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INTEGRABLE TOLKYNAY EQUATIONS AND RELATED YAJIMA-OIKAWA TYPE EQUATIONS

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Abstract. We consider some nonlinear models describing resonance interactions of long waves and short-waves (shortly, the LS waves models). Such LS models were derived and proposed due to various motivations, which mainly come from the different branches of modern physics, especially, from the fluid and plasma physics. In this paper, we study some of integrable LS models, namely, the Yajima-Oikawa equation, the Newell equation, the Ma equation, the Geng-Li equation and their different modifications and extensions. In particular, the gauge equivalent counterparts of these integrable LS models (equations), namely, different integrable spin systems are constructed. In fact, these gauge equivalent counterparts of these LS equations are integrable generalized Heisenberg ferromagnet type models (equations) (HFE) with self-consistent potentials (HFESCP). The associated Lax representations of these HFESCP are presented. Using these Lax representations of these HFESCP, they can be studied by the inverse scattering method. For instance, the equivalence established using the Lax representation also makes it possible to find a connection between the solutions of the corresponding integrable equations.

Keywords: Integrable equations, Heisenberg ferromagnet equation, Yajima-Oikawa equation, gauge equivalent, Lax representation.

Mathematics Subject Classification: 35C08, 35Q51

1. Introduction

Nonlinear models of long wave-short wave resonant interactions, the so-called LS equations, play an important role in modern physics as well as in modern mathematics [1]-[3]. Some of these models are integrable nonlinear partial differential equations in 1+1 and 2+1 dimensions (see, e.g. [4]-[13] and references therein). In 1+1 dimensions, such LS equations of long wave-short wave resonant interactions, formally in general, can be written as

$$iq_t + q_{xx} + f_1(v, v_x, q, \bar{q}, \ldots) = 0, \qquad v_t + \delta(|q|^2)_x = 0, \tag{1.1}$$

where $q(x,t)$ represents an envelope (complex-valued) of the short wave and $v(x,t)$ denotes a real-valued amplitude of the long wave (potential), $\delta = const$. Here f_1 is some function of its arguments. System of equations (1.1) can be considered as the nonlinear Schrödinger type equations with self-consistent potentials. It combines all well-known integrable LS models, namely, the Yajima-Oikawa equation (YOE), the Newell equation (NE), the Geng-Li equation (GLE) etc. This outcome is similar to the one that proves that the Korteweg-de Vries and the modified KdV equations are just two particular cases of the Gardner equation. There exists another interesting class of integrable systems, namely, integrable generalized spin systems or

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Heisenberg ferromagnet equations with self-consistent potentials (GHFESCP) (see, e.g. [14]–[18] and references therein). Such spin systems in general can be written as

$$iS_t + f_2(S, S_x, S_{xx}, u, u_x, \ldots) = 0, \qquad u_t + \nu\,\mathrm{tr}(S[S_x, S_t])_x = 0, \tag{1.2}$$

where a 2×2 spin matrix S reads as

$$S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix} \tag{1.3}$$

and satisfies the following condition

$$S^2 = I. \tag{1.4}$$

Some time we use the following form of integrable generalized spin systems

$$iR_t + f_3(R, R_x, R_{xx}, u, u_x, \ldots) = 0, \tag{1.5}$$

where, in contrast to (1.3) and (1.4), the symbol R denotes a 3×3 spin matrix satisfying the condition

$$R^3 = \epsilon R, \qquad (\epsilon = \pm 1).$$

In the above equations, f_2 and f_3 are some matrix functions of their arguments, u is a real function (potential) and (ν, δ) are real constants.

The set of equations (1.1) is some kind generalizations of the nonlinear Schrödinger equation (NLSE):

$$iq_t + q_{xx} + 2\nu|q|^2 q = 0. \tag{1.6}$$

At the same time system (1.2) and (1.5) are certain extensions of the following Heisenberg ferromagnet equation (HFE)

$$iS_t + \frac{1}{2}[S, S_{xx}] = 0. \tag{1.7}$$

Both NLSE (1.6) and HFE (1.7) are integrable and admit some integrable extentions in 1+1 and 2+1 dimensions (see, e.g., [19]–[38] and references therein). It is well-known that between NLSE (1.6) and HFE (1.7) are related by a gauge and geometrical equivalence [39]-[40]. The aim of this paper is to find such gauge equivalence between some particular reductions of systems of equations (1.1), (1.2) and (1.5).

