ISSN 2304-0122 Ufa Mathematical Journal. Vol. 15. No 1 (2023). P. 34-42.

doi:10.13108/2023-15-1-34

PARTIAL ORDERS ON $*$-REGULAR RINGS

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Abstract. In this work we consider some new partial orders on $*$-regular rings. Let $\mathcal{A}$ be a $*$-regular ring, $P(\mathcal{A})$ be the lattice of all projectors in $\mathcal{A}$ and μ be a sharp normal normalized measure on $P(\mathcal{A})$. Suppose that $(\mathcal{A}, \rho)$ is a complete metric $*$-ring with respect to the rank metric ρ on $\mathcal{A}$ defined as $\rho(x, y) = \mu(l(x-y)) = \mu(r(x-y))$, $x, y \in \mathcal{A}$, where $l(a)$, $r(a)$ is respectively the left and right support of an element a. On $\mathcal{A}$ we define the following three partial orders: $a \prec_s b \Longleftrightarrow b = a + c$, $a \perp c$; $a \prec_l b \Longleftrightarrow l(a)b = a$; $a \prec_r b \Longleftrightarrow br(a) = a$, $a \perp c$ means algebraic orthogonality, that is, $ac = ca = a^*c = ac^* = 0$. We prove that the order topologies associated with these partial orders are stronger than the topology generated by the metric ρ. We consider the restrictions of these partial orders on the subsets of projectors, unitary operators and partial isometries of $*$-regular algebra $\mathcal{A}$. In particular, we show that these three orders coincide with the usual order $\leqslant$ on the lattice of the projectors of $*$-regular algebra. We also show that the ring isomorphisms of $*$-regular rings preserve partial orders $\prec_l$ and $\prec_r$.

Keywords: partial order, $*$-regular ring, von Neumann algebra, order topology.

Mathematics Subject Classification: 46L10, 46L51, 16E50

1. Introduction

In their famous work [1], G. Birkhoff and J. von Neumann showed that the set of statements of quantum mechanics possesses algebraic properties different from boolean algebra, namely, they have a structure of orthomodular lattice. The field of regular von Neumann rings is a part of non-commutative ring theory, which was originally introduced by von Neumann for clarifying some aspects of operator algebras [2], [3]. This mostly motivated the developing of regular rings as well as a series of other connections with the functional analysis of two main types: constructions of regular rings associated with operator algebras and complete complemented modular lattices, as well as by structural analogies between regular rings and operator algebras. The survey of modern state-of-art of orthomodular algebras can be found in works [4], [5]. There is a series of works devoted to the star order and topologies of von Neumann algebras [6], [7]. In works [6] a topology generated by the star order on von Neumann algebras was studied and it was proved that the order topology is finer that σ-strong* topology. It was also shown that the order topology coincides with the convergence topology in the norm if and only if the von Neumann algebra is finite-dimensional. In work [8] the authors studied the order topology on the Hermitian part of the von Neumann algebra and they provided the characterization of many important properties of von Neumann algebras like finiteness, sigma-finiteness, finiteness and atomicity, from the point of view how the order topology is compared with other known topologies on the von Neumann algebras.

In the present work we introduce some new order relations on $*$-regular in von Neumann sense rings. The work is organized as follows. In the second section we collect some basic

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K.K. Kudaybergenov, B.O. Nurjanov, Partial orders on $*$-regular rings.

The research by the first author was partially supported by the Ministry of Science and Higher Education of Russian Federation, agreement no. 075-02-2022-896.

Submitted December 25, 2021.

facts on $*$-regular in the von Neumann sense rings, the measures on the lattice of projectors of $*$-regular algebras and Murray-von Neumann algebras. In the third section we introduce some new partial orders on $*$-regular rings. Let $\mathcal{A}$ be a $*$-regular ring and $a, b \in \mathcal{A}$. On $\mathcal{A}$ we define the following three partial orders:

(1) $a \prec_s b \Longleftrightarrow b = a + c,\ a \perp c$;
(2) $a \prec_l b \Longleftrightarrow l(a)b = a$;
(3) $a \prec_r b \Longleftrightarrow br(a) = a$.

