ISSN 2304-0122 Ufa Mathematical Journal. Vol. 15. No 1 (2023). P. 43-55.

doi:10.13108/2023-15-1-43

GROUND STATES OF ISING-POTTS MODEL ON CAYLEY TREE

M.M. RAHMATULLAEV, B.M. ISAKOV

Abstract. It is known that for low temperatures, a ground state is associated with a limiting Gibbs measure. This is why, the studying of the sets of ground states for a given physical system is a topical issue.

We consider a model of mixed type on the Cayley tree, which is referred to as Ising-Potts model, that is, the Ising and Potts models are related with the parameter α, where $\alpha \in [0, 1]$. In the paper we study the ground state for the Ising-Potts model with three states on the Cayley tree. It is known that there exists a one-to-one correspondence between the set of the vertices V of the Cayley tree of order k and a group G_k being a free product of $k+1$ cyclic groups of second order. We define periodic and weakly periodic ground states corresponding to normal divisors of the group G_k. For the Ising-Potts model we describe the set of periodic and weakly periodic ground states corresponding to normal divisors of index 2 of the group G_k. We prove that for some values of the parameters there exist no such periodic (non translation-invariant) ground states. We also prove that for a normal subgroup consisting of even layers there exist periodic (non translation-invariant) ground states and we also prove the existence of weakly-periodic (non periodic) ground states.

Keywords: Cayley tree, Ising-Potts model, periodic and weakly periodic ground states.

Mathematics Subject Classification: 82B26, 60K35

1. Introduction

With each Gibbs measure, one phase of a physical system is associated. If there exists more than one Gibbs measure, one says that there are phase transitions. A main problem for a given Hamiltonian is to describe all associated limiting Gibbs measures.

It is known that a phase diagram of Gibbs measures for a given Hamiltonian is close to a phase diagram of ground isolated (stable) states of this Hamiltonian. For low temperatures, with the ground state a limiting Gibbs measure is associated, see [1], [2]. This is a problem on describing ground states naturally arises.

In work [5] there were studied translation-invariant and periodic ground states for the Ising model on the Cayley tree. In work [6] a notion of weakly periodic ground states was introduced. Weakly periodic ground states for the Ising model with competing interactions were described in works [6] and [7]. Periodic ground states for the Potts model with competing interactions on the Cayley tree of order $k = 2$ were studied in works [8] and [9]. In work [10], for the Potts model, weakly periodic ground states were studied for a normal divisor of index 2. In work [11] for the Potts model with competing interactions on the Cayley tree of order $k \geqslant 2$ there was described a set of periodic and weakly periodic ground states corresponding to normal divisors of index 4 of the group representation of the Cayley tree. In works [12] and [13] periodic and weakly periodic ground states for a λ-model on the Cayley tree were studied. In work [14], for the Ising model, periodic ground states with respect to a subgroup of index three were studied.

M.M. Rahmatullaev, B.M. Isakov, Ground states of Ising-Potts model on Cayley tree.

Submitted February 10, 2022.

In work [16], on the base of the replica algorithm, by the Monte Carlo method, the phase transitions of the antiferromagnetic layered Ising model on a cubic lattice were studied taking into consideration intralayer interactions of the second nearest neighbors in small ranges. In work [17] there were studied various phase transitions and interactions of two and three nearest neighbors for the Potts model as $q = 3$ by the Monte Carlo method. It was shown in work [18] that the transition from an antiferromagnetic and collinear phases into a paramagnetic one is a phase transition of the first kind, while the transition from a frustrated domain into the paramagnetic one is a phase transition of second kind for the Potts model as $q = 3$ on a triangular lattice. In work [19] a ferro- and antiferromagnetic three-vertices ($q = 3$) Potts model were studied on a triangular lattice taking into consideration the interactions of second nearest neighbors.

In recent work [20] the authors considered the Potts model on the Cayley tree and proved the existence of new classes of Gibbs measures differing from one known earlier.

In the present work we consider a model of a mixed type and in what follows we call it an Ising-Potts model in the Cayley tree of order $k \geqslant 2$. Ising and Potts models are related with a parameter α, where $\alpha \in [0, 1]$. If $\alpha = 0$, then the model coincides with the Potts model, while for $\alpha = 1$ the model coincides with the Ising model. For the Ising-Potts model with respect a normal divisor the group G_k, which a free product of $k+1$ cyclic groups $\{e, a_i\}$ of second order with generators $a_1, a_2, \ldots, a_{k+1}$, respectively, that is, $a_i^2 = e$ (see [15]), we determine periodic and weakly periodic ground states.

In conclusion we briefly describe the structure of the paper. In the next section we introduce main definitions and recall known facts. In the third section we study periodic and weakly periodic ground states.

2. Definitions and known facts

Let $\tau^k = (V, L)$, $k \geqslant 1$, be a Cayley tree of order k, that is, an infinite tree, each vertex of which is an origin for exactly $k + 1$ edges, where V is the set of vertices and L is the set of edges τ^k.

Let G_k be a free product of $k + 1$ cyclic groups $\{e, a_i\}$ of second order with generators a_1, $a_2, \ldots, a_{k+1}$, respectively, that is, $a_i^2 = e$, see [15]. There exists a one-to-one correspondence between the set of the vertices V of the Cayley tree of order k and the group G_k, see [10], [4].

Two vertices $x, y \in V$ are called neighbors if they the end points of some edge $l \in L$; in this case we write $l = \langle x, y\rangle$.

For an arbitrary point $x^0 \in V$ we let

$$W_n = \{x \in V |\ d(x^0, x) = n\}, \qquad V_n = \bigcup_{m=0}^{n} W_m, \qquad L_n = \{\langle x, y\rangle \in L |\ x, y \in V_n\},$$

where $d(x, y)$ is the distance between x and y on the Cayley tree, that is, the number of edges in a path connecting x and y.