This paper is organized as follows. In Section 2, the integrable Tolkynay equation (TE) is presented. The Yajima-Oikawa-Mewell equation (YONE) is considered in Section 3 and its gauge equivalence with the TE is studied in Section 4. In the next two Sections 5 and 6 the gauge equivalent counterparts of the Yajima-Oikawa equation (YOE) and the Ma equation (ME) are presented. The relation between the TE and NE is established in Section 7. The same problem was studied in Section 8 for the Geng-Li equation (GLE). In Section 9, the M-XXXIV equation is investigated. In Section 10, the M-V equation and its relation with the LS equations were considered. The last section is devoted to some conclusions and discussions.

2. The Tolkynay equation

In this paper, in particular, we study the following Tolkynay equation (TE)

$$iR_t + [R^2, R_x]_x = 0. \tag{2.1}$$

Here the spin matrix R has the form

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix},$$

and satisfies the following conditions

$$R^3 = R, \qquad tr(R) = 0, \qquad det(R) = 0.$$

The TE is one of integrable generalized Heisenberg ferromagnet type equation. The Lax representation (LR) of the TE is given by

$$\Psi_x = U_1\Psi, \qquad \Psi_t = V_1\Psi,$$

where

$$U_1 = i\lambda R, \qquad V_1 = -i\lambda^2\left(R^2 - \frac{2}{3}I\right) - \lambda[R^2, R_x].$$

3. The YONE

One of most general integrable LS equations is the Yajima-Oikawa-Newell equation (YONE) [8]

$$iq_t + q_{xx} + (i\alpha v_x + \alpha^2 v^2 - \beta v - 2\alpha|q|^2)q = 0, \qquad v_t - 2(|q|)_x = 0, \tag{3.1}$$

where the parameters α, β are arbitrary real constants. These parameters may be considered as independent constants describing a long-short wave cross-interaction. This system reduces to the YOE for $\alpha = 0$, $\beta = 1$ and to the Newell equation as $\alpha = \sigma$, $\beta = 0$. System YONE (3.1) is integrable. Its LR has the form [8]

$$\Phi_x = U_2\Phi, \qquad \Phi_t = V_2\Phi,$$

where

$$U_2 = i\lambda\Sigma + Q, \qquad V_2 = -\lambda^2 B_2 + i\lambda B_1 + B_0.$$

Here

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & q & iv \\ \alpha\bar{q} & 0 & \bar{q} \\ i(\alpha^2 v - \beta) & \alpha q & 0 \end{pmatrix},$$

$$B_2 = \frac{i}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & iq & 0 \\ i\alpha\bar{q} & 0 & -i\bar{q} \\ 0 & -i\alpha q & 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} -i\alpha|q|^2 & -\alpha v q + iq_x & i|q|^2 \\ -\alpha^2 v\bar{q} + \beta\bar{q} - i\alpha\bar{q}_x & 2i\alpha|q|^2 & -\alpha v\bar{q} - i\bar{q}_x \\ i\alpha^2|q|^2 & -\alpha^2 vq + \beta q + i\alpha q_x & -i\alpha|q|^2 \end{pmatrix}.$$

4. Gauge equivalence between TE and YONE

In this section we establish the gauge equivalence between TE (2.1) and YONE (3.1). In order to do this, we consider the gauge transformation

$$\Psi = g^{-1}\Phi, \qquad g = \Phi_{\lambda=0}.$$

We obtain

$$\Psi_x = U_1\Psi, \qquad \Psi_t = V_1\Psi,$$

where

$$U_1 = g^{-1}(U_2 - g_x g^{-1})g, \qquad V_1 = g^{-1}(V_2 - g_t g^{-1})g.$$

As a result, we have

$$U_1 = i\lambda R, \qquad V_1 = -\lambda^2 g^{-1}B_2 g + i\lambda g^{-1}B_1 g,$$

where

$$R = g^{-1}\Sigma g.$$

After some calculations, we obtain

$$Q\Sigma + \Sigma Q = \begin{pmatrix} 0 & q & 0 \\ \alpha\bar{q} & 0 & -\bar{q} \\ 0 & -\alpha q & 0 \end{pmatrix} = -iB_1,$$

or

$$g^{-1}Qgg^{-1}\Sigma g + g^{-1}\Sigma gg^{-1}Qg = g^{-1}\begin{pmatrix} 0 & q & 0 \\ \alpha\bar{q} & 0 & -\bar{q} \\ 0 & -\alpha q & 0 \end{pmatrix} = -ig^{-1}B_1 g = g^{-1}QgR + Rg^{-1}Qg.$$