We prove that if $\mathcal{A}$ is a $*$-regular algebra with a rank-metrics ρ, then the order topologies associated with these partial orders are stronger than the topology generated by the metrics ρ. We also consider the restrictions of these partial orders on the subsets of projectors, unitary operators and partial isometries of the $*$-regular algebra $\mathcal{A}$.

2. $*$-REGULAR RINGS

In this section we provide a preliminary information on $*$-regular rings and Murray-von Neumann algebras from works [3], [9], [10].

A ring $\mathcal{A}$ is called a $*$-ring (or a ring with an involution) if there exists an operation $* : \mathcal{A} \to \mathcal{A}$ such that for all $a, b \in \mathcal{A}$ the following identities hold:

$$(a^*)^* = a, \qquad (a+b)^* = a^* + b^*, \qquad (ab)^* = b^*a^*.$$

We recall that an element e of a $*$-ring $\mathcal{A}$ is called a projector if $e^2 = e = e^*$.

A ring $\mathcal{A}$ is called regular if for each $x \in \mathcal{A}$ there exists an element $y \in \mathcal{A}$ such that $xyx = x$. An involution $*$ in $\mathcal{A}$ is called proper if the identity $x^*x = 0$ implies $x = 0$ for each $x \in \mathcal{A}$. A $*$-ring $\mathcal{A}$ is called $*$-regular if this is a regular ring with a proper involution.

Let $\mathcal{A}$ be a $*$-regular ring. Then there exists a unique projector $r(x)$ such that

(1) $xr(x) = x$;
(2) $xy = 0$ if and only if $r(x)y = 0$.

Similarly, there exists a unique projector $l(x)$ such that

(3) $l(x)x = x$;
(4) $yx = 0$ if and only if $yl(x) = 0$.

The projectors $r(x)$ and $l(x)$ are respectively called a right and a left projector of x. A projector $s(x) = l(x) \vee r(x)$ is called a support of an element x.

Let $\mathcal{A}$ be a $*$-regular ring and let $P(\mathcal{A})$ be a lattice of all projectors in $\mathcal{A}$, that is, $P(\mathcal{A}) = \{p \in \mathcal{A} : p^2 = p = p^*\}$. A real-valued function μ on $P(\mathcal{A})$ is called a normal exact normalized measure if

(1) $0 \leqslant \mu(p) \leqslant 1$;
(2) $\mu(0) = 0$, $\mu(\mathbf{1}) = 1$;
(3) $\mu(p \vee q) + \mu(p \wedge q) = \mu(p) + \mu(q)$;
(4) $p \leqslant q \Rightarrow \mu(p) \leqslant \mu(q)$;
(5) if $p_i \uparrow p$, then $\mu(p_i) \uparrow \mu(p)$.

We consider a so-called rank-metrics ρ on $\mathcal{A}$ defined as follows

$$\rho(x, y) = \mu(l(x-y)), \qquad x, y \in \mathcal{A}, \tag{2.1}$$

see [3, Lm. 18.1].

One of the important classes of $*$-regular algebras are Murray-von Neumann algebras, for more details see [11]–[15].

Let H be a Hilbert space, $B(H)$ be a $*$-algera of all bounded linear operators in H and $\mathcal{M}$ be a finite von Neumann algebra in $B(H)$.

A closed linear operator $x : \mathrm{dom}(x) \to H$ with a dense domain (here the domain $\mathrm{dom}(x)$ of an operator x is a dense linear subspace in H) is called adjoint to $\mathcal{M}$ if $yx \subset xy$ for each y from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. We denote by $S(\mathcal{M})$ the set of all operators adjoint to $\mathcal{M}$. It is well known that $S(\mathcal{M})$ is a unital $*$-regular algebra over $\mathbb{C}$, see [16], [17]. An algebra $S(\mathcal{M})$ is called Murray-von Neumann algebra associated with $\mathcal{M}$ [12].

Let τ be the exact normal finite trace on $\mathcal{M}$ and ρ be the rank metrics on $S(\mathcal{M})$ defined as in (2.1). According to [18, Prop. 2.1], the algebra $S(\mathcal{M})$ with the metrics ρ is a complete topological $*$-ring. Ring isomorphisms of the algebra $S(\mathcal{M})$ and of their $*$-subalgebras in the case of algebras of type II_1 were described in works [19], [20].