We denote by $S(x)$ the set of 'direct descendants' of a point $x \in G_k$, that is, if $x \in W_n$, then $S(x) = \{y \in W_{n+1} : d(x, y) = 1\}$. By $S_1(x)$ we denote the set of all nearest neighbors of a point $x \in G_k$, that is, $S_1(x) = \{y \in G_k : \langle x, y\rangle\}$ and by $x_\downarrow$ we denote a unique element in the set $S_1(x) \setminus S(x)$.

We consider a model, where the spin ranges in the set $\Phi = \{-1, 0, 1\}$. A configuration σ on V is defined as a function $x \in V \to \sigma(x) \in \Phi$; the set of all configuration coincides with $\Omega = \Phi^V$.

Let $G_k/G_k^* = \{H_1, \ldots, H_r\}$ be a quotient group, where G_k^* is a normal divisor of index $r \geqslant 1$.

Definition 2.1. *A configuration $\sigma(x)$ is called G_k^*-periodic if $\sigma(x) = \sigma_i$ as $x_i \in H_j$, for all $x \in G_k$. A G_k-periodic configuration is called translation-invariant.*

For a given periodic configuration the index of the normal divisor is called a period of the configuration.

Definition 2.2. *A configuration $\sigma(x)$ is called G_k^*-weakly periodic if $\sigma(x) = \sigma_{ij}$ as $x_\downarrow \in H_i$, $x \in H_j$ for all $x \in G_k$.*

The Hamiltonian of the Ising-Potts model reads as

$$H(\sigma) = -\alpha J_1 \sum_{\langle x,y\rangle \in\ L} \sigma(x)\sigma(y) - (1-\alpha)J_2 \sum_{\langle x,y\rangle \in\ L} \delta_{\sigma(x)\sigma(y)}, \tag{2.1}$$

where $J = (J_1, J_2) \in \mathbb{R}^2$, $0 \leqslant \alpha \leqslant 1$ and $\delta_{i,j}$ is the Kronecker delta

$$\delta_{i,j} = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i\ = j. \end{cases} \tag{2.2}$$

3. Ground states

For a pair of configurations σ and φ coinciding almost everywhere, that is, except for finitely many points, we consider a relative Hamiltonian $H(\sigma, \varphi)$, which is a difference between the energies of the configurations σ, φ, that is,

$$H(\sigma,\varphi) = -\alpha J_1 \sum_{\substack{\langle x,y\rangle, \\ x,y\in V}} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)) - (1-\alpha)J_2 \sum_{\substack{\langle x,y\rangle, \\ x,y\in V}} (\delta_{\sigma(x)\sigma(y)} - \delta_{\varphi(x)\varphi(y)}), \tag{3.1}$$

where $J = (J_1, J_2) \in \mathbb{R}^2$ is an arbitrary fixed parameter.

Let M be a set of unit balls with vertices at V. A restriction of the configuration σ on a ball $b \in M$ is called a bounded configuration σ_b. We define the energy of the configuration σ_b on b as follows:

$$U(\sigma_b) = -\frac{1}{2}\alpha J_1 \sum_{<x,y>\in L} \sigma(x)\sigma(y) - \frac{1}{2}(1-\alpha)J_2 \sum_{<x,y>\in L} \delta_{\sigma(x)\sigma(y)}. \tag{3.2}$$

The following lemma is known, see [6], [8].

Lemma 3.1. *Relative Hamiltonian* (2.2) *is of the form*

$$H(\sigma,\varphi) = \sum_{b\in M} (U(\sigma_b) - U(\varphi_b)).$$

In this work we consider the case $k = 2$. By c_b we denote the center of a unit ball b. Let

$$B_- = \{x \in S_1(c_b) : \varphi_b(x) = -1\},$$
$$B_0 = \{x \in S_1(c_b) : \varphi_b(x) = 0\},$$
$$B_+ = \{x \in S_1(c_b) : \varphi_b(x) = 1\}.$$

Let $\varphi_b(c_b) = z$, $z \in \Phi$ and $|B_-| = d$, $|B_+| = t$. Then $|B_0| = 3 - d - t$. It is easy to prove the following statement.

Lemma 3.2. *For each configuration σ_b the belonging*

$$U(\sigma_b) \in \{U_1, U_2, U_3, \dots, U_{12}\}$$

is true, where

$$U_1 = 0, \qquad U_2 = -\frac{3\alpha}{2}J_1 - \frac{3}{2}(1-\alpha)J_2, \qquad U_3 = \frac{3\alpha}{2}J_1,$$

$$U_4=-\alpha J_1-(1-\alpha)J_2, \qquad U_5=-\frac{\alpha}{2}J_1-\frac{1-\alpha}{2}J_2, \qquad U_6=-\frac{\alpha}{2}J_1-(1-\alpha)J_2,$$
$$U_7=\frac{\alpha}{2}J_1-\frac{1-\alpha}{2}J_2, \qquad U_8=\alpha J_1, \qquad U_9=\frac{\alpha}{2}J_1,$$
$$U_{10}=-\frac{1-\alpha}{2}J_2, \qquad U_{11}=-\frac{3-3\alpha}{2}J_2, \qquad U_{12}=-(1-\alpha)J_2.$$

Definition 3.1. *A configuration φ is called a ground state of the Hamiltonian H if*
$$U(\varphi_b)=\min\{U_1,U_2,U_3,\dots,U_{12}\}$$
for each $b\in M$.

A periodic (weakly periodic, translation-invariant) configuration being a ground state is called *periodic (weakly periodic, translation-invariant) ground state.*

The aim of the present work is to describe the set of periodic and weakly periodic ground states for the Ising-Potts model on the Cayley tree of order $k=2$ corresponding to normal divisors of index 2 of the group representations of the Cayley tree.

We denote $C_i=\{\varphi_b : U(\varphi_b)=U_i\}$ and
$$A_i=\left\{J\in\mathbb{R}^2 : U_i=\min\{U_1,U_2,U_3,\dots,U_{12}\}\right\}, \tag{3.3}$$
where $i=1,2,\dots,12$.

