doi:10.13108/2023-15-1-21

LOCAL AND NONLOCAL BOUNDARY VALUE PROBLEMS FOR GENERALIZED ALLER–LYKOV EQUATION

S.KH. GEKKIEVA, M.A. KEREFOV, F.M. NAKHUSHEVA

Abstract. In mathematical modelling of solid media with memory there arise equations describing a new type of wave motion, which is between the usual diffusion and classical waves. Here we mean differential equations with fractional derivatives both in time and spatial variables, which are a base for most part of mathematical models in mechanics of liquids, viscoelasticity as well as in processes in media with fractal structure and memory.

In the present work we present a qualitatively new moisture transfer equation being a generalization of Aller-Lykov equation. This generalization provides an opportunity to reflect specific features of the studied objects in the nature of the equation such, namely, the structure and physical properties of the going processes, by means of introducing a fractal velocity of moisture varying.

The work is devoted to studying local and nonlocal boundary value problems for inhomogeneous Aller-Lykov moisture transfer equation with variable coefficients and Riemann-Liouville fractional time derivative. For a generalized equation of Aller-Lykov type we consider initial boundary value problems with Dirichlet and Robin boundary conditions as well as nonlocal problems involving nonlocality in time in the boundary conditions. Assuming the existence of regular solutions, by the method of energy inequalities, we obtain apriori estimates in terms of Riemann-Liouville fractional derivative, which imply the uniqueness of the solutions to the considered problems as well as their stability in the right hand side and initial data.

Keywords: Aller-Lykov water transfer equation, Riemann-Liouville fractional derivative, Fourier method, apriori estimate.

Mathematics Subject Classification: 35E99

1. Introduction

The issues of heat and moisture transfer in soils are fundamental in solving many problems in hydrology, agricultural physics, building physics and other fields in science. The researchers focus their attention on the possibility of describing specific features of the studied objects, their structure, physical properties of the ongoing processes and others by mathematical equations [1, Ch. 6]. Because of this there arises a qualitatively new class of differential equations of state and transfer with fractional derivatives being the base for many mathematical models describing a wide class of physical and mathematical processes with a fractal structure and memory [2, Ch. 5].

As an example of such model describing the transfer of soil moisture with taking into consideration the motion against the moisture gradient as well as of the fractal structure of the soil, a generalized Aller-Lykov transfer equation can serve:

$$A_1 D_{0t}^{\alpha+1} u + D_{0t}^{\alpha} u = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) + A D_{0t}^{\alpha}\frac{\partial^2 u}{\partial x^2} + f(x,t),$$

S.Kh. Gekkieva, M.A. Kerefov, F.M. Nakhusheva, Local and nonlocal boundary value problems for generalized Aller–Lykov equation.

Submitted December 2, 2021.

where D_{0t}^{γ} is an operator of the fractional Riemann-Liouville differentiation [2], $0 < \alpha < 1$, A_1, $A = const > 0$, $k(x,t)$ is a diffusion coefficient, $f(x,t)$ is the density of the moisture sources. For a function $u(x,t)$ depending on two variables, the operator of partial integral-differentiation $D_{0t}^{\alpha}u(x,\tau)$ in the variable t is described in the same way as for a function of one variable, while the second variable x is treated as a parameter. For instance, in the case $0 < \alpha < 1$ we have:

$$D_{0t}^{\alpha}u(x,\tau) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha}}d\tau.$$

The moisture transfer equation in the local setting (as $\alpha = 1$) were considered in works by many authors and were solved by the separation of variables method, by the method of apriori estimates as well as by numerical methods. Among recent works we mention [3], [4], in which there were obtained apriori estimates for solutions to nonlocal problems for Aller-Lykov moisture transfer equation in differential and difference setting, as well as works [5]–[7], in which the Aller-Lykov moisture transfer with a fractional time derivative was studied with boundary conditions o various types.

In [8] a unique solvability of Dirichlet boundary value problem for the Aller-Lykov equation with constant coefficients was proved. In work [9] there was studied the Neumann boundary value problem.

In [10] for generalized Aller and Aller-Lykov equations with Dirichlet boundary conditions there were obtained solutions for a system of difference equation with constant coefficients arising while using the methods of straight lines. There were obtained apriori estimates, which implied the convergence of solutions to a system of ordinary fractional differential equations with varying coefficients.

The present paper is devoted to studying local and nonlocal boundary value problems for moisture transfer of Aller-Lykov type with varying coefficients and with the fractional Riemann-Liouville derivative

$$A_1 D_{0t}^{\alpha+1}u + D_{0t}^{\alpha}u = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) + D_{0t}^{\alpha}\frac{\partial}{\partial x}\left(\eta(x)\frac{\partial u}{\partial x}\right) - q(x,t)u + f(x,t). \tag{1.1}$$

We shall study equation (1.1) by the method of apriori estimates.