Let $\mathcal{M}$ be a finite von Neumann algebra. Let $a = v|a|$ be a polar decomposition of an element $a \in S(\mathcal{M})$. Then $l(a) = vv^*$ and $r(a) = v^*v$ are the left and right support of the element a, respectively. The projector $s(a) = l(a) \vee r(a)$ is the support of the element a. There exists a unique element $i(a)$ in $S(\mathcal{M})$ such that $ai(a) = l(a)$, $i(a)a = r(a)$, $ai(a)a = a$, $i(a)l(a) = i(a)$ and $r(a)i(a) = i(a)$. An element $i(a)$ is called partially inverse to the element a, see [9], [15].

3. Partial orders on $*$-regular rings

3.1. Partial orders. In this subsection $\mathcal{A}$ is a $*$-regular ring with a rank-metrics ρ. Moreover, we assume that $(\mathcal{A}, \rho)$ is a complete metric $*$-ring.

Let $a, b \in \mathcal{A}$. We say that a is algebraically orthogonal to b if

$$ab = ba = a^*b = ab^* = 0;$$

this is denoted as $a \perp b$. In particular, if $a, b \in \mathcal{A}_h = \{x \in \mathcal{A} : x = x^*\}$, then a is algebraically orthogonal to b if and only if $ab = 0$.

We observe that a is algebraically orthogonal to b if and only if $s(a)s(b) = 0$. Indeed, assume that $s(a)s(b) = 0$. Then

$$r(a)l(b) = r(b)l(a) = l(a)l(b) = r(a)r(b) = 0$$

and therefore, $ab = ba = a^*b = ab^* = 0$.

If a and b are algebraically orthogonal, this implies that

$$r(a)l(b) = r(b)l(a) = l(a)l(b) = r(a)r(b) = 0.$$

Thus, $l(a) \leqslant \mathbf{1} - l(b)$ and $r(a) \leqslant \mathbf{1} - l(b)$ and therefore, $s(a) \leqslant \mathbf{1} - l(b)$. Hence, $s(a)l(b) = 0$. Similarly, $s(a)r(b) = 0$. Thus, $l(b) \leqslant \mathbf{1} - s(a)$ and $r(b) \leqslant \mathbf{1} - s(a)$ and therefore $s(b) \leqslant \mathbf{1} - s(a)$. Hence, $s(a)s(b) = 0$.

For elements $a, b \in \mathcal{A}$ we let

$$a \prec_s b \Longleftrightarrow b = a + c,\ a \perp c,$$
$$a \prec_l b \Longleftrightarrow l(a)b = a,$$
$$a \prec_r b \Longleftrightarrow br(a) = a.$$

Lemma 3.1. *Let $a, b \in \mathcal{A}$. Then*

$$a \prec_s b \quad \text{if and only if} \quad s(a)b = bs(a) = a. \tag{3.1}$$

Proof. Suppose that $a \prec_s b$. In this case $b = a + c$, where $a, c \in \mathcal{A}$ are such that $s(a)s(c) = 0$. Then

$$s(a)b = s(a)(a + c) = s(a)a + s(a)c = a$$

and

$$bs(a) = (a + c)s(a) = as(a) + cs(a) = a.$$

Now we assume that $s(a)b = bs(a) = a$. Then

$$l(a) = l(bs(a)) \leqslant l(b) \leqslant s(b) \qquad \text{and} \qquad r(a) \leqslant r(s(a)b) \leqslant r(b) \leqslant s(b).$$

Therefore, $s(a) = l(a) \vee r(a) \leqslant s(b) \vee s(b) = s(b)$.

We consider a Pierce decomposition $b = ebe+ebf+fbe+fbf$, where $e = s(a)$ and $f = \mathbf{1}-s(a)$. Since $s(a)b = bs(a) = a$, this implies that $ebe = a$, $ebf = fbe = 0$. Thus, $b = a + c$, where $c = fbf$. Since $s(a)s(c) = es(c) = e(fs(c)) = efs(c) = 0$, we have $a \prec_s b$. The proof is complete. □

Lemma 3.2. *Let $a, b \in \mathcal{A}$. Then*

$$a \prec_s b \quad \Rightarrow \quad a \prec_l b \quad \Leftrightarrow \quad a^* \prec_r b^*. \tag{3.2}$$

Proof. Let $a \prec_s b$, that is, $s(a)b = bs(a) = a$. Hence, we have

$$a = l(a)a = l(a)s(a)b = l(a)b,$$

that is, $a \prec_l b$.