On the other hand,

$$R_x = g^{-1}[\Sigma, Q]g = g^{-1}\begin{pmatrix} 0 & q & 2iv \\ -\alpha\bar{q} & 0 & \bar{q} \\ -2i(\alpha^2 v - \beta) & -\alpha q & 0 \end{pmatrix} g.$$

Hence, we get

$$[R, R_x] = g^{-1}\begin{pmatrix} 0 & q & 4iv \\ \alpha\bar{q} & 0 & \bar{q} \\ 4i(\alpha^2 v - \beta) & \alpha q & 0 \end{pmatrix} g = g^{-1}Qg + 3ig^{-1}\begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ (\alpha^2 v - \beta) & 0 & 0 \end{pmatrix} g,$$

and therefore,

$$g^{-1}Qg = [R, R_x] - 3ig^{-1}\begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ (\alpha^2 v - \beta) & 0 & 0 \end{pmatrix} g.$$

Taking into consideration the formula

$$\Sigma\begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ (\alpha^2 v - \beta) & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ (\alpha^2 v - \beta) & 0 & 0 \end{pmatrix}\Sigma = 0,$$

we obtain

$$g^{-1}B_1 g = i(g^{-1}Qgg^{-1}\Sigma g + g^{-1}\Sigma gg^{-1}Qg) = i([R, R_x]R + R[R, R_x]) = i[R^2, R_x].$$

Thus, we have shown that

$$g^{-1}B_2 g = -\frac{2i}{3}I + iR^2, \qquad g^{-1}B_1 g = i[R^2, R_x].$$

Finally, we have the following Lax pair for TE (2.1)

$$U_1 = i\lambda R, \qquad V_1 = -i\lambda^2(R^2 - \frac{2}{3}I) - \lambda[R^2, R_x].$$

We provide some useful formulas:

$$\operatorname{tr}(R_x^2) = -4\alpha|q|^2 + 8v(\alpha^2 v - \beta), \qquad \det(R_x) = 2i\beta|q|^2.$$

As integrable equation, the YONE admits the infinity number of integrals of motion. For example, here we present the following integral of motion for the YONE (3.1):

$$P = \int J\, dx,$$

where a, b are some constants and

$$J = 4a[\alpha(5\beta - 1)|q|^2 + 2v(\alpha^2 v - \beta)] = a\operatorname{tr}(S_x^2) + b\det(S_x), \qquad b = \frac{10\alpha a}{i\beta}.$$

In fact, the quantity J satisfies the following conservation equation

$$J_t = 16a[i\alpha(\bar{q}_x q - \bar{q}q_x) - \beta|q|^2 + 2\alpha^2 v|q|^2]_x,$$

and hence,

$$P_t = \left(\int J\,dx\right)_t = 0$$

with the boundary conditions

$$\lim q(x,t) \to 0, \qquad \lim v(x,t) \to 0 \qquad as \qquad x \to \pm\infty.$$

5. Gauge equivalence between the YOE and the MM-IIE

The first example of integrable long-short waves interactions models is the following Yajima-Oikawa equation (YOE) [1]

$$iq_t + \frac{1}{2}q_{xx} - uq = 0, \qquad u_t + u_x + |q|^2_x = 0. \tag{5.1}$$

Its Lax representation reads as

$$U_3 = A_0 + 2i\lambda\Sigma + (2\lambda)^{-1}A_{-1},$$

$$V_3 = -U + 2i\lambda^2 A_1^2 + \lambda B_1 + B_0 + i(4\lambda)^{-1}\begin{pmatrix} |\Phi|^2 & 0 & |\Phi|^2 \\ \Phi_x & 0 & \Phi_x \\ -|\Phi|^2 & 0 & -|\Phi|^2 \end{pmatrix},$$