2. Local boundary value problems for inhomogeneous Aller-Lykov equation with fractional time derivative

In a rectangle $\Omega_T = \{(x,t): \ 0 < x < l, \ 0 < t \leqslant T\}$ we consider equation (1.1). A regular solution of equation (1.1) in Ω_T is a function $u = u(x,t)$ in the class

$$D_{0t}^{\alpha-1}u(x,t), \quad D_{0t}^{\alpha}u(x,t) \in C\left(\overline{\Omega}_T\right); \quad D_{0t}^{\alpha+1}u(x,t), \quad u_{xx}(x,t), \quad D_{0t}^{\alpha}u_{xx}(x,t) \in C\left(\Omega_T\right),$$

which satisfies equation (1.1) at all $(x,t) \in \Omega_T$.

We consider the Dirichlet boundary value problem for equation (1.1) in a closed rectangle $\overline{\Omega}_T = \{(x,t): \ 0 \leqslant x \leqslant l, \ 0 \leqslant t \leqslant T\}$:

$$u(0,t) = u(l,t) = 0, \qquad 0 \leqslant t \leqslant T, \tag{2.1}$$

$$\lim_{t\to 0} D_{0t}^{\alpha-1}u(x,t) = u_0(x), \qquad \lim_{t\to 0} D_{0t}^{\alpha}u(x,t) = u_1(x), \qquad 0 \leqslant x \leqslant l, \tag{2.2}$$

where $u_0(x)$, $u_1(x)$ are given functions.

In what follows we assume the existence of regular solutions for the considered problems. By M_i, $i = 1, 2, \ldots$, we shall denote positive constants depending only on the given data of a considered problem.

We define a scalar product and a norm as follows:

$$(a,b)=\int\limits_0^l ab dx, \qquad (a,a)=\|a\|_0^2,$$

where a, b are given functions on the segment $[0,l]$.

2.1. Apriori estimate for solution to Dirichlet boundary value problem.

Theorem 2.1. *If*

$$k_x(x,t),\ \eta_x(x),\ k_t(x,t),\ q_t(x,t),\ f(x,t)\in C\left(\overline{\Omega}_T\right);\quad u_1(x)\in C[0,l];\quad u_0(x)\in C^2[0,l]$$

and

$$0<c_1\leqslant k(x,t);\quad \eta(x),\quad q(x,t)\leqslant c_2;\quad k_t,\quad q_t\leqslant 0 \quad \text{everywhere in} \quad \overline{\Omega}_T$$

and the condition $u_0(0)=u_0(l)=0$ is obeyed, then for a solution of problem (1.1)–(2.2) an apriori estimate

$$\|D_{0t}^{\alpha}u\|_0^2+\|D_{0t}^{\alpha}u\|_{2,\Omega_t}^2+\|D_{0t}^{\alpha}u_x\|_{2,\Omega_t}^2\leqslant M\left(\|f\|_{2,\Omega_t}^2+\|u_0(x)\|_{W_2^2(0,l)}^2+\|u_1(x)\|_0^2\right) \tag{2.3}$$

holds, where

$$\|D_{0t}^{\alpha}u\|_{2,\Omega_t}^2=\int\limits_0^t\|D_{0\tau}^{\alpha}u(x,\tau_1)\|_0^2\,d\tau,\qquad \|u_0(x)\|_{W_2^2(0,l)}^2=\|u_0(x)\|_0^2+\|u_0'(x)\|_0^2+\|u_0''(x)\|_0^2.$$

Proof. We introduce a new unknown function $g(x,t)$ by letting

$$u(x,t)=g(x,t)+\frac{t^{\alpha-1}}{\Gamma(\alpha)}u_0(x) \tag{2.4}$$

so that $g(x,t)$ is a deviation of the function $u(x,t)$ from a known function $\frac{t^{\alpha-1}}{\Gamma(\alpha)}u_0(x)$. In view of [11],

$$D_{0t}^{\alpha+1}t^{\alpha-1}=0,\qquad D_{0t}^{\alpha}t^{\alpha-1}=0,\qquad D_{0t}^{\alpha-1}t^{\alpha-1}=\Gamma(\alpha)$$

and the function $g(x,t)$ is determined as a solution of the equation

$$A_1D_{0t}^{\alpha+1}g+D_{0t}^{\alpha}g-(kg_x)_x-D_{0t}^{\alpha}(\eta g_x)_x+qg=F(x,t),\quad 0<x<l,\quad 0<t\leqslant T, \tag{2.5}$$

with the initial conditions

$$\begin{aligned}
&\lim_{t\to0}D_{0t}^{\alpha-1}g(x,t)=\lim_{t\to0}D_{0t}^{\alpha-1}\left(u(x,t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}u_0(x)\right)=u_0(x)-\frac{u_0(x)}{\Gamma(\alpha)}\lim_{t\to0}D_{0t}^{\alpha-1}t^{\alpha-1}=0,\\
&\lim_{t\to0}D_{0t}^{\alpha}g(x,t)=\lim_{t\to0}D_{0t}^{\alpha}\left(u(x,t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}u_0(x)\right)=u_1(x)-\frac{u_0(x)}{\Gamma(\alpha)}\lim_{t\to0}D_{0t}^{\alpha}t^{\alpha-1}=u_1(x)
\end{aligned} \tag{2.6}$$

and boundary conditions

$$g(0,t)=g(l,t)=0,\qquad 0\leqslant t\leqslant T, \tag{2.7}$$

where

$$F(x,t)=f(x,t)+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\left(k_xu_0'(x)+ku_0''(x)+\eta_xu_0'(x)+\eta u_0''(x)-qu_0(x)\right).$$