Since $r(a^*) = l(a)$, it follows that $a \prec_l b \Leftrightarrow a^* \prec_r b^*$. The proof is complete. □

Lemma 3.3. *A relation $\prec$, where $\prec \in \{\prec_s, \prec_l, \prec_r\}$, is a partial order on $\mathcal{A}$, that is,*

(1) $x \prec x$;
(2) $x \prec y$, $y \prec x \Rightarrow x = y$;
(3) $x \prec y$, $y \prec z \Rightarrow x \prec z$.

Proof. We check the statement of the lemma for the case $\prec_l$. The cases $\prec_s$ and $\prec_r$ are similar.

Property (1) is obvious. We note that it follows from $a \prec_l b$ that

$$l(a) \leqslant l(b). \tag{3.3}$$

Indeed, $l(a) = l(br(a^*)) \leqslant l(b)$.

We take elements $x, y \in \mathcal{A}$ such that $x \prec_l y$, $y \prec_l x$. Then $l(x) \leqslant l(y)$ and $l(y) \leqslant l(x)$, that is, $l(x) = l(y)$. We then have

$$x = l(x)y = l(y)y = y.$$

Let $x \prec_l y$ and $y \prec_l z$. Then

$$l(x) \leqslant l(y), \quad l(y) \leqslant l(z).$$

Thus,

$$x = l(x)y = l(x)(l(y)z) = l(x)l(y)z = l(x)z.$$

Hence, $x \prec_l z$. The proof is complete. □

In the general case, the opposite to the first implication in (3.2) is wrong. Let $\mathcal{A}$ be a $*$-regular ring containing the ring of the matrices of order 2. We take nonzero mutually orthogonal equivalent projectors $p, q \in \mathcal{A}$ and an element $u \in \mathcal{A}$ such that $u^*u = p$, $uu^* = q$. We let $a = u^*$ and $b = u^* + u$. Then

$$l(a) = p, \quad r(a) = q, \quad s(a) = p + q.$$

This implies

$$l(a)b = l(a)(u^* + u) = p(u^* + u) = pu^* = a.$$

Thus, $a \prec_l b$. But

$$s(a)b = (p + q)b = b \neq a,$$

and hence, $a \prec_s b$ is wrong.

We observe that
$$a \prec_s b \Rightarrow ab = ba = a^2. \tag{3.4}$$
Indeed, for elements a and b obeying the condition $a \prec_s b$ by employing (3.1) we have
$$ab = (as(a))b = a(s(a)b) = a^2 = (bs(a))a = ba.$$
We mention that the binary relation $\leqslant$ on $\mathcal{A}$ is a partial order [4, Exm. 1.6.7]:
$$a \leqslant b \qquad \text{if and only if} \qquad ab = ba = a^2.$$

In the general case, the opposite for the implication in (3.4) is wrong. Let $\mathcal{A}$ be a $*$-regular ring containing a factor of type I_3. We take nonzero mutually orthogonal equivalent projectors $p, q, r \in \mathcal{A}$ and elements $a, b \in \mathcal{A}$ such that $a^*a = p$, $aa^* = q$, $b = r$. Then
$$ab = ba = 0 = a^2,$$
but the inequality $a \prec_s b$ is wrong.

It should be noted that
$$(a \prec_l b) \wedge (a \prec_r b) \Rightarrow aa^* = ba^*, \quad a^*a = a^*b.$$
Indeed, for elements a and b obeying the condition $a \prec_l b$ we have
$$ba^* = bl(a^*)a^* = br(a)a^* = aa^*,$$
and it follows from inequality $a \prec_r b$ that
$$a^*b = a^*r(a^*)b = a^*l(a)b = a^*a.$$
We note that the binary relation $\preceq$ on $\mathcal{A}$ defined as
$$a \preceq b \quad \Leftrightarrow \quad aa^* = ba^*, \quad a^*a = a^*b$$
is a partial order [6], [7], [21], [22]. This order is called a star order, which comes from the matrix analysis and it was introduced for $*$-semigroups by M.P. Drazin in work [21].

