proof of Theorem 3.1. Let $A_1 < 0$ and $|B_2| < 1$ and let $h > 0$ be fixed. It is sufficient to show that if the quantities $\|A_2\|$ and $\|B_1\|$ are small enough, the zero equilibrium of system (2.11) is asymptotically stable. In order to do this, by Lemma 2.5 it is sufficient to show that if $\|A_2\|$ and $\|B_1\|$ are small, matrix (2.12) satisfies the condition $|A(h)| < 1$.

For small $\|A_2\|$ and $\|B_1\|$ the matrix $A(h)$ can be treated as a perturbation of the matrix

$$A_0(h) = \begin{bmatrix} e^{A_1 h} & 0 \\ 0 & B_2 \end{bmatrix}.$$

Since $A_1 < 0$, $|B_2| < 1$ and $h > 0$, then $|A_0(h)| < 1$. By the perturbation theory for linear operators, see, for instance, [26], this yields that for sufficiently small $\|A_2\|$ and $\|B_1\|$ we have $|A(h)| < 1$.

7.5. Proof of Theorem 3.2. It is sufficient to observe that by Lemma 2.5 an analogue of Theorem 3.2 holds for the zero equilibrium of system (2.11).

7.6. Proof of Theorem 3.3. It is sufficient to show that if the zero solution of system (2.11) is stable for small $h > 0$, then $|B_2| \leqslant 1$.

We argue by contradiction assuming that the zero solution of system (2.11) is stable for small $h > 0$, but the matrix B_2 possesses at least one eigenvalue μ_0 such that $|\mu_0| > 1$. Then matrix (3.4) possesses an eigenvalue μ_0. As $h = 0$, matrix (2.12) coincides with matrix (3.4). This is why for small $h > 0$ matrix (3.4) possesses an eigenvalue $\mu(h)$ such that $|\mu(h)| > 1$. Then the zero solution of system (2.11) is unstable for small $h > 0$. We have obtained a contradiction.

7.7. Proof of Theorem 3.4. It is sufficient to show that the zero solution of system (2.11) is asymptotically stable for small $h > 0$.

Since $|B_2| < 1$, by Lemma 3.1 matrix (3.4) possesses a semi-simple eigenvalue $\lambda_0 = 1$ of multiplicity n. This is why the stability nature of the zero solution of discrete system (2.11) for small $h > 0$ is determined by the behavior of the part of the spectrum of the matrix (2.12), which is obtained as a perturbation of the eigenvalue $\lambda_0 = 1$ of the matrix (3.4).

In accordance with the perturbation theory for linear operators, see, for instance, [26, Ch. II, Sect. 5, Thm. 5.4], for small $|h|$ matrix (2.12) possesses n eigenvalues $\lambda^{(j)}(h)$, $j = 1, 2, \dots, n$, such that the functions $\lambda^{(j)}(h)$ are differentiable at the point $h = 0$ and $\lambda^{(j)}(0) = 1$. These functions are represented as $\lambda^{(j)}(h) = 1 + h\lambda_1^{(j)} + o(h)$, where the coefficients $\lambda_1^{(j)}$ are the eigenvalues of the matrix D, see [17, Thm. 3.5].

Then for $D < 0$ we obtain that $|\lambda^{(j)}(h)| < 1$ for all small $h > 0$. Therefore, for small $h > 0$ matrix (2.12) satisfies the condition $|A(h)| < 1$, that is, the zero solution of system (2.11) is asymptotically stable for small $h > 0$.

7.8. Proof of Theorem 4.1. It is sufficient to prove the statement of Theorem 4.1 for the zero equilibrium of discrete system (2.11) as $n = m = 1$.

Matrix (2.12) here reads as

$$A(h) = \begin{bmatrix} e^{a_1 h} & \frac{b_1}{a_1}(e^{a_1 h} - 1) \\ a_2 e^{a_1 h} & \frac{a_2 b_1}{a_1}(e^{a_1 h} - 1) + b_2 \end{bmatrix}. \tag{7.14}$$

The corresponding characteristic equation is

$$\lambda^2 + a\lambda + b = 0, \tag{7.15}$$

where

$$a = \frac{a_2 b_1}{a_1}(1 - e^{a_1 h}) - b_2 - e^{a_1 h}, \qquad b = b_2 e^{a_1 h}.$$

It is known, see, for instance, [21, Sect. 10.1], that both roots λ_1 and λ_2 of square equation (7.15) satisfies the inequalities $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $|b| < 1$ and $|a| < b + 1$. This statement and Theorem 3.2 yield the validity of Theorem 4.1.