where

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} 0 & -\Phi^* & 0 \\ 0 & 0 & 0 \\ 0 & \Phi^* & 0 \end{pmatrix}, \qquad A_{-1} = \begin{pmatrix} -ni & 0 & -ni \\ \Phi & 0 & \Phi \\ ni & 0 & ni \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & -\Phi^* & 0 \\ 0 & 0 & 0 \\ 0 & \Phi^* & 0 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & \frac{i}{2}\Phi_x^* + \Phi^* & 0 \\ \frac{1}{2}\Phi & 0 & -\frac{1}{2}\Phi \\ 0 & -\frac{i}{2}\Phi_x^* - \Phi^* & 0 \end{pmatrix},$$

$$B_{-1} = \begin{pmatrix} |\Phi|^2 & 0 & |\Phi|^2 \\ \Phi_x & 0 & \Phi_x \\ -|\Phi|^2 & 0 & -|\Phi|^2 \end{pmatrix}, \qquad \Phi = qe^{i(\frac{t}{2}-x)}.$$

We consider a gauge transformation

$$\phi = g\bar{\phi}, \qquad g(x,t,\lambda_0) = \phi(x,t,\lambda)|_{\lambda=\lambda_0} \subset GL(\xi,\ C)$$

as $\lambda = \lambda_0$, where

$$g_x = U_3(x,t,\lambda_0)g, \qquad g_t = V_3(x,t,\lambda_0)g,$$

Then

$$U_4 = g^{-1}U_3 g - g^{-1}g_x, \qquad V_4 = g^{-1}V_3 g - g^{-1}g_t.$$

We obtain:

$$U_4 = 2i(\lambda-\lambda_0)g^{-1}A_1 g - \frac{\lambda-\lambda_0}{4\lambda_0\lambda}g^{-1}A_{-1}g,$$

$$V_4 = 2i(\lambda^2-\lambda_0^2)g^{-1}A_1^2 g - U_4 - \frac{i(\lambda-\lambda_0)}{4\lambda_0\lambda}g^{-1}B_{-1}g + (\lambda-\lambda_0)B_1.$$

We introduce two new matrices R and σ

$$R = g^{-1}\Sigma g, \qquad \sigma = g^{-1}A_{-1}g, \qquad R^3 = R.$$

Then

$$U_4 = 2i(\lambda-\lambda_0)[R + (4\pi\lambda_0\lambda)^{-1}\sigma],$$

$$V_4 = -\tilde{U} + (\lambda-\lambda_0)\{2i(\lambda+\lambda_0)R^2 + R(R^2)_x + \lambda^{-1}[i(4\lambda_0)^{-1}[\sigma, R^2] + \frac{1}{2}[\sigma, R^2]R]\}.$$

This Lax pair gives the following Makhankov-Myrzakulov-II equation (MM-IIE) [24]

$$R_t + R_x + (\frac{i}{2}RR_x^2 - 2\lambda_0 R_x^2)_x + (4i\lambda_0)^{-1}h = 0,$$
$$\sigma_t + \sigma_x + (\frac{1}{2}[\sigma, R^2]_x - i\lambda_0[\sigma, R^2])_x + i\lambda_0 h = 0,$$

where

$$h = [\sigma(R_x - 2i\lambda_0 I), R^2] + ([R^2, \sigma]R)_x,$$
$$\mathrm{tr}(U_4) = \mathrm{tr}(V_4) = \mathrm{tr}(U_3) = \mathrm{tr}(V_3) = \mathrm{tr}(R) = \mathrm{tr}(\sigma) = 0.$$