A simple but bulky analysis show that A_i read as
$$A_1=\left\{(J_1,J_2)\in\mathbb{R}^2 \mid J_1\geqslant 0;\ \frac{\alpha-1}{\alpha}J_2\geqslant J_1\right\},$$
$$A_2=\left\{(J_1,J_2)\in\mathbb{R}^2 \mid J_1\geqslant 0;\ \frac{\alpha-1}{\alpha}J_2\leqslant J_1\right\},$$
$$A_3=\left\{(J_1,J_2)\in\mathbb{R}^2 \mid J_1\leqslant 0;\ \frac{\alpha-1}{\alpha}J_2\geqslant J_1\right\},$$
$$A_4=A_5=\left\{(J_1,J_2)\in\mathbb{R}^2 \mid J_1=-\frac{(1-\alpha)}{\alpha}J_2;\ J_2\leqslant 0\right\},$$
$$A_6=A_7=A_{10}=\left\{(J_1,J_2)\in\mathbb{R}^2 \mid J_1=0;\ J_2=0\right\},$$
$$A_8=A_9=\left\{(J_1,J_2)\in\mathbb{R}^2 \mid J_1=0;\ J_2\leqslant 0\right\},$$
$$A_{11}=\left\{(J_1,J_2)\in\mathbb{R}^2 \mid J_1\geqslant \frac{-(1-\alpha)}{\alpha}J_2;\ J_2\geqslant 0\right\},$$
$$A_{12}=\left\{(J_1,J_2)\in\mathbb{R}^2 \mid J_1\leqslant 0;\ J_2=0\right\}.$$

It is easy to confirm
$$\mathbb{R}^2=\bigcup_{i=1}^{12}A_i.$$

On Figure 1 we show the location of A_i, $i=1,2,\dots,12$, in the plane.

3.1. Translation-invariant ground states. In this subsection we study translation-invariant ground states. We recall that a configuration $\sigma(x)$ is called translation-invariant if $\sigma(x)=i$ for all $x\in V$, $i\in\Phi$.

We prove the following theorem.

Theorem 3.1. *Let $k=2$. The following statements hold true for the Ising-Potts model:*

1) On the set A_{11} there exists a unique translation-invariant ground state and it reads as $\varphi(x)=0$ for all $x\in G_k$.

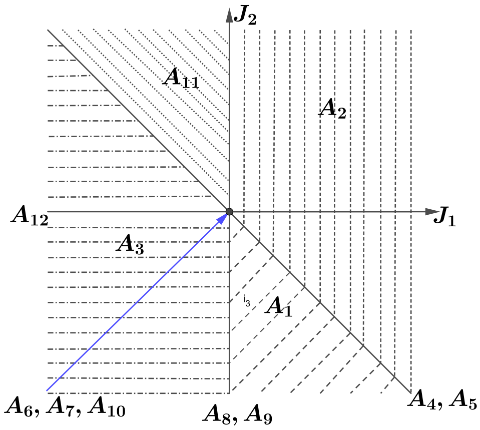

FIGURE 1. Location of A_i, $i = 1, 2, \dots, 12$ in the plane.

2) On the set A_2 there exist two ground states and they read as $\varphi(x) = \pm 1$ for all $x \in G_k$.
3) If $(J_1, J_2) \in \mathbb{R}^2 \setminus (A_{11} \cup A_2)$, then there exist no translation-invariant ground states.

Proof. 1. Let $\varphi(x) = 0$ for all $x \in G_k$. Then for all $b \in M$ we have

$$\varphi_b(c_b) = 0, \qquad |B_-| = 0, \qquad |B_+| = 0, \qquad |B_0| = 3,$$

and therefore, $\varphi_b \in C_{11}$. Hence, the configuration $\varphi(x) = 0$ for all $x \in G_k$ is a translation-invariant ground state on the set A_{11}.

2. Let $\varphi(x) = 1$ for all $x \in G_k$. Then for all $b \in M$ we have

$$\varphi_b(c_b) = 1, \qquad |B_-| = 0, \qquad |B_+| = 3, \qquad |B_0| = 0$$

and hence, $\varphi_b \in C_2$. This gives that the configuration $\varphi(x) = 1$ for all $x \in G_k$ is a translation-invariant ground state on the set A_2.

Let $\varphi(x) = -1$ for all $x \in G_k$. Then for all $b \in M$ we have

$$\varphi_b(c_b) = -1, \qquad |B_-| = 3, \qquad |B_+| = 0, \qquad |B_0| = 0$$

and hence $\varphi_b \in C_2$. This implies that the configuration $\varphi(x) = -1$ for all $x \in G_k$ is a translation-invariant ground state on the set A_2.

3. The proof is obvious. □

Remark 3.1. *We note that as $\alpha = 0$, Hamiltonian* (2.1) *describes the Potts model. Then Theorem 3.1 coincides with Item B.1 in Theorem 2 in work* [8] *in the case $J_2^* = 0$ ($J_2^* = J_2$, where is from work* [8]*).*

3.2. Periodic ground states. Let $A \subset \{1, 2, \dots, k+1\}$. It is known that each normal divisor of index of the group G_k reads as (see [10])

$$H_A = \left\{ x \in G_k : \sum_{i \in A} \omega_x(a_i) \text{ is even} \right\}.$$

We consider a quotient group $G_k/H_A = \{H_0, H_1\}$, where $H_0 = H_A$, $H_1 = G_k \setminus H_0$. In this subsection we study an H_A-periodic ground state. An H_A-periodic configuration reads as

$$\varphi(x) = \begin{cases} \sigma_1 & \text{if} \quad x \in H_0, \\ \sigma_2 & \text{if} \quad x \in H_1, \end{cases} \tag{3.4}$$

where $\sigma_1, \sigma_2 \in \Phi$.