We shall obtain an apriori estimate in terms of the fractional Riemann-Liouville derivative; in order to do this, we calculate the scalar product of equation (2.5) with

$$D_{0t}^{\alpha}g=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int\limits_0^t\frac{g(x,\tau)d\tau}{(t-\tau)^{\alpha}}.$$

We get:

$$\begin{aligned}\left(A_1 D_{0t}^{\alpha+1} g, D_{0t}^{\alpha} g\right) + \left(D_{0t}^{\alpha} g, D_{0t}^{\alpha} g\right) - \left((k g_x)_x, D_{0t}^{\alpha} g\right) \\ - \left(D_{0t}^{\alpha} (\eta g_x)_x, D_{0t}^{\alpha} g\right) + \left(q g, D_{0t}^{\alpha} g\right) = \left(F, D_{0t}^{\alpha} g\right).\end{aligned} \tag{2.8}$$

We transform the terms in identity (2.8) by using (2.6), (2.7):

$$\begin{aligned}
\left(A_1 D_{0t}^{\alpha+1} g, D_{0t}^{\alpha} g\right) &= A_1 \int_0^l \frac{1}{\Gamma(1-\alpha)} \frac{\partial^2}{\partial t^2} \int_0^t \frac{g(x,\tau)d\tau}{(t-\tau)^\alpha} \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{g(x,\tau)d\tau}{(t-\tau)^\alpha} dx \\
&= \frac{A_1}{2} \int_0^l \frac{\partial}{\partial t} \left(D_{0t}^{\alpha} g\right)^2 dx = \frac{A_1}{2} \frac{\partial}{\partial t} \left\| D_{0t}^{\alpha} g \right\|_0^2, \\
\left(D_{0t}^{\alpha} g, D_{0t}^{\alpha} g\right) &= \left\| D_{0t}^{\alpha} g \right\|_0^2, \\
\left((k g_x)_x, D_{0t}^{\alpha} g\right) &= \frac{1}{\Gamma(1-\alpha)} \int_0^l (k g_x)_x \frac{\partial}{\partial t} \int_0^t \frac{g(x,\tau)d\tau}{(t-\tau)^\alpha} dx \\
&= \frac{1}{\Gamma(1-\alpha)} \left\{ k g_x(x,t) \frac{\partial}{\partial t} \int_0^t \frac{g(x,\tau)d\tau}{(t-\tau)^\alpha} \Bigg|_0^l - \int_0^l k g_x(x,t) \frac{\partial}{\partial t} \int_0^t \frac{g_x(x,\tau)d\tau}{(t-\tau)^\alpha} dx \right\} \\
&= -\frac{1}{\Gamma(1-\alpha)} \int_0^l k g_x(x,t) \frac{\partial}{\partial t} \int_0^t \frac{g_x(x,\tau)d\tau}{(t-\tau)^\alpha} dx, \\
\left(D_{0t}^{\alpha} (\eta g_x)_x, D_{0t}^{\alpha} g\right) &= \frac{1}{\Gamma^2(1-\alpha)} \int_0^l \frac{\partial}{\partial t} \int_0^t \frac{(\eta g_x)_x d\tau}{(t-\tau)^\alpha} \frac{\partial}{\partial t} \int_0^t \frac{g(x,\tau)d\tau}{(t-\tau)^\alpha} dx \\
&= \frac{1}{\Gamma^2(1-\alpha)} \left\{ \frac{\partial}{\partial t} \eta \int_0^t \frac{g_x(x,\tau)d\tau}{(t-\tau)^\alpha} \frac{\partial}{\partial t} \int_0^t \frac{g(x,\tau)d\tau}{(t-\tau)^\alpha} \Bigg|_0^l \right. \\
&\quad \left. - \int_0^l \frac{\partial}{\partial t} \eta \int_0^t \frac{g_x(x,\tau)d\tau}{(t-\tau)^\alpha} \frac{\partial}{\partial t} \int_0^t \frac{g_x(x,\tau)d\tau}{(t-\tau)^\alpha} dx \right\} \\
&\leqslant -c_1 \int_0^l \left(\frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{g_x(x,\tau)d\tau}{(t-\tau)^\alpha} \right)^2 dx = -c_1 \left\| D_{0t}^{\alpha} g_x \right\|_0^2.
\end{aligned}$$

To estimate the right hand side in (2.8), we use the Cauchy-Schwarz inequality and ε-inequality [12]:

$$\left(F, D_{0t}^{\alpha} g\right) \leqslant \frac{1}{4\varepsilon} \|F\|_0^2 + \varepsilon \left\| D_{0t}^{\alpha} g \right\|_0^2, \quad \varepsilon > 0.$$