6. The MM-IE and its relation with the Ma equation

Let us consider the following Makhankov-Myrzakulov-I equation (MM-IE) [24]

$$iR_t + 2[R, R_{xx}] - 4(R_xR)_x = 0, \tag{6.1}$$

where the spin matrix R satisfies the following condition

$$R^3 = R.$$

The MM-IE is integrable. The corresponding LR has the form

$$\Psi_x = U_5\Psi,$$
$$\Psi_t = V_5\Psi,$$

where

$$U_5 = i\lambda R, \qquad V_5 = 2i\lambda^2 R^2 + 2\lambda(R^2)_x.$$

Note that the MM-IE is gauge equivalent to the following Ma equation (ME) [24]

$$iq_t - 2q_{xx} + 2uq = 0, \qquad u_t + |q|_x^2 = 0. \tag{6.2}$$

After some simple scale transformations, the ME takes becomes YOE (5.1). In contrast to the YOE, for the ME the LR takes a simpler form [3]

$$\Phi_x = U_6\Phi, \qquad \Phi_t = V_6\Phi,$$

where

$$U_6 = C_0 + i\lambda\Sigma, \qquad V_6 = D_0 + \lambda D_1 + 2i\lambda^2 D_2.$$

Here

$$C_0 = \begin{pmatrix} 0 & \frac{E}{2} & in \\ 0 & 0 & \frac{E^*}{2} \\ -i & 0 & 0 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$D_0 = \begin{pmatrix} 0 & -iE_x & -\frac{i}{2}|E|^2 \\ -E^* & 0 & iE_x^* \\ 0 & -E & 0 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 0 & E & 0 \\ 0 & 0 & -E^* \\ 0 & 0 & 0 \end{pmatrix}, \qquad D_2 = \Sigma^2.$$

As we mentioned, the MM-IE and the ME are gauge equivalent one to other. In order to prove this, let us consider the gauge transformation $\Psi = \omega^{-1}\Phi$, where $\omega = \Phi(x, t, \lambda)|_{\lambda=0}$. Then we have

$$\omega_x = U_{6_0}\omega, \qquad \omega_t = V_{6_0}\omega,$$

where

$$U_{6_0} = U_1 \mid_{\lambda=0}, \qquad V_{6_0} = V_1 \mid_{\lambda=0}.$$

As a result, we get

$$U_5 = \omega^{-1}U_6\omega - \omega^{-1}\omega_x, \quad V_5 = \omega^{-1}V_6\omega - \omega^{-1}\omega_t,$$

or

$$U_6 = i\lambda\omega^{-1}\Sigma\omega, \quad V_6 = \lambda\omega^{-1}D_1\omega + 2i\lambda^2\omega^{-1}D_2\omega.$$

After some calculation we obtain

$$\omega^{-1}D_1\omega = 2(R^2)_x, \quad \omega^{-1}D_2\omega = R^2,$$

where

$$R = \omega^{-1}\Sigma\omega.$$

The matrix function R satisfies the following conditions

$$\mathrm{tr}(R) = 0, \quad \det(R) = \frac{i}{2}|E|^2, \qquad \det(R_x) = \det(\omega^{-1}[C_1, C_0]\omega) = \frac{i}{2}|E|^2.$$

It is not difficult to confirm that the matrix function R satisfies MM-IE (6.1). This is proves that MM-IE (6.1) and ME (6.2) are gauge equivalent one to the other.

7. The TE and its relation with the Newell equation

The TE reads as

$$iR_t + [R^2, R_x]_x = 0, \tag{7.1}$$

where the spin matrix R is of the form

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & iv \\ -\sigma\bar{R}_{12} & 0 & \sigma\bar{R}_{12} \\ -iv & -R_{12} & -R_{11} \end{pmatrix} = \begin{pmatrix} R_3 & R^- & iv \\ -\sigma R^+ & 0 & \sigma R^+ \\ -iv & -R^- & -R_3 \end{pmatrix},$$

and satisfies the following conditions

$$R^3 = R, \qquad \mathrm{tr}(R) = 0, \qquad \det(R) = 0.$$

Here $\sigma = \pm 1$, $R_{12} = R^- = R_1 - iR_2$ is a complex function and $R_{11} = R_3, R_1, R_2, v$ are real functions. Let us calculate some useful expressions

$$R^2 = \begin{pmatrix} R_{11}^2 - \sigma|R_{12}|^2 + v^2 & (R_{11} - iv)R_{12} & \sigma|R_{12}|^2 \\ -(R_{11} + iv)\sigma\bar{R}_{12} & -2\sigma|R_{12}|^2 & -(R_{11} + iv)\sigma\bar{R}_{12} \\ \sigma|R_{12}|^2 & (R_{11} - iv)R_{12} & R_{11}^2 - \sigma|R_{12}|^2 + v^2 \end{pmatrix},$$

$$R^3 = R^2R = R = (R_{11}^2 - 2\sigma|R_{12}|^2 + v^2)\begin{pmatrix} R_{11} & R_{12} & iv \\ -\sigma\bar{R}_{12} & 0 & \sigma\bar{R}_{12} \\ -iv & -R_{12} & -R_{11} \end{pmatrix} = (R_{11}^2 - 2\sigma|R_{12}|^2 + v^2)R.$$