Theorem 3.2. *Let $k = 2$ and $|A| = 1$. Then each H_A-periodic ground state is translation-invariant.*

Proof. For $\sigma_1 = \sigma_2$ we have that H_A-periodic configuration (3.4) is a translation-invariant state, which was studied in Theorem 2.1.

We consider the case when $\sigma_1 \neq \sigma_2$. Let

$$\varphi_1(x) = \begin{cases} -1 & \text{if} \quad x \in H_0, \\ 0 & \text{if} \quad x \in H_1. \end{cases}$$

If $c_b \in H_0$ then for all $b \in M$ we have

$$\varphi_{1,b}(c_b) = -1, \qquad |B_-| = 2, \qquad |B_0| = 1, \qquad |B_+| = 0,$$

and therefore, $\varphi_{1,b} \in C_4$. If $c_b \in H_1$, then

$$\varphi_{1,b}(c_b) = 0, \qquad |B_-| = 1, \qquad |B_0| = 2, \qquad |B_+| = 0$$

and hence, $\varphi_{1,b} \in C_{12}$. Since $A_4 \cap A_{12} = \{0, 0\}$, this implies that the corresponding configuration is not a ground state on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Let

$$\varphi_2(x) = \begin{cases} -1 & \text{if} \quad x \in H_0, \\ 1 & \text{if} \quad x \in H_1. \end{cases}$$

If $c_b \in H_0$ then for all $b \in M$ we have

$$\varphi_{2,b}(c_b) = -1, \qquad |B_-| = 2, \qquad |B_0| = 0, \qquad |B_+| = 1$$

and hence, $\varphi_{2,b} \in C_6$. If $c_b \in H_1$, then for all $b \in M$ we have

$$\varphi_{2,b}(c_b) = 1, \qquad |B_-| = 1, \qquad |B_0| = 0, \qquad |B_+| = 2$$

and hence, $\varphi_{2,b} \in C_7$. Since $A_6 \cap A_7 = \{0, 0\}$, this implies that the corresponding configuration is not a ground state on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Let

$$\varphi_3(x) = \begin{cases} 0 & \text{if} \quad x \in H_0, \\ -1 & \text{if} \quad x \in H_1. \end{cases}$$

If $c_b \in H_0$, then for all $b \in M$ we have

$$\varphi_{3,b}(c_b) = 0, \qquad |B_-| = 1, \qquad |B_0| = 2, \qquad |B_+| = 0,$$

and therefore, $\varphi_{3,b} \in C_{12}$. If $c_b \in H_1$, then for all $b \in M$ we have

$$\varphi_{3,b}(c_b) = -1, \qquad |B_-| = 2, \qquad |B_0| = 1, \qquad |B_+| = 0$$

and hence, $\varphi_{3,b} \in C_4$. Since $A_4 \cap A_{12} = \{0, 0\}$, the corresponding configuration is not a ground state on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

For the configuration $\varphi_4(x) = -\varphi_1(x)$, $\varphi_5(x) = -\varphi_2(x)$ and $\varphi_6(x) = -\varphi_3(x)$, by similar method we prove that the corresponding configurations are also not ground states on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

We note that apart of $\varphi_i(x)$, $i = 1, \ldots, 6$, there exist no H_0-periodic and not translation-invariant configuration. The proof is complete. □

By a similar method for $|A| = 2$ we can prove the following theorem.

Theorem 3.3. *Let $k = 2$ and $|A| = 2$. Then each H_A-periodic ground state are translation invariant.*

We observe that as $k=2$ and $|A|=3$, the normal divisor of H_A is of the form

$$G_k^{(2)}=\{x:\ |x| \text{ is even}\},$$

see [10]. For $G_k^{(2)}$-periodic ground state the following theorem holds.

Theorem 3.4. *Let $k=2$. Then for the Ising-Potts model the following statements hold true:*

I) On the set A_3 there exist two $G_k^{(2)}$-periodic ground states and they read as

$$\sigma(x)=\pm\begin{cases} 1 & \text{if} \quad x\in H_0,\\ -1 & \text{if} \quad x\in H_1.\end{cases}$$

II) On the set A_1 there exist four $G_k^{(2)}$-periodic ground states and they read as

$$\varphi(x)=\begin{cases} i & \text{if} \quad x\in H_0,\\ j & \text{if} \quad x\in H_1,\end{cases}$$

where $|i-j|=1$ $(i,j\in\Phi)$.

Proof. We begin with proving Statement I. We consider the configurations

$$\sigma_1(x)=\begin{cases} 1 & \text{if} \quad x\in H_0,\\ -1 & \text{if} \quad x\in H_1.\end{cases}$$

If $c_b\in H_0$, then for all $b\in M$ we have

$$\varphi_{1,b}(c_b)=1, \qquad |B_-|=3, \qquad |B_0|=0, \qquad |B_+|=0,$$

and hence, $\varphi_{1,b}\in C_3$. If $c_b\in H_1$, then

$$\varphi_{1,b}(c_b)=-1, \qquad |B_-|=0, \qquad |B_0|=0, \qquad |B_+|=3$$

and hence, $\varphi_{1,b}\in C_3$. Then we see that on $G_k^{(2)}$ the configuration $\sigma_1(x)$ is a periodic ground state on A_3.

Now we consider

$$\sigma_2(x)=\begin{cases} -1 & \text{if} \quad x\in H_0,\\ 1 & \text{if} \quad x\in H_1.\end{cases}$$

A configuration $\sigma_2(x)$ is also a periodic ground state on A_3; this can be proved similar to the same fact for $\sigma_1(x)$.