3.2. Order topology. In this subsection we consider order topology on a $*$-regular ring $\mathcal{A}$ generated by partial orders $\prec_s$, $\prec_l$ and $\prec_r$.

The notion of order convergence of a net was introduced by G. Birkhoff, see [1]. We recall the notion of the order topology or (o)-topology, for more details see [1], [23]. Let $\prec \in \{\prec_s, \prec_l, \prec_r\}$. Fro the net $\{x_\alpha\}_{\alpha\in A} \subset \mathcal{A}$ the notion $x_\alpha \uparrow x$ (respectively, $x_\alpha \downarrow x$), where $x \in \mathcal{A}$, means that $x_\alpha \prec x_\beta$ (respectively, $x_\beta \prec x_\alpha$) for $\alpha \leqslant \beta$ and $x = \sup\limits_{\alpha\in A} x_\alpha$ (respectively, $x = \inf\limits_{\alpha\in A} x_\alpha$). A net $\{x_\alpha\}_{\alpha\in A} \subset \mathcal{A}$ is called (o)-converging to an element x in $\mathcal{A}$, which is denoted as $x_\alpha \stackrel{(o)}{\to} x$, if there exist nets $\{y_\alpha\}_{\alpha\in A}$ and $\{z_\alpha\}_{\alpha\in A}$ from $\mathcal{A}$ such that $y_\alpha \prec x_\alpha \prec z_\alpha$ for each $\alpha \in A$ and $y_\alpha \uparrow x$, $z_\alpha \downarrow x$. A strongest topology on $\mathcal{A}$, for which (o)-convergence of the nets implies their convergence in the topology is called an order topology or (o)-topology and is denoted $t_o(\prec)$.

Let t_ρ be a topology on $\mathcal{A}$ generated by the rank metrics ρ.

Theorem 3.1. *Let $\mathcal{A}$ be a $*$-regular ring with a rank metrics ρ such that $(\mathcal{A}, \rho)$ is a complete metric $*$-ring. Then the order topology $t_o(\prec)$ is stronger than t_ρ.*

In order to prove the theorem, it is sufficient to show that each net $\{x_\alpha\} \subset \mathcal{A}$, which (o)-converges to zero, also converges to zero in the topology t_ρ. We are going to show this in the following two lemmata.

Lemma 3.4. *Let $\{x_\alpha\}_{\alpha\in A} \subset (\mathcal{A}, \prec)$ be an increasing (respectively, decreasing) net, where $\prec \in \{\prec_s, \prec_l, \prec_r\}$. Then there exists $x \in \mathcal{A}$ such that*

(1) *$x = \sup\limits_{\alpha\in A} x_\alpha \in \mathcal{A}$ (respectively, $x = \inf\limits_{\alpha\in A} x_\alpha \in \mathcal{A}$);*

(2) $s(x) = \sup_{\alpha\in A} s(x_\alpha)$ *(respectively,* $s(x) = \inf_{\alpha\in A} s(x_\alpha)$*);*

(3) $x_\alpha \xrightarrow{\rho} x$.

Proof. Let us prove the three required statements in the case $\prec_l$. The cases $\prec_s$ and $\prec_r$ can be studied in a similar way.

Let $\{x_\alpha\}_{\alpha\in A} \subset (\mathcal{A}, \prec)$ be an increasing net. Since $x_\alpha \prec x_\beta$ for all $\alpha \leqslant \beta$, this implies that $l(x_\alpha) \leqslant l(x_\beta)$, see (3.3). Therefore, there exists a projector $l = \sup_{\alpha\in A} l(x_\alpha)$. Then

$$\begin{aligned}\tau(l(x_\beta - x_\alpha)) &= \tau(l(x_\beta - l(x_\alpha)x_\beta)) = \tau(l((l(x_\beta) - l(x_\alpha))x_\beta))\\ &\leqslant \tau(l(x_\beta) - l(x_\alpha)) \leqslant \tau(l - l(x_\alpha)) \to 0.\end{aligned}$$

We then have

$$\rho(x_\beta, x_\alpha) = \tau(l(x_\beta - x_\alpha)) \to 0.$$

This means that $\{x_\alpha\}_{\alpha\in A}$ is a Cauchy net. Since $(\mathcal{A}, \rho)$ is complete, there exists an element $x \in \mathcal{A}$ such that $\rho(x_\alpha, x) \to 0$, that is, $x_\alpha \xrightarrow{\rho} x$.