In view of the obtained inequalities by (2.8) we find:

$$\begin{aligned}\frac{A_1}{2} \frac{\partial}{\partial t} \left\| D_{0t}^{\alpha} g \right\|_0^2 + \left\| D_{0t}^{\alpha} g \right\|_0^2 + \frac{1}{\Gamma(1-\alpha)} \int_0^l k g_x(x,t) \frac{\partial}{\partial t} \int_0^t \frac{g_x(x,\tau)d\tau}{(t-\tau)^\alpha} dx \\ + c_1 \left\| D_{0t}^{\alpha} g_x \right\|_0^2 + \left(q g, D_{0t}^{\alpha} g\right) \leqslant \frac{1}{4\varepsilon} \|F\|_0^2 + \varepsilon \left\| D_{0t}^{\alpha} g \right\|_0^2.\end{aligned} \tag{2.9}$$

We integrate (2.9) over τ from 0 to t :

$$
\begin{aligned}
&\frac{A_1}{2}\left\|D_{0t}^{\alpha}g\right\|_0^2+\int\limits_0^t\left\|D_{0\tau}^{\alpha}g(x,\tau_1)\right\|_0^2d\tau+\frac{1}{\Gamma(1-\alpha)}\int\limits_0^t d\tau\int\limits_0^l kg_x(x,\tau)\frac{\partial}{\partial\tau}\int\limits_0^\tau\frac{g_x(x,\tau_1)d\tau_1}{(\tau-\tau_1)^\alpha}dx\\
&\qquad+c_1\int\limits_0^t\left\|D_{0\tau}^{\alpha}g_x(x,\tau_1)\right\|_0^2d\tau+\frac{1}{\Gamma(1-\alpha)}\int\limits_0^t d\tau\int\limits_0^l qg(x,\tau)\frac{\partial}{\partial\tau}\int\limits_0^\tau\frac{g(x,\tau_1)d\tau_1}{(\tau-\tau_1)^\alpha}dx \qquad (2.10)\\
&\qquad\leqslant\frac{1}{4\varepsilon}\left\|F\right\|_{2,\Omega_t}^2+\varepsilon\int\limits_0^t\left\|D_{0\tau}^{\alpha}g(x,\tau_1)\right\|_0^2d\tau+\frac{A_1}{2}\left\|(D_{0t}^{\alpha}g)(x,0)\right\|_0^2.
\end{aligned}
$$

We are going to prove that the triple integrals in the left hand side of the latter inequalities [13, Ch. 2]. We introduce the notations:

$$
J=\int\limits_0^l dx\int\limits_0^t kg_x(x,\tau)\frac{\partial}{\partial\tau}\int\limits_0^\tau\frac{g_x\left(x,\tau_1\right)d\tau_1}{\left(\tau-\tau_1\right)^\alpha}d\tau,
$$

$$
F_1(x,t)=\frac{\sin\pi(1-\alpha)}{\pi}\frac{\partial}{\partial t}\int\limits_0^t\frac{g_x(x,\tau_1)d\tau_1}{(t-\tau_1)^\alpha}.
$$

Then in view of the resolving formula for the integral Abel equation [14] we have

$$
J=\frac{\pi}{\sin\pi(1-\alpha)}\int\limits_0^l dx\int\limits_0^t kF_1(x,\tau)d\tau\int\limits_0^\tau\frac{F_1(x,\tau_1)d\tau_1}{(t-\tau_1)^{1-\alpha}}.
$$

Using the formula for the Gamma function

$$
\int\limits_0^\infty t^{\mu-1}\cos ktdt=\frac{\Gamma(\mu)}{k^\mu}\cos\frac{\mu\pi}{2},\quad k>0,\quad 0<\mu<1,
$$

for $k=(\tau-\tau_1)$, $\mu=1-\alpha$ we obtain

$$
\frac{1}{(t-\tau_1)^{1-\alpha}}=\frac{1}{\Gamma(1-\alpha)\cos\frac{(1-\alpha)\pi}{2}}\int\limits_0^\infty\xi^{-\alpha}\cos\xi(t-\tau_1)d\xi,
$$

while for the initial integral we get

$$
\begin{aligned}
J=&\frac{\pi}{\sin\pi(1-\alpha)\Gamma(1-\alpha)\cos\frac{(1-\alpha)\pi}{2}}\int\limits_0^l dx\int\limits_0^t kF_1(x,\tau)d\tau\int\limits_0^\tau F_1(x,\tau_1)d\tau_1\int\limits_0^\infty\xi^{-\alpha}\cos\xi(t-\tau_1)d\xi\\
=&\frac{\Gamma(\alpha)}{\cos\frac{(1-\alpha)\pi}{2}}\int\limits_0^l dx\int\limits_0^t kF_1(x,\tau)d\tau\int\limits_0^\tau F_1(x,\tau_1)d\tau_1\int\limits_0^\infty\xi^{-\alpha}\cos\xi(\tau-\tau_1)d\xi.
\end{aligned}
$$