We hence obtain the following condition

$$R_{11}^2 - 2\sigma|R_{12}|^2 + v^2 = R_3^2 - 2\sigma|R^+|^2 + v^2 = R_3^2 - 2\sigma(R_1^2 + R_2^2) + v^2 = 1.$$

The TE is one of integrable generalized Heisenberg ferromagnet type equation. The LR of the TE is given by

$$\Psi_x = U_7\Psi, \qquad \Psi_t = V_7\Psi,$$

where

$$U_7 = i\lambda R, \qquad V_7 = -i\lambda^2\left(R^2 - \frac{2}{3}I\right) - \lambda[R^2, R_x].$$

It is not difficult to confirm that the gauge equivalent of TE (7.1) is a Newell equation (NE) which reads as [2]

$$iq_t + q_{xx} + (iu_x + u^2 - 2\sigma|q|^2)q = 0, \qquad u_t - 2\sigma(|q|^2)_x = 0,$$

where $\sigma = \pm 1$. Note that in addition to a long wave-short wave coupling, the short wave has the same self-interaction as the NLSE (1.6). The NE is integrable. Its Lax representation is [2], [11]

$$\Phi_x = U_8\Phi, \qquad \Phi_t = V_8\Phi,$$

where

$$U_8 = \begin{pmatrix} i\lambda & q & iv \\ \sigma\bar{q} & 0 & \sigma\bar{q} \\ iv & q & -i\lambda \end{pmatrix}, \quad V_8 = \begin{pmatrix} -\frac{1}{3}i\lambda^2 - i\sigma|q|^2 & -\lambda q + iq_x - vq & i\sigma|q|^2 \\ -\sigma(\lambda\bar{q} + \bar{q}_x + v\bar{q}) & \frac{2}{3}i\lambda^2 + 2i\sigma|q|^2 & \sigma(\lambda\bar{q} - i\bar{q}_x - v\bar{q}) \\ i\sigma|q|^2 & \lambda q + iq_x - vq & -\frac{1}{3}i\lambda^2 - i\sigma|q|^2 \end{pmatrix}.$$

It is not difficult to verify that these matrices satisfy the following conditions

$$U^+(\lambda) = -AU(\bar{\lambda})A, \qquad V^+(\lambda) = -AV(\bar{\lambda})A,$$
$$U^+(\lambda) = -BV(-\lambda)B, \qquad U^+(\lambda) = -BV(-\lambda)B,$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

8. Geng-Li equation

Again we consider TE

$$iR_t + [R^2, R_x]_x = 0. \tag{8.1}$$

In contrast to the previous cases, now we assume that the spin matrix R satisfies the conditions

$$R^3 = -R, \qquad \mathrm{tr}(R) = 0, \qquad \det(R) = 0. \tag{8.2}$$

Then the LR for TE (8.1) becomes

$$\Psi_x = U_9\Psi, \qquad \Psi_t = V_9\Psi, \tag{8.3}$$

where

$$U_9 = \lambda R, \qquad V_9 = -i\lambda^2 R^2 + i\lambda[R^2, R_x].$$

Now we assume that the matrix R can be written as

$$R = g^{-1}Jg,$$

where

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We consider the gauge transformation

$$\Phi = g\Psi.$$

Hence, it follows from equations (8.3) that the new matrix function Φ satisfies the equations

$$\Phi_x = U_8\Phi, \qquad \Phi_t = V_8\Phi,$$

where

$$U_8 = \begin{pmatrix} iv & -\lambda & 0 \\ \lambda & 0 & -\bar{q} \\ 0 & q & 0 \end{pmatrix}, \qquad V_8 = \begin{pmatrix} i\lambda^2 - 2i|q|^2 & 0 & -i\lambda\bar{q} \\ 0 & i\lambda^2 - i|q|^2 & i\bar{q}_x + v\bar{q} \\ -i\lambda q & iq_x - vq & i|q|^2 \end{pmatrix}.$$