First we are going to show that $x_\alpha \prec x$ for all $\alpha \in A$.

Let $\gamma \in A$ be a fixed index. For all $\alpha \geqslant \gamma$ we have $l(x_\gamma)x_\alpha = x_\gamma$ since $x_\gamma \prec x_\alpha$. Since the multiplication in $\mathcal{A}$ is ρ-continuous, this implies that $l(x_\gamma)x_\alpha \xrightarrow{\rho} l(x_\gamma)x$. Thus, $l(x_\gamma)x = x_\gamma$ and therefore $x_\gamma \prec x$, in particular, $l(x_\alpha) \leqslant l(x)$ for all $\alpha \in A$.

Since $x_\alpha \xrightarrow{\rho} x$, $l(x_\alpha) \leqslant l(x)$ for all $\alpha \in A$, this implies that $l(x) = \sup_{\alpha\in A} l(x_\alpha)$.

We take an element $y \in \mathcal{A}$ such that $x_\alpha \prec y$ and let us show that $x \prec y$. We have

$$l(x_\alpha)y = x_\alpha$$

for all $\alpha \in A$. Thus,

$$l(x_\alpha)y = l(x_\alpha)x$$

for all $\alpha \in A$. Since $l(x) = \sup_{\alpha\in A} l(x_\alpha)$, then

$$l(x)y = x.$$

This means that $x \prec y$ and therefore, $x = \sup_{\alpha\in A} x_\alpha$. The proof is complete. $\square$

Let $y \prec x \prec z$. Then

$$\rho(x, z) \leqslant \rho(y, z) \quad \text{and} \quad \rho(y, x) \leqslant \rho(y, z). \tag{3.5}$$

Let us prove the first inequality. Using the identity

$$l(x)y = x,$$

we obtain that

$$\begin{aligned}\rho(x, z) &= \tau(l(z - x)) = \tau(l(z - l(x)z)) = \tau(l((l(z) - l(x))z))\\ &\leqslant \tau(l((l(z) - l(y))z)) \leqslant \tau(l(z - y)) = \rho(y, z).\end{aligned}$$

Lemma 3.5. *If* $x_\alpha \xrightarrow{(o)} x$*, then* $x_\alpha \xrightarrow{\rho} x$.

Proof. Let $x_\alpha \xrightarrow{(o)} x$. Then there exist nets $\{y_\alpha\}_{\alpha\in A}$ and $\{z_\alpha\}_{\alpha\in A}$ such that $y_\alpha \prec x_\alpha \prec z_\alpha$ for each $\alpha \in A$ and $y_\alpha \uparrow x$, $z_\alpha \downarrow x$. By Statement (3) in Lemma 3.4 we have $y_\alpha \xrightarrow{\rho} x$ and $z_\alpha \xrightarrow{\rho} x$. Employing these relations, we obtain

$$\rho(x_\alpha, x) \leqslant \rho(x_\alpha, y_\alpha) + \rho(y_\alpha, x) \overset{(3.5)}{\leqslant} \rho(z_\alpha, y_\alpha) + \rho(y_\alpha, x)$$

$$\leqslant \rho(z_\alpha, x) + \rho(x, y_\alpha) + \rho(y_\alpha, x) \to 0,$$

that is, $x_\alpha \xrightarrow{\rho} x$. The proof is complete. $\square$

Remark 3.1. *We observe that if $\mathcal{A}$ is a finite-dimensional $*$-regular algebra with a rank-metrics ρ, then both topologies t_o and t_ρ are discrete. Indeed, since $\mathcal{A}$ is finite-dimensional, then the set of all values of the function ρ is finite. Therefore, the topology t_ρ is discrete. Moreover, since t_o should be stronger than t_ρ, this implies that the order topology is also discrete.*

3.3. Restriction of order on lattice of projectors and set of partial isometries. Let $\mathcal{A}_h$ be the subset of all Hermitian elements in $\mathcal{A}$ and $\mathcal{A}_+$ be the cone of all positive elements in $\mathcal{A}_h$, that is, $\mathcal{A}_+ = \{x \in \mathcal{A}_h : x = y^2,\ y \in \mathcal{A}_h\}$. Let $\leqslant$ be the usual order on $\mathcal{A}_h$, that is, for $x, y \in \mathcal{A}_h$ the inequality $x \leqslant y$ means that $y - x \in \mathcal{A}_+$.