Interchanging the integration order, we have:

$$
J=\frac{\Gamma(\alpha)}{\cos\frac{(1-\alpha)\pi}{2}}\left[\int\limits_0^l dx\int\limits_0^\infty\xi^{-\alpha}d\xi\int\limits_0^t kF_1(x,t)\cos\xi\tau d\tau\int\limits_0^\tau F_1(x,\tau_1)\cos\xi\tau_1d\tau_1\right.
$$

$$+\int\limits_0^l dx\int\limits_0^\infty \xi^{-\alpha}d\xi\int\limits_0^t kF_1(x,\tau)\sin\xi\tau d\tau\int\limits_0^\tau F_1(x,\tau_1)\sin\xi\tau_1 d\tau_1\Bigg]$$

$$=\frac{\Gamma(\alpha)}{2\cos\frac{(1-\alpha)\pi}{2}}\left\{\int\limits_0^l dx\int\limits_0^\infty \xi^{-\alpha}d\xi\int\limits_0^t k\,(x,\tau)\left[\left(\int\limits_0^\tau F_1(x,\tau_1)\cos\tau_1\xi d\tau_1\right)^2\right]_\tau d\tau\right.$$

$$\left.+\int\limits_0^l dx\int\limits_0^\infty \xi^{-\alpha}d\xi\int\limits_0^t k(x,\tau)\left[\left(\int\limits_0^\tau F_1(x,\tau)\sin\tau_1\xi d\tau_1\right)^2\right]_\tau dt\right\}.$$

By integration by parts the latter identity is transformed to the form

$$J=\frac{\Gamma(\alpha)}{2\cos\frac{(1-\alpha)\pi}{2}}\left\{\int\limits_0^l dx\int\limits_0^\infty \xi^{-\alpha}k\left[\left(\int\limits_0^\tau F_1(x,\tau_1)\cos\tau_1\xi d\tau_1\right)^2\right.\right.$$

$$\left.+\left(\int\limits_0^\tau F_1(x,\tau_1)\sin\tau_1\xi d\tau_1\right)^2\right]d\xi-\int\limits_0^l dx\int\limits_0^\infty \xi^{-\alpha}d\xi\int\limits_0^t k_\tau\left[\left(\int\limits_0^\tau F_1(x,\tau_1)\cos\tau_1\xi d\tau_1\right)^2\right.$$

$$\left.\left.+\left(\int\limits_0^\tau F_1(x,\tau_1)\sin\tau_1\xi d\tau_1\right)^2\right]\right\}.$$

Under the condition $k_t\leqslant 0$, since $\Gamma(\alpha)>0$, $0<\cos\frac{(1-\alpha)\pi}{2}<1$, we obtain that $J\geqslant 0$.

Thus, under the conditions $k_t,\ q_t\leqslant 0$, the triple integrals in the left hand side in inequality (2.10) are negative.

Strengthening inequality (2.10), we obtain

$$A_1\left\|D_{0t}^\alpha g\right\|_0^2+2(1-\varepsilon)\int\limits_0^t\left\|D_{0\tau}^\alpha g(x,\tau_1)\right\|_0^2 d\tau+2c_1\int\limits_0^t\left\|D_{0\tau}^\alpha g_x(x,\tau_1)\right\|_0^2 d\tau$$

$$\leqslant\frac{1}{2\varepsilon}\left\|F\right\|_{2,Q_t}^2+A_1\|u_1(x)\|_0^2,$$

and this implies the estimate

$$\|D_{0t}^\alpha g\|_0^2+\|D_{0t}^\alpha g\|_{2,\Omega_t}^2+\|D_{0t}^\alpha g_x\|_{2,\Omega_t}^2\leqslant M_1\left(\|F\|_{2,\Omega_t}^2+\|u_1(x)\|_0^2\right)$$

or, returning back to $u(x,t)$, we arrive at (2.3). The proof is complete. □

Remark 2.1. *Inequality (2.3) implies the uniqueness of solutio to problem (1.1)–(2.2). Indeed, let u be a solution to the inhomogeneous problem, that is, $f=u_0=u_1=0$. Then by (2.3) we have*

$$\|D_{0t}^\alpha u\|_0^2+\|D_{0t}^\alpha u\|_{2,Q_t}^2+\|D_{0t}^\alpha u_x\|_{2,Q_t}^2=0.$$

Applying the generalized Newton-Leibnitz formula [11]

$$D_{0t}^{-\alpha}D_{0t}^\alpha u(x,t)=u(x,t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}\lim_{t\to 0}D_{0t}^{\alpha-1}u(x,t),$$

in particular, we get

$$u(x,t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}\lim_{t\to 0}D_{0t}^{\alpha-1}u(x,t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}u_0(x)=0\quad in\quad \Omega_T.$$

2.2. Apriori estimate for solution to Robin boundary value problem. We consider a Robin boundary value problem for equation (1.1) in the domain Ω_T with the boundary conditions

$$\begin{cases} \Pi(0,t) = \beta_1(t)u(0,t) - \mu_1(t), \\ -\Pi(l,t) = \beta_2(t)u(l,t) - \mu_2(t) \end{cases} \tag{2.11}$$

and initial conditions (2.2), where $\Pi(x,t) = k(x,t)u_x + D_{0t}^{\alpha}(\eta u_x)$.