7.9. Proof of Theorem 4.2. Under the assumptions of this theorem matrix (3.7) is the number $\lambda_1 = (A'(0)e, g)$; here $A'(0)$ is the derivative defined by identity (7.14) of the matrix $A(h)$ at $h = 0$ and e, g are non-zero vectors such that the identities $A(0)e = e$, $(A(0))^*g = g$, $(e, g) = 1$ hold. Simple calculations show that the identity $\lambda_1 = \dfrac{\gamma}{1 - b_2}$ holds. By Theorem 3.4 this implies the validity of Theorem 4.2.

7.10. Proof of Theorem 5.1. By Theorem 2.1 and Lemmas 2.2 and 2.3 it is sufficient to show analogues of Theorems 5.1 for discrete system (2.11). In their turn, the validity of these analogues follows from the obtained in [23, Sect. 2.2] and [24, Sects. 3.2, 3.3] sufficient conditions for multiple equilibrium bifurcation (case of eigenvalue 1) and the doubling period bifurcation (case of eigenvalue -1) for discrete systems of form (2.11).

In view of this we note that the results of works [23], [24] can be also employed for a detailed study of main bifurcation scenarios of hybrid system (1.1) including the analysis of the stability of arising solutions, determining the bifurcation form (soft or hard), calculation of Lyapunov quantities, etc.

BIBLIOGRAPHY

1. V.P. Maksimov. *Continuous-discrete dynamic models* // Ufimskij Matem. Zhurn. **13**:3, 97–106 (2021). [Ufa Math. J. **13**:3, 95–103 (2021).]
2. V.P. Maksimov, A.L. Chadov. *Hybrid models in problems of economical dynamics* // Vestn. Perm. Univ. **2**:9, 13–23 (2011). (in Russian).
3. O.Ya. Shpilevaya, K.Yu. Kotov. *Switched systems: Stability and design* // Avtometriya. **44**:5, 71–87 (2008). [Optoelectronics, Instrumentation and Data Processing. **44**:5, 439–449 (2008).]
4. O.S. Logunova, E.B. Agapitov, I.I. Barankova, S.M. Andreev, G.N. Chusavitina. *Mathematical models for studying bodies heat states and control of heat processes* // Elektrotekhnicheskie sistemy i kompleksy. **2**:43, 25–34 (2019). (in Russian).
5. S.N. Vasiliev, A.I. Malikov. *On some results on stability of switching and hybrid systems* // in Coll. Sci. Pap. “Actual problems in solid state mechanics. To 20th anniversary of IMM KazSC RAS”. Foliant, Kazan **1**, 23–81 (2011). (in Russian).
6. M.S. Branicky. *Multiple Lyapunov functions and other analysis tools for switched and hybrid systems* // IEEE Trans. Automat. Contr. **43**:4, 475–482 (1998).
7. R.A. Decarlo, M.S. Branicky, S. Pettersson, B. Lennartson. *Perspectives and results on the stability and Stabilizeability of hybrid systems* // Proc. IEEE. **88**:7, 1069–1082 (2000).
8. D. Liberzon. *Switching in systems and control.* Birkhauser, Boston (2003).
9. P.A. Lakrisenko. *On stability of equilibria of nonlinear hybrid mechanical systems* // Vestn. St.-Peterb. Univ. Ser. 10. Prikl. Matem. Inform. Prots. Uprav. **3**, 116–125 (2015).
10. A.V. Platonov. *Stability analysis of nonstationary switched systems* // Izv. VUZov. Matem. **2**, 63–73 (2020). [Russ. Math. (Iz. VUZ). **64**:2, 56–65 (2020).]
11. A.Yu. Aleksandrov, A.V. Platonov. *On the asymptotic stability of solutions of hybrid multivariable systems* // Avtomat. Telemekh. **5**, 18–30 (2014). [Autom. Remote Control. **75**:5, 818–828 (2014).]
12. R. Shorten, F. Wirth, O. Mason, K. Wulff, C. King. *Stability criteria for switched and hybrid systems* // SIAM Rev. **49**:4, 545–592 (2007).
13. L. Hou, A. Michel. *Unifying theory for stability of continuous, discontinuous, and discrete-time dynamical systems* // Nonl. Anal. Hybrid Systems. **1**:2, 154–172 (2007).
14. I.L.D. Santos, G.N. Silva. *Some results in stability analysis of hybrid dynamical systems* // Tend. Mat. Apl. Comput. **8**:3, 453–462 (2007).
15. V.M. Marchenko, J.-J. Loiseau. *On the stability of hybrid difference-differential systems* // Diff. Uravn. **45**:5, 728–740 [Diff. Equat. **45**:5, 743–756 (2009).]