Partial orders $\prec_s$, $\prec_l$ and $\prec_r$ on the set $P(\mathcal{A})$ of all projectors from $\mathcal{A}$ coincide with the usual order $\leqslant$.

Indeed, let $p, q \in P(\mathcal{A})$ be such that $p \prec_i q$, where $i \in \{s, l, r\}$. Since $s(p) = l(p) = r(p) = p$, it follows from identities $l(p)q = qr(p) = p$ that $pq = qp = p$ and this means that $p \leqslant q$.

And vice versa, if $p \leqslant q$ for $p, q \in P(\mathcal{A})$, this implies that $pq = qp = p$. Hence, $p \prec_i q$, $i \in \{s, l, r\}$.

We denote by $\mathcal{GU}(\mathcal{A})$ the set of all partial isometries in $\mathcal{A}$, that is,

$$\mathcal{GU}(\mathcal{A}) = \{w \in \mathcal{A} : w = ww^*w\}.$$

We observe that $l(w) = ww^*$ and $r(w) = w^*w$ are left and right supports for $w \in \mathcal{GU}(\mathcal{A})$.

On the set $\mathcal{GU}(\mathcal{A})$ we can define a partial order as follows:

$$u \leqslant_l v \Leftrightarrow uu^* \leqslant vv^*, \quad u = uu^*v.$$

It is clear that

$$u \leqslant_r v \Leftrightarrow u^*u \leqslant v^*v, \quad u = vu^*u$$

also defines a partial order on the set $\mathcal{GU}(\mathcal{A})$ and

$$u \leqslant_l v \Leftrightarrow u^* \leqslant_r v^*.$$

This means that the restrictions of partial orders $\prec_l$ and $\prec_r$ on $\mathcal{GU}(\mathcal{A})$ coincide with partial orders $\leqslant_l$ and $\leqslant_r$, respectively.

3.4. Ring isomorphisms: order preserving mappings. Now we are going to show that the ring isomorphisms preserve the partial orders $\prec_l$ and $\prec_r$.

Proposition 3.1. *Let $\mathcal{A}$, $\mathcal{B}$ be $*$-regular rings and $\Phi : \mathcal{A} \to \mathcal{B}$ be a ring isomorphism. Then*

1. *$x \prec_l y$ if and only if $\Phi(x) \prec_l \Phi(y)$.*
2. *$x \prec_r y$if and only if $\Phi(x) \prec_r \Phi(y)$.*

Proof. It is sufficient to consider the case $\prec_l$. The case $\prec_r$ is similar.

Let $x \prec_l y$, that is,

$$l(x)y = x. \tag{3.6}$$

We have

$$\Phi(l(x)) = \Phi(xi(x)) = \Phi(x)\Phi(i(x))$$

and this is why

$$l\left(\Phi(l(x))\right) \leqslant l\left(\Phi(x)\right).$$

Then

$$\Phi(x) = \Phi(l(x)x) = \Phi(l(x))\Phi(x).$$

Therefore,

$$l\left(\Phi(x)\right) \leqslant l\left(\Phi(l(x))\right)$$

and

$$l\left(\Phi(x)\right) = l\left(\Phi(l(x))\right). \tag{3.7}$$

Finally, we have

$$\begin{aligned} l\left(\Phi(x)\right)\Phi(y) &\overset{(3.7)}{=} l\left(\Phi(l(x))\right)\Phi(y) = l\left(\Phi(l(x))\right)\Phi\left(l(x)\right)\Phi(y) \\ &= l\left(\Phi(l(x))\right)\Phi\left(l(x)y\right) \overset{(3.6)}{=} l\left(\Phi(l(x))\right)\Phi\left(x\right) \\ &\overset{(3.7)}{=} l\left(\Phi(x)\right)\Phi\left(x\right) = \Phi\left(x\right). \end{aligned}$$

This means that $\Phi(x) \prec_l \Phi(y)$.

Since Φ is a ring isomorphism, this yields that the inverse implication also holds. The proof is complete. $\square$

Acknowledgments

The authors express a deep gratitude to the referee for useful remarks and comments.

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