We proceed to proving Statement II. We consider the configurations

$$\varphi_1(x)=\begin{cases} 0 & \text{if} \quad x\in H_0,\\ -1 & \text{if} \quad x\in H_1.\end{cases}$$

If $c_b\in H_0$, then for all $b\in M$ we have

$$\varphi_{1,b}(c_b)=0, \qquad |B_-|=3, \qquad |B_0|=0, \qquad |B_+|=0$$

and hence, $\varphi_{1,b}\in C_1$. If $c_b\in H_1$, then for all $b\in M$ we have

$$\varphi_{1,b}(c_b)=-1, \qquad |B_-|=0, \qquad |B_0|=3, \qquad |B_+|=0$$

and hence, $\varphi_{1,b}\in C_1$. This shows that on G_k^2 the configuration σ_1 is a periodic ground state on A_1.

Now we consider

$$\varphi_2(x)=\begin{cases} -1 & \text{if} \quad x\in H_0,\\ 0, & \text{if} \quad x\in H_1.\end{cases}$$

A configuration $\varphi_2(x)$ is also a periodic gorund state on A_1 that can be proved as the same for $\varphi_1(x)$.

We consider
$$\varphi_3(x) = \begin{cases} 0 & \text{if} \quad x \in H_0, \\ 1 & \text{if} \quad x \in H_1. \end{cases}$$
If $c_b \in H_0$, then for all $b \in M$ we have
$$\varphi_{3,b}(c_b) = 0, \qquad |B_-| = 0, \qquad |B_0| = 0, \qquad |B_+| = 3$$
and hence, $\varphi_{3,b} \in C_1$. If $c_b \in H_1$, then for all $b \in M$ we have
$$\varphi_{3,b}(c_b) = 1, \qquad |B_-| = 0, \qquad |B_0| = 3, \qquad |B_+| = 0$$
and hence, $\varphi_{3,b} \in C_1$. This show that on $G_k^{(2)}$ the configuration φ_3 is a periodic ground state on A_1.

We consider
$$\varphi_4(x) = \begin{cases} 1 & \text{if} \quad x \in H_0, \\ 0 & \text{if} \quad x \in H_1. \end{cases}$$
The configuration $\varphi_4(x)$ is also a periodic ground state on A_1 and this can be proved as the same has been done for $\varphi_3(x)$. This shows that on $G_k^{(2)}$ the configurations $\varphi_i(x)$, $i = 1, \ldots, 4$, are periodic ground states on A_1. The proof is complete. □

Remark 3.2. *We note that as $\alpha = 0$, Hamiltonian* (2.1) *describes the Potts model. Then Theorem 3.4 coincides with Statement B.3 in Theorem 2 from work* [8] *in the case $J_2 = 0$.*

3.3. Weakly periodic ground states. We are going to study H_A-weakly periodic ground state. An H_A-weakly periodic configuration reads as
$$\varphi_(x) = \begin{cases} \sigma_{0,0} & \text{if} \quad x_\downarrow \in H_0, \quad x \in H_0, \\ \sigma_{0,1} & \text{if} \quad x_\downarrow \in H_0, \quad x \in H_1, \\ \sigma_{1,0} & \text{if} \quad x_\downarrow \in H_1, \quad x \in H_0, \\ \sigma_{1,1} & \text{if} \quad x_\downarrow \in H_1, \quad x \in H_1, \end{cases}$$
where $\sigma_{i,j} \in \Phi$, $i, j = 0, 1$.

In what follows for the sake of convenience we write a weakly periodic configuration $\varphi(x)$, $x \in G_k$, as $\varphi = (\sigma_{0,0}, \sigma_{0,1}, \sigma_{1,0}, \sigma_{1,1})$.

Theorem 3.5. *Let $k = 2$ and $|A| = 1$. Then for the Ising-Potts model the following statements hold true:*

I) On the set
$$\left\{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = -\frac{(1-\alpha)}{\alpha} J_2;\ J_2 \leqslant 0\right\}$$
there exist six H_A-weakly periodic (non-periodic) ground states and they are of the form
$$(1, 1, 0, 1), (-1, -1, 0, -1), (1, 0, 1, 1), (-1, 0, -1, -1), (-1, 0, 0, 1), (1, 0, 0, -1).$$
II) Each H_A-weakly periodic ground state except for the configurations provided in Statement I are translation-invariant.

Proof. We begin with proving Statement I. We consider a configuration
$$\varphi_1 = (1, 1, 0, 1).$$
1. Let $c_b \in H_0$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 0$, $|B_+| = 3$ and hence, $\varphi_{1,b} \in C_4$.

b) $c_{b\downarrow} \in H_0$ and $\varphi_{1,b}(c_{b\downarrow}) = 0$, then $\varphi_{1,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 1$, $|B_+| = 2$ and hence, $\varphi_{1,b} \in C_4$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 0$, $|B_-| = 0$, $|B_0| = 0$, $|B_+| = 3$ and hence, $\varphi_{1,b} \in C_1$.

2. Let $c_b \in H_1$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 0|$, $|B_+| = 3$ and hence, $\varphi_{1,b} \in C_4$.

b) $c_{b\downarrow} \in H_1$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 1$, $|B_+| = 2$ and hence, $\varphi_{1,b} \in C_4$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 0$, $|B_+| = 3$ and hence, $\varphi_{1,b} \in C_1$.

This implies that on the set

$$A_1 \cap A_4 = \left\{ (J_1, J_2) \in \mathbb{R}^2 \mid J_1 = -\frac{(1-\alpha)}{\alpha} J_2;\ J_2 \leqslant 0 \right\}$$

a weakly periodic configuration φ_1 is a $G_k^{(2)}$-weakly periodic ground state.

We consider

$$\varphi_2 = (-1, -1, 0, -1).$$

1. Let $c_b \in H_0$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then $\varphi_{2,b}(c_b) = -1$, $|B_-| = 3$, $|B_0| = 0$, $|B_+| = 0$ and hence, $\varphi_{2,b} \in C_4$.

b) $c_{b\downarrow} \in H_0$ and $\varphi_{2,b}(c_{b\downarrow}) = 0$, then $\varphi_{2,b}(c_b) = -1$, $|B_-| = 2$, $|B_0| = 1$, $|B_+| = 0$ and hence, $\varphi_{2,b} \in C_4$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then $\varphi_{2,b}(c_b) = 0$, $|B_-| = 3$, $|B_0| = 0$, $|B_+| = 0$ and hence, $\varphi_{2,b} \in C_1$.