The compatibility condition

$$U_{8t} - V_{8x} + [U_8, V_8] = 0$$

gives the Geng-Li equation (GLE) [4]

$$iq_t + q_{xx} + 2|q|^2q + i(vq)_x = 0, \qquad v_t + 2(|q|^2)_x = 0. \tag{8.4}$$

This proves that TE (8.1) with conditions (8.2) and LR (8.3) is gauge equivalent to GLE (8.4).

9. The M-XXXIV equation

One of integrable HFE with self-consistent potentials (HFESCP) is the following Myrzakulov-XXXIV (M-XXXIV) equation

$$\mathbf{S}_t + \mathbf{S}\wedge\mathbf{S}_{xx} - u\mathbf{S} = 0, \qquad u_t + \frac{1}{2}(\mathbf{S}_x^2)_x = 0,$$

where $\mathbf{S} = (S_1, S_2, S_3)$ is the unit spin vector that is $\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2 = 1$ and u is a real function (potential). The M-XXXIV equation is integrable. The corresponding LR has the form

$$\alpha\Psi_y = \frac{1}{2}[S+I]\Psi_x, \qquad \Psi_t = \frac{i}{2}[S+(2b+1)I]\Psi_{xx} + \frac{i}{2}W\Phi_x, \tag{9.1}$$

where $W = W_1 + W_2$ and

$$W_1 = (2b+1)E + (2b+\frac{1}{2})SS_x + (2b+1)FS,$$

$$W_2 = FI + \frac{1}{2}S_x + ES + \alpha SS_y, \qquad S^{\pm} = S_1 \pm iS_2,$$

$$E = -\frac{i}{2\alpha}u_x, \qquad F = \frac{i}{2}\left(\frac{u_x}{\alpha} - 2u_y\right), \qquad S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}.$$

In fact, the compatibility condition $\Psi_{yt} = \Psi_{ty}$ gives the following system of equations

$$iS_t + \frac{1}{2}[S, S_{\xi\xi}] - iwS_\xi = 0,$$

$$w_\eta - \frac{1}{4i}tr(S[S_\xi, S_\eta]) = 0,$$

where

$$\xi = x + \frac{1}{\alpha}y, \qquad \eta = -x, \quad w = u_\xi.$$

Hence after the simple transformation $\eta = t$, $w \to u$, $\xi \to x$, we obtain

$$iS_t + \frac{1}{2}[S, S_{xx}] - iuS_x = 0, \qquad u_t - \frac{1}{4i}tr(S[S_x, S_t]) = 0. \tag{9.2}$$

The following equations hold:

$$[S_x, S_t] = -i(S_x^2)_x S, \qquad S[S_x, S_t] = -i(S_x^2)_x I, \qquad S_x^2 = \mathbf{S}_x^2 I, \qquad S[S_x, S_t] = -i(\mathbf{S}_x^2)_x I,$$

and

$$tr(S[S_x, S_t]) = -2i(\mathbf{S}_x^2)_x.$$

Hence, M-XXXIV equation (9.2) can be written as

$$iS_t + \frac{1}{2}[S, S_{xx}] - iuS_x = 0, \qquad u_t + \frac{1}{2}(\mathbf{S}_x^2)_x = 0. \tag{9.3}$$

Let us find the equation which is gauge equivalent to M-XXXIV equation (9.3). To this end, we consider the following tranformation

$$\Phi = g^{-1}\Psi, \tag{9.4}$$

where Ψ is the matrix solution of linear problem (9.1), Φ and g are a temporally unknown matrix functions. Substituting (9.4) into (9.1), after some calculations we get

$$\alpha\Phi_y = B_1\Phi_x + B_0\Phi, \qquad \Phi_t = iC_2\Phi_{xx} + C_1\Phi_x + C_0\Phi, \tag{9.5}$$

with

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix},$$
$$C_2 = \begin{pmatrix} b+1 & 0 \\ 0 & b \end{pmatrix}, \qquad C_1 = \begin{pmatrix} 0 & iq \\ ir & 0 \end{pmatrix}, \qquad C_0 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$
$$c_{12} = i(2b+1)q_x + i\alpha q_y, \qquad c_{21} = -2ibr_x - i\alpha r_y.$$