Theorem 2.2. *If, in additions to the assumptions of Theorem 2.1, the relations*

$$\beta_1,\ \beta_2 \in C^1[0,T]; \qquad \mu_1,\ \mu_2 \in C[0,T];$$

$$\beta_1,\ \beta_2 \geqslant c_0 > 0; \qquad \beta_{1t} \leqslant 0; \qquad \beta_{2t} \leqslant 0 \quad \text{everywhere in} \overline{\Omega}_t$$

hold as well as the condition $u_0(0) = u_0(l) = u_0'(0) = u_0'(l) = 0$, then a solution of problem (1.1), (2.11), (2.2) satisfies an apriori estimate

$$\begin{aligned} &\|D_{0t}^{\alpha}u\|_0^2 + \|D_{0t}^{\alpha}u\|_{2,Q_t}^2 + \|D_{0t}^{\alpha}u_x\|_{2,Q_t}^2 \\ &\leqslant M\left(\|f\|_{2,Q_t}^2 + \int_0^t \left(\mu_1^2 + \mu_2^2\right) d\tau + \|u_0(x)\|_{W_2^2(0,l)}^2 + \|u_1(x)\|_0^2 \right). \end{aligned} \tag{2.12}$$

Proof. We introduce a new unknown function $g(x,t)$ by formula (2.4). As a result we obtain that the function $g(x,t)$ should solve equation (2.5) with initial conditions (2.6) and boundary conditions

$$\begin{cases} \Pi_1(0,t) = \beta_1(t)g - \mu_1(t), & x = 0, \\ -\Pi_1(l,t) = \beta_2(t)g - \mu_2(t), & x = l, \end{cases} \tag{2.13}$$

where $\Pi_1(x,t) = k(x,t)g_x + D_{0t}^{\alpha}(\eta g_x)$.

Then we transform the terms in (2.8) by using (2.6), (2.13) and we obtain:

$$\begin{aligned} \left(A_1 D_{0t}^{\alpha+1} g, D_{0t}^{\alpha} g\right) =& \frac{A_1}{2}\frac{\partial}{\partial t}\|D_{0t}^{\alpha}g\|_0^2, \\ (D_{0t}^{\alpha}g, D_{0t}^{\alpha}g) =& \|D_{0t}^{\alpha}g\|_0^2, \\ ((kg_x)_x, D_{0t}^{\alpha}g) =& kg_x(l,t)\, D_{0t}^{\alpha}g(l,\tau) - kg_x(0,t)\, D_{0t}^{\alpha}g(0,\tau) \\ &- \frac{1}{\Gamma(1-\alpha)} \int_0^l kg_x(x,t) \frac{\partial}{\partial t} \int_0^t \frac{g_x(x,\tau)\, d\tau}{(t-\tau)^{\alpha}} dx, \\ \left(D_{0t}^{\alpha}(\eta g_x)_x, D_{0t}^{\alpha}g\right) \leqslant& D_{0t}^{\alpha}(\eta(l)\, g_x(l,\tau))\, D_{0t}^{\alpha}g(l,\tau) \\ &- D_{0t}^{\alpha}(\eta(0)\, g_x(0,\tau))\, D_{0t}^{\alpha}g(0,\tau) - c_1\|D_{0t}^{\alpha}g_x\|_0^2, \\ (F, D_{0t}^{\alpha}g) \leqslant& \frac{1}{4\varepsilon}\|F\|_0^2 + \varepsilon\|D_{0t}^{\alpha}g\|_0^2, \quad \varepsilon > 0. \end{aligned}$$

In view of the obtained inequalities in (2.8) we have

$$\begin{aligned} &\frac{A_1}{2}\frac{\partial}{\partial t}\|D_{0t}^{\alpha}g\|_0^2 + \|D_{0t}^{\alpha}g\|_0^2 + \frac{1}{\Gamma(1-\alpha)}\int_0^l kg_x(x,t)\frac{\partial}{\partial t}\int_0^t \frac{g_x(x,\tau)d\tau}{(t-\tau)^{\alpha}}dx \\ &+ c_1\|D_{0t}^{\alpha}g_x\|_0^2 + \frac{1}{\Gamma(1-\alpha)}\int_0^l qg(x,t)\frac{\partial}{\partial t}\int_0^t \frac{g(x,\tau)d\tau}{(t-\tau)^{\alpha}}dx \\ &\leqslant \frac{1}{4\varepsilon}\|F\|_0^2 + \varepsilon\|D_{0t}^{\alpha}g\|_0^2 + \Pi_1(x,t)D_{0t}^{\alpha}g(x,\tau)\big|_0^l. \end{aligned} \tag{2.14}$$

Let us estimate the last term in the right hand side of inequality (2.14):