2. Let $c_b \in H_1$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then $\varphi_{2,b}(c_b) = -1$, $|B_-| = 3$, $|B_0| = 0$, $|B_+| = 0$ and hence, $\varphi_{2,b} \in C_4$.

b) $c_{b\downarrow} \in H_1$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then $\varphi_{2,b}(c_b) = -1$, $|B_-| = 2$, $|B_0| = 1$, $|B_+| = 0$ and hence, $\varphi_{2,b} \in C_4$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then $\varphi_{2,b}(c_b) = -1$, $|B_-| = 3$, $|B_0| = 0$, $|B_+| = 0$ and hence, $\varphi_{2,b} \in C_1$.

This yields that on the set

$$A_1 \cap A_4 = \left\{ (J_1, J_2) \in \mathbb{R}^2 \mid J_1 = -\frac{(1-\alpha)}{\alpha} J_2;\ J_2 \leqslant 0 \right\}$$

are weakly periodic configuration φ_2 is a $G_k^{(2)}$-weakly periodic ground state.

We consider

$$\varphi_3 = (1, 0, 1, 1).$$

1. Let $c_b \in H_0$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{3,b}(c_{b\downarrow}) = 1$, then $\varphi_{3,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 1$, $|B_+| = 2$ and hence, $\varphi_{3,b} \in C_4$.

b) $c_{b\downarrow} \in H_1$ and $\varphi_{3,b}(c_{b\downarrow}) = 1$, then $\varphi_{3,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 0$, $|B_+| = 3$ and hence, $\varphi_{3,b} \in C_4$.

2. Let $c_b \in H_1$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{3,b}(c_{b\downarrow}) = 1$, then $\varphi_{3,b}(c_b) = 0$, $|B_-| = 0$, $|B_0| = 0$, $|B_+| = 3$ and hence, $\varphi_{3,b} \in C_2$.

b) $c_{b\downarrow} \in H_1$ and $\varphi_{3,b}(c_{b\downarrow}) = 1$, then $\varphi_{3,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 0$, $|B_+| = 3$ and hence, $\varphi_{3,b} \in C_2$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{3,b}(c_{b\downarrow}) = 0$, then $\varphi_{3,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 1$, $|B_+| = 2$ and hence, $\varphi_{3,b} \in C_4$.

This yields that on the set

$$A_2 \cap A_4 = \left\{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = -\frac{(1-\alpha)}{\alpha} J_2;\ J_2 \leqslant 0\right\}$$

a weakly periodic configuration φ_3 is a $G_k^{(2)}$-weakly periodic ground state.

We consider

$$\varphi_4 = (-1, 0, -1, -1).$$

1. Let $c_b \in H_0$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{4,b}(c_{b\downarrow}) = -1$, then $\varphi_{4,b}(c_b) = -1$, $|B_-| = 2$, $|B_0| = 1$, $|B_+| = 0$ and hence, $\varphi_{4,b} \in C_4$.

b) $c_{b\downarrow} \in H_1$ and $\varphi_{4,b}(c_{b\downarrow}) = -1$, then $\varphi_{4,b}(c_b) = -1$, $|B_-| = 3$, $|B_0| = 0$, $|B_+| = 0$ and hence, $\varphi_{4,b} \in C_4$.

2. Let $c_b \in H_1$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{4,b}(c_{b\downarrow}) = -1$, then $\varphi_{4,b}(c_b) = 0$, $|B_-| = 3$, $|B_0| = 0$, $|B_+| = 0$ and hence, $\varphi_{4,b} \in C_2$.

b) $c_{b\downarrow} \in H_1$ and $\varphi_{4,b}(c_{b\downarrow}) = -1$, then $\varphi_{4,b}(c_b) = -1$, $|B_-| = 3$, $|B_0| = 0$, $|B_+| = 0$ and hence, $\varphi_{4,b} \in C_2$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{4,b}(c_{b\downarrow}) = 0$, then $\varphi_{4,b}(c_b) = -1$, $|B_-| = 2$, $|B_0| = 1$, $|B_+| = 0$ and hence, $\varphi_{4,b} \in C_4$.

This yields that on the set

$$A_2 \cap A_4 = \left\{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = -\frac{(1-\alpha)}{\alpha} J_2;\ J_2 \leqslant 0\right\}$$

a weakly periodic configuration φ_4 is a $G_k^{(2)}$-weakly periodic ground state

We consider

$$\varphi_5 = (-1, 0, 0, 1).$$

1. Let $c_b \in H_0$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{5,b}(c_{b\downarrow}) = -1$, then $\varphi_{5,b}(c_b) = -1$, $|B_-| = 2$, $|B_0| = 1$, $|B_+| = 0$ and hence, $\varphi_{5,b} \in C_4$.

b) $c_{b\downarrow} \in H_0$ and $\varphi_{5,b}(c_{b\downarrow}) = -1$, then $\varphi_{5,b}(c_b) = 0$, $|B_-| = 1$, $|B_0| = 0$, $|B_+| = 2$ and hence, $\varphi_{5,b} \in C_1$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{5,b}(c_{b\downarrow}) = 1$, then $\varphi_{5,b}(c_b) = 0$, $|B_-| = 2$, $|B_0| = 0$, $|B_+| = 1$ and hence, $\varphi_{5,b} \in C_1$.