16. P.M. Simonov. *Stability and asymptotically periodic solutions of hybrid systems with aftereffect* // Itogi Nauki Tekhn. Ser. Sovrem. Matem. Pril. Temat. Obz. **168**, 91–98 (2019). [J. Math. Sci. **262**:6, 855-862 (2022)]
17. M.G. Yumagulov, L.S. Ibragimova, A.S. Belova. *Methods for studying the stability of linear periodic systems depending on a small parameter* // Itogi Nauk. Tekhn. Ser. Sovrem. Mat. Pril. Temat. Obz. **163**, 113–126 (2019). [J. Math. Sci. **258**:1, 115–127 (2021).]
18. M.G. Yumagulov, S.V. Akmanova. *Stability and bifurcations of continuous-discrete dynamical systems with a constant discretization step* // Vestnik BashGU. **26**:4, 862–865 (2021). (in Russian).
19. A.S. Bortakovskii. *Sufficient optimality conditions for controlling switched systems* // Izv. RAN. Teor. Sist. Upravl. 4, 86–103 (2017). [J. Comput. Syst. Sci. Int. **56**:4, 636–651 (2017).]
20. M.A. Krasnosel'skii. *The operator of translation along the trajectories of differential equations.* Nauka, Moscow (1966). [Amer. Math. Soc., Providence, RI (1968).]
21. L.P. Shilnikov, A.L. Shilnikov, D.V. Turaev, L.O. Chua. *Methods of qualitative theory in nonlinear dynamics. II.* Inst. Komput. Issl. Moscow (2009). [World Scientific, Singapore (2001).]
22. B.S. Bardin, A.P. Markeev. *The stability of the equilibrium of a pendulum for vertical oscillations of the point of suspension* // Prikl. Matem. Mekh. **59**:6, 922–929 (1995). [J. Appl. Math. Mech. **59**:6, 879–886 (1995).]
23. A.A. Vyshinskiy, L.S. Ibragimova, S.A. Murtazina, M.G. Yumagulov. *An operator method for approximate investigation of a regular bifurcation in multiparameter dynamical systems* // Ufimskij Matem. Zhurn. **2**:4, 3–26 (2010). (in Russian).
24. N.I. Gusarova, S.A. Murtazina, M.F. Fazlytdinov, M.G. Yumagulov. *Operator methods for calculating Lyapunov values in problems on local bifurcations of dynamical systems* // Ufimskij Matem. Zhurn. **10**:1, 25–49 (2018). [Ufa Math. J. **10**:1, 25–48 (2018).]
25. M. Roseau. *Nonlinear oscillations and stability theory.* Nauka, Moscow (1971). [*Vibrations non linéaires et théorie de la stabilité.* Springer-Verlag, Berlin (1966). (in French).]
26. T. Kato. *Perturbation theory for linear operators.* Springer-Verlag, Berlin (1966).

Marat Gayazovich Yumagulov,
Bashkir State University,
Zaki Validi str. 32,
450074, Ufa, Russia
E-mail: yum_mg@mail.ru

Svetlana Vladimirovna Akmanova,
Nosov Magnitogorsk State Technical University,
Lenin av. 38,
450000, Magnitogorsk, Russia
E-mail: svet.akm_74@mail.ru