$$\begin{aligned}\Pi_1(x,t)D_{0t}^{\alpha}g(x,\tau)\big|_0^l =& D_{0t}^{\alpha}g(l,\tau)\left(kg_x\left(l,t\right)+D_{0t}^{\alpha}\left(\eta\left(l\right)g_x(l,\tau)\right)\right)\\ &-D_{0t}^{\alpha}g(0,\tau)\left(kg_x\left(0,t\right)+D_{0t}^{\alpha}\left(\eta(0)g_x(0,\tau)\right)\right)\\ =& D_{0t}^{\alpha}g(l,\tau)\left(\mu_2(t)-\beta_2 g\right)-D_{0t}^{\alpha}g(0,\tau)\left(\beta_1 g-\mu_1(t)\right)\\ =& D_{0t}^{\alpha}g(l,\tau)\mu_2(t)-D_{0t}^{\alpha}g(l,\tau)\beta_2 g\left(l,t\right)\\ &-D_{0t}^{\alpha}g\left(0,\tau\right)\beta_1 g\left(0,t\right)+D_{0t}^{\alpha}g\left(0,\tau\right)\mu_1\left(t\right)\\ \leqslant& -D_{0t}^{\alpha}g(l,\tau)\beta_2 g\left(l,t\right)-D_{0t}^{\alpha}g\left(0,\tau\right)\beta_1 g\left(0,t\right)+\frac{1}{2}(\mu_1^2+\mu_2^2)\\ &+\frac{1}{2}\left(\left(D_{0t}^{\alpha}g\left(0,\tau\right)\right)^2+\left(D_{0t}^{\alpha}g\left(l,\tau\right)\right)^2\right).\end{aligned}$$

Taking into consideration the obtained estimate, by (2.14) we arrive at the inequality

$$\begin{aligned}\frac{A_1}{2}\frac{\partial}{\partial t}\left\|D_{0t}^{\alpha}g\right\|_0^2+\left\|D_{0t}^{\alpha}g\right\|_0^2+D_{0t}^{\alpha}\left(0,\tau\right)\beta_1 g\left(0,t\right)+D_{0t}^{\alpha}g(l,\tau)\beta_2 g\left(l,t\right)&\\ +\left(kg_x,D_{0t}^{\alpha}g_x\right)+c_1\left\|D_{0t}^{\alpha}g_x\right\|_0^2+\left(qg,D_{0t}^{\alpha}g\right)&\\ \leqslant\frac{1}{4\varepsilon}\|F\|_0^2+\varepsilon\left\|D_{0t}^{\alpha}g\right\|_0^2+\frac{1}{2}(\mu_1^2+\mu_2^2)&\\ +\frac{1}{2}\left(\left(D_{0t}^{\alpha}g\left(0,\tau\right)\right)^2+\left(D_{0t}^{\alpha}g\left(l,\tau\right)\right)^2\right).&\end{aligned}$$

We integrate the latter inequality over τ from 0 to t :

$$\begin{aligned}\frac{A_1}{2}\left\|D_{0t}^{\alpha}g\right\|_0^2+\int_0^t\left\|D_{0\tau}^{\alpha}g\right\|_0^2d\tau+\int_0^t\beta_1 g\left(0,\tau\right)D_{0\tau}^{\alpha}g\left(0,\tau_1\right)d\tau+\int_0^t\beta_2 g\left(l,\tau\right)D_{0\tau}^{\alpha}g\left(l,\tau_1\right)d\tau&\\ +\int_0^t\left(kg_x,D_{0\tau}^{\alpha}g_x\right)d\tau+c_1\int_0^t\left\|D_{0\tau}^{\alpha}g_x\right\|_0^2d\tau+\int_0^t\left(qg,D_{0\tau}^{\alpha}g\right)d\tau&\\ \leqslant\frac{1}{4\varepsilon}\int_0^t\|F\|_0^2d\tau+\varepsilon\int_0^t\left\|D_{0\tau}^{\alpha}g\right\|_0^2d\tau+\frac{1}{2}\int_0^t\left(\mu_1^2+\mu_2^2\right)d\tau&\\ +\frac{1}{2}\int_0^t\left(\left(D_{0\tau}^{\alpha}g\left(0,\tau_1\right)\right)^2+\left(D_{0\tau}^{\alpha}g\left(l,\tau_1\right)\right)^2\right)d\tau+\frac{A_1}{2}\left\|\left(D_{0t}^{\alpha}g\right)(x,0)\right\|_0^2.&\end{aligned}$$

Under the conditions $k_t\leqslant 0$, $q_t\leqslant 0$, $\beta_{1t}\leqslant 0$, $\beta_{2t}\leqslant 0$, strengthening this inequality, we obtain

$$\begin{aligned}\frac{A_1}{2}\left\|D_{0t}^{\alpha}g\right\|_0^2+\int_0^t\left\|D_{0\tau}^{\alpha}g\right\|_0^2d\tau+c_1\int_0^t\left\|D_{0\tau}^{\alpha}g_x\right\|_0^2d\tau\leqslant\frac{1}{4\varepsilon}\int_0^t\|F\|_0^2d\tau+\varepsilon\int_0^t\left\|D_{0\tau}^{\alpha}g\right\|_0^2d\tau&\\ +\frac{1}{2}\int_0^t\left(\mu_1^2+\mu_2^2\right)d\tau+\frac{1}{2}\int_0^t\left(\left(D_{0\tau}^{\alpha}g\left(0,\tau_1\right)\right)^2+\left(D_{0\tau}^{\alpha}g\left(l,\tau_1\right)\right)^2\right)d\tau&\\ +\frac{A_1}{2}\left\|\left(D_{0t}^{\alpha}g\right)(x,0)\right\|_0^2.&\end{aligned}$$