2. Let $c_b \in H_1$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{5,b}(c_{b\downarrow}) = -1$, then $\varphi_{5,b}(c_b) = 0$, $|B_-| = 1$, $|B_0| = 0$, $|B_+| = 2$ and hence, $\varphi_{5,b} \in C_1$.

b) $c_{b\downarrow} \in H_1$ and $\varphi_{5,b}(c_{b\downarrow}) = 0$, then $\varphi_{5,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 2$, $|B_+| = 1$ and hence, $\varphi_{5,b} \in C_5$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{5,b}(c_{b\downarrow}) = 1$, then $\varphi_{5,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 1$, $|B_+| = 2$ and hence, $\varphi_{5,b} \in C_4$.

This yields that on the set

$$A_1 \cap A_4 \cap A_5 = \left\{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = -\frac{(1-\alpha)}{\alpha} J_2;\ J_2 \leqslant 0\right\}$$

a weakly periodic configuration φ_5 is a $G_k^{(2)}$-weakly periodic ground state.

We consider

$$\varphi_6 = (1, 0, 0, -1).$$

1. Let $c_b \in H_0$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{6,b}(c_{b\downarrow}) = 1$, then $\varphi_{6,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 1$, $|B_+| = 2$ and hence, $\varphi_{6,b} \in C_4$.

b) $c_{b\downarrow} \in H_0$ and $\varphi_{6,b}(c_{b\downarrow}) = 1$, then $\varphi_{6,b}(c_b) = 0$, $|B_-| = 2$, $|B_0| = 0$, $|B_+| = 1$ and hence, $\varphi_{6,b} \in C_1$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{6,b}(c_{b\downarrow}) = -1$, then $\varphi_{6,b}(c_b) = 0$, $|B_-| = 1$, $|B_0| = 0$, $|B_+| = 2$ and hence, $\varphi_{6,b} \in C_1$.

2. Let $c_b \in H_1$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{6,b}(c_{b\downarrow}) = 1$, then $\varphi_{6,b}(c_b) = 0$, $|B_-| = 2$, $|B_0| = 0$, $|B_+| = 1$ and hence, $\varphi_{6,b} \in C_1$.

b) $c_{b\downarrow} \in H_1$ and $\varphi_{6,b}(c_{b\downarrow}) = 0$, then $\varphi_{6,b}(c_b) = -1$, $|B_-| = 1$, $|B_0| = 2$, $|B_+| = 0$ and hence, $\varphi_{6,b} \in C_5$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{6,b}(c_{b\downarrow}) = -1$, then $\varphi_{6,b}(c_b) = -1$, $|B_-| = 2$, $|B_0| = 1$, $|B_+| = 0$ and hence, $\varphi_{6,b} \in C_4$.

This yields that on the set

$$A_1 \cap A_4 \cap A_5 = \left\{ (J_1, J_2) \in \mathbb{R}^2 \mid J_1 = -\frac{(1-\alpha)}{\alpha} J_2;\ J_2 \leqslant 0 \right\}$$

a weakly periodic configuration φ_4 is a $G_k^{(2)}$-weakly periodic ground state.

We observe that in all cases the domain of the intersections of A_1, A_2, A_4 and A_5 is equal to the set

$$\left\{ (J_1, J_2) \in \mathbb{R}^2 \mid J_1 = -\frac{(1-\alpha)}{\alpha} J_2;\ J_2 \leqslant 0 \right\}.$$

Hence, on the set

$$\left\{ (J_1, J_2) \in \mathbb{R}^2 \mid \alpha J_1 = -(1-\alpha) J_2;\ (1-\alpha) J_2 \leqslant 0 \right\}$$

the configurations φ_i, $i = 1, \ldots, 6$ are weakly periodic ground states. This completes the proof of the first part of the theorem.

We proceed to proving Statement II. We consider an arbitrary configuration differing from the configurations in Statement I:

$$\varphi_7 = (1, 1, 0, -1).$$

1. Let $c_b \in H_0$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{7,b}(c_{b\downarrow}) = 1$, then $\varphi_{7,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 0$, $|B_+| = 3$ and hence, $\varphi_{7,b} \in C_2$.

b) $c_{b\downarrow} \in H_0$ and $\varphi_{7,b}(c_{b\downarrow}) = 0$, then $\varphi_{7,b}(c_b) = 1$, $|B_-| = 0$, $|B_0| = 1$, $|B_+| = 2$ and hence, $\varphi_{7,b} \in C_4$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{7,b}(c_{b\downarrow}) = -1$, then $\varphi_{7,b}(c_b) = 0$, $|B_-| = 1$, $|B_0| = 0$, $|B_+| = 2$ and hence, $\varphi_{7,b} \in C_1$.

2. Let $c_b \in H_1$, then the following cases are possible:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{7,b}(c_{b\downarrow}) = 1$, then $\varphi_{7,b}(c_b) = 1$, $|B_-| = 2$, $|B_0| = 0$, $|B_+| = 1$ and hence, $\varphi_{7,b} \in C_7$.

b) $c_{b\downarrow} \in H_1$ and $\varphi_{7,b}(c_{b\downarrow}) = 1$, then $\varphi_{7,b}(c_b) = -1$, $|B_-| = 1$, $|B_0| = 1$, $|B_+| = 1$ and hence, $\varphi_{7,b} \in C_{10}$.

c) $c_{b\downarrow} \in H_1$ and $\varphi_{7,b}(c_{b\downarrow}) = -1$, then $\varphi_{7,b}(c_b) = -1$, $|B_-| = 2$, $|B_0| = 1$, $|B_+| = 0$ and hence, $\varphi_{7,b} \in C_4$.

We note that the intersection $A_1 \cap A_2 \cap A_4 \cap A_7 \cap A_{10}$ produces the points $(0, 0)$.

This gives that the configuration $\varphi_{7,b}$ is translation-invariant. By a similar method we can treat the remaining cases. Hence, all configurations excepts ones mentioned in Statement I are translation-invariant on $\mathbb{R}^2 \setminus (0, 0)$. □

Remark 3.3. *It was proved in work* [10] *that for the Potts model on the Cayley tree of order two, as* $|A| = 1$*, there exist no* H_A*-weakly periodic (non-periodic) ground states. At the same time, for the Ising-Potts model on the Cayley tree of order two, as* $|A| = 1$*, there exist* H_A*-weakly periodic (non-periodic) ground states.*

Acknowledgments

The authors express a deep gratitude to professors N.N. Ganikhodzhaev and U.A. Rozikov for useful advices on the work.