Here c_{jj} satisfy the following system of equations

$$c_{11x} - \alpha c_{11y} = iqr_x + rc_{12} - qc_{21}, \qquad \alpha c_{22y} = -irq_x + rc_{12} - qc_{21}.$$

The compatibility condition of equations (9.5) gives the following (2+1)-dimensional nonlinear Schrödinger equation:

$$iq_t + q_{\xi\xi} + vq = 0, \qquad ir_t - r_{\xi\xi} - vr = 0, \qquad v_\eta + 2(rq)_\xi = 0,$$

or, after $\eta \to t$, we have

$$iq_t + q_{xx} + vq = 0, \qquad ir_t - r_{xx} - vr = 0, \qquad v_t + 2(rq)_x = 0.$$

It coincide with ME (6.2). Thus, we have presented a new LR for the ME or for the YOE. Consequently, we have found a new form of the gauge equivalent counterpart of the ME and/or the YOE, namely, the M-XXXIV equation.

10. The M-V equation

Our next example of integrable generalized HFE is the so-called Myrzakulov-V (M-V) equation. The M-V equation reads as

$$iR_t + 2[R, R_y]_x + 3(R^2R_yR)_x = 0$$

or

$$iR_t + \frac{1}{2}[R, R_y]_x + \frac{3}{2}[R^2, (R^2)_y]_x = 0, \tag{10.1}$$

where the spin matrix R satisfies the conditions

$$R^3 = R, \qquad \mathrm{tr}(R) = 0, \qquad \det(R) = 0.$$

M-V equation (10.1) is a (2+1)-dimensional integrable equation. Its LR reads as

$$\Psi_x = U_1\Psi, \qquad \Psi_t = V_1\Psi,$$

where

$$U_1 = -i\lambda R, \qquad V_1 = -2i\lambda^2 R + \frac{\lambda}{2}\left([R, R_y] + 3[R^2, (R^2)_y]\right).$$

In order to find its gauge equivalent equation, we consider the transformation

$$R = \Phi^{-1}\Sigma\Phi,$$

where $\Sigma = \mathrm{diag}(1, 0, -1)$. Let us we assume that Φ satisfies the following equations

$$\Phi_x = -i\lambda\Sigma + Q, \qquad \Phi_t = (\mu_2\lambda^2 + \mu_1\lambda + \mu_0)\Phi_y + V\Phi,$$

with a given matrix Q; here V is an unknown matrix and $\mu_j = consts$. The compatibility condition $\Phi_{xt} = \Phi_{tx}$ gives the following two equations:

$$Q_t - V_x + [U, V] - (\mu_2\lambda^2 + \mu_1\lambda + \mu_0)Q_y = 0 \tag{10.2}$$

and

$$\lambda_t - (\mu_2\lambda^2 + \mu_1\lambda + \mu_0)\lambda_y = 0.$$

Equation (10.2) is the desired nonlinear Schrödinger type equation coupled with the equation for the potential $v(x,t)$. At the same time, equation (10.2) indicates that in this case, we have a nonisospectral problem, where $\lambda = \lambda(y,t)$.

11. Conclusions

Nonlinear models describing interactions of long and short (LS) waves are given by the Yajima-Oikawa type equations. These long wave–short wave interaction models were derived and proposed with various motivations, which mainly come from fluid and plasma physics. It is well known that in these long wave–short wave equations is that a long wave always arises as generated by short waves. In this paper, we study some of integrable LS models, namely, the Yajima-Oikawa equation, the Newell equation, the Ma equation, the Geng-Li equation and etc. Any integrable equations admitting the Lax representations, generally speaking, are gauge equivalent to some integrable generalized HFE. In this context, it is interesting to find the gauge equivalent counterparts of the above mentioned integrable LS models. In this paper, the gauge equivalent counterparts of integrable LS models (equations) are found. In fact, these gauge equivalents of the LS equations are integrable generalized Heisenberg ferromagnet equations with self-consistent potentials (HFESCP). The associated Lax representations of these HFESCP are given.

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