BIBLIOGRAPHY

1. Ya.G. Sinaj. *Theory of phase transitions: rigorous results.* Nauka, Moscow (1980). [Pergamon Press, Oxford (1982).]
2. R.A. Minlos. *Introduction to mathematical statistical physics.* Amer. Math. Soc., Providence, RI. (2000).
3. U.A. Rozikov. *Gibbs measures on Cayley trees: results and open problems* // Rev. Math. Phys. **25**:1, 112 pp. (2013).
4. N.N. Ganikhodzhaev, U.A. Rozikov. *Discription of periodic extreme Gibbs measures of some lattice models on the Cayley tree* // Teor. Matem. Fiz. **111**:1, 109–117 (1997). [Theor. Math. Phys. **111**:1, 480–486 (1997).]
5. U.A. Rozikov. *A Constructive description of ground states and Gibbs measures for Ising model with two-step interactions on Cayley tree* // J. Stat. Phys. **122**:2, 217–235 (2006).
6. U.A. Rozikov, M.M. Rakhmatullaev. *Weakly periodic ground states and Gibbs measures for the Ising model with competing interactions on the Cayley tree* // Teor. Matem. Fiz. **160**:3, 507–516 (2009). [Theor. Math. Phys. **160**:3, 1292–1300 (2009).]
7. M.M. Rahmatullaev. *Description of weak periodic ground states of Ising model with competing interactions on Cayley tree* // Appl. Math. Inf. Sci. **4**:2, 237–241 (2010).
8. G.I. Botirov, U.A. Rozikov. *Potts model with competing interactions on the Cayley tree: the contour method* // Teor. Matem. Fiz. **153**:1, 86–97 (2007). [Theor. Math. Phys. **153**:1, 1423–1433 (2007).]
9. F.M. Mukhamedov, U.A. Rozikov, F.F. Mendes. *On contour arguments for the three state Potts model with competing interactions on a semi-infinite Cayley tree* // J. Math Phys. **48**:1, 013301 (2007).
10. M.M. Rakhmatullaev. *Weakly periodic Gibbs measures and ground states for the Potts model with competing interactions on the Cayley tree* // Teor. Matem. Fiz. **176**:3, 477–493 (2013). [Theor. Math. Phys. **176**:3, 1236–1251 (2013).]
11. M.M. Rahmatullaev, M.A. Rasulova. *Periodic and weakly periodic ground states for the Potts model with competing interactions on the Cayley tree* // Matem. Trudy. **18**:2, 112–132 (2015). [Sib. Adv. Math. **26**:3, 215–229 (2016).]
12. F. Mukhamedov, Chin Hee Pah, and H. Jamil. *Ground states and phase transition of the* λ*-model on the Cayley tree* // Teor. Matem. Fiz. **194**:2, 304–319 (2018). [Theor. Math. Phys. **194**:2, 260–273 (2018).]
13. F.M. Mukhamedov, M.M. Rahmatullaev, M.A. Rasulova. *Weakly periodic ground states for the* λ*-model* // Ukr. Math. Zhur. **72**:5, 667–678 (2020). [Ukr. Math. J. **72**:5, 771–784 (2020).
14. M.M. Rahmatullaev, D.O. Egamov, F.H. Haydarov. *Periodic and weakly periodic ground states corresponding to subgroups of index three for the ising model on Cayley tree* // Rep. Math. Phys. **88**:2, 247–257 (2021).
15. M.I. Kargapolov, Ju.I. Merzljakov. *Fundamentals of the theory of groups.* Nauka, Moscow (1982). [Springer-Verlag, New York (1979).]
16. M.K. Ramazanova, A.K. Murtazaeva. *Phase transitions in the antiferromagnetic layered Ising model on a cubic Lattice* // Pisma ZhETF. **103**:7, 522–526 (2016). [JETP Letters. **103**:7, 460–464 (2016).]

17. F.A. Kassan-Ogly, A.K. Murtazaev, A.K. Zhuravleva, M.K. Ramazanov, A.I. Proshkina. *Ising model on a square lattice with second-neighbor and third-neighbor interactions* // J. Magn. Magn. Mat. **384**, 247–254 (2015).
18. M.A. Magomedov, A.K. Murtazaev, L.K. Magomedova. *Phase transitions in the Potts model on triangular lattice* // Vestn. Dagestan. Gosud. Univ. Ser. 1. Estestvenye nauki. **32**:4, 14–24 (2017). (in Russian).
19. A.B. Babaeva, M.A. Magomedov, A.K. Murtazaev, F.A. Kassan-Ogly, A. I. Proshki. *Phase transitions in a two-dimensional antiferromagnetic Potts model on a triangular lattice with next-nearest neighbor interactions* // ZhETF. **149**:2, 357–366 (2016). [J. Exper. Theor. Phys. **122**:2, 310–317 (2016).]
20. M.M. Rahmatullaev, D.J. Dekhkonov. *The Potts model on a Cayley tree: the new class of Gibbs measures* // Preprint: arXiv:2211.12128 (2022).

Muzaffar Mukhamadzhanovich Rahmatullaev,
Insitute of Mathematics named after V.I. Romanovsky
of the Academy of Sciences of the Republic of Uzbekistan,
Universitetstkaya str. 9,
100174, Tashkent, Uzbekistan
Namangan State University,
Uyci str. 316,
160136, Namangan, Uzbekistan
E-mail: `mrahmatullaev@rambler.ru`

Begzod Mukhtorzhonovich Isakov,
Andijan State University,
Universitetstkaya str. 129,
170100, Andijan, Uzbekistan
E-mail: `begzod1981bek@gmail.com`