In view of the estimates [15]

$$\left(D_{0\tau}^{\alpha}g\left(0,\tau_1\right)\right)^2\leqslant\varepsilon\left\|\left(D_{0t}^{\alpha}g\right)_x\right\|_0^2+\left(\frac{2}{\varepsilon}+\frac{1}{l}\right)\left\|D_{0t}^{\alpha}g\right\|_0^2,$$

$$\left(D_{0\tau}^{\alpha} g\left(l,\tau_1\right)\right)^2 \leqslant \varepsilon \left\|\left(D_{0t}^{\alpha} g\right)_x\right\|_0^2 + \left(\frac{2}{\varepsilon}+\frac{1}{l}\right)\left\|D_{0t}^{\alpha} g\right\|_0^2$$

we obtain

$$\frac{A_1}{2}\left\|D_{0t}^{\alpha} g\right\|_0^2 + \left(1-\varepsilon-\left(\frac{2}{\varepsilon}+\frac{1}{l}\right)\right)\int_0^t \left\|D_{0\tau}^{\alpha} g\right\|_0^2 d\tau + (c_1-\varepsilon)\int_0^t \left\|D_{0\tau}^{\alpha} g_x\right\|_0^2 d\tau$$

$$\leqslant \frac{1}{4\varepsilon}\int_0^t \|F\|_0^2 d\tau + \frac{1}{2}\int_0^t (\mu_1^2+\mu_2^2) d\tau + \frac{A_1}{2}\left\|\left(D_{0t}^{\alpha} g\right)(x,0)\right\|_0^2$$

or

$$\left\|D_{0t}^{\alpha} g\right\|_0^2 + \int_0^t \left\|D_{0\tau}^{\alpha} g\right\|_0^2 d\tau + \int_0^t \left\|D_{0\tau}^{\alpha} g_x\right\|_0^2 d\tau \leqslant M_2\left(\int_0^t \|F\|_0^2 d\tau + \int_0^t \left(\mu_1^2+\mu_2^2\right) d\tau + \|u_1(x)\|_0^2\right).$$

Returning back to $u(x,t)$, we obtain (2.12), and this implies the uniqueness of the solution to problem (1.1), (2.11), (2.2). The proof is complete. □

3. Nonlocal boundary value problem for inhomogeneous Aller-Lykov equation with fractional time derivative

Many works were devoted to problems with nonlocal boundary conditions including work [16], which has already become classical. It is conventional [17] that nonlocal problems are ones involving conditions relating the value of the values of the sought solution and/or its derivatives at different points on the boundary or on the boundary and at some internal points.

Problem 3.1. *We consider a nonlocal boundary value problem for equation (1.1) in the domain Ω_T with the boundary conditions*

$$\begin{cases} \Pi_2(0,t) = \beta_1(t)u(0,t) + D_{0t}^{\alpha} u(0,\tau) - \mu_1(t), \\ -\Pi_2(l,t) = \beta_2(t)u(l,t) + D_{0t}^{\alpha} u(l,\tau) - \mu_2(t) \end{cases} \tag{3.1}$$

and initial conditions (2.2), where $\Pi_2(x,t) = k(x,t)u_x + D_{0t}^{\alpha}\left(\eta u_x\right)$.

Nonlocal boundary value problems involving time nonlocality in the boundary conditions were first studied by A.I. Kozhanov [18]. Boundary value problems for the moisture transfer with such boundary condition were also considered in works [19], [20].

Theorem 3.1. *Under the assumptions of Theorem 2.2 a solution of problem (1.1), (3.1), (2.2) satisfies the apriori estimate*

$$\begin{aligned} &\left\|D_{0t}^{\alpha} u\right\|_0^2 + \int_0^t \left\|D_{0\tau}^{\alpha} u\right\|_0^2 d\tau + \int_0^t \left\|D_{0\tau}^{\alpha} u_x\right\|_0^2 d\tau + \int_0^t \left(D_{0\tau}^{\alpha} u(l,\tau_1)\right)^2 + \left(D_{0\tau}^{\alpha} u(0,\tau_1)\right)^2 d\tau \\ &\leqslant M_3\left(\int_0^t \|f\|_0^2 d\tau + \int_0^t \left(\mu_1^2+\mu_2^2\right) d\tau + \|u_0(x)\|_{W_2^2(0,l)}^2 + \|u_1(x)\|_0^2\right). \end{aligned} \tag{3.2}$$

Proof. For a new unknown function $g(x,t)$ we obtain equation (2.5) with initial conditions (2.6) and boundary conditions

$$\begin{cases} \Pi_3(0,t) = \beta_1(t)g(0,t) + D_{0t}^{\alpha} g(0,t) - \mu_1(t), \\ -\Pi_3(l,t) = \beta_2(t)g(l,t) + D_{0t}^{\alpha} g(l,t) - \mu_2(t), \end{cases} \tag{3.3}$$

where $\Pi_3(x,t) = k(x,t)g_x + D_{0t}^{\alpha}(\eta g_x)$. In view of (3.3), the latter term in the right hand side in inequality (2.14) is represented as

$$\begin{aligned}\Pi_3 D_{0t}^{\alpha} g\big|_0^l =& D_{0t}^{\alpha} g(l,\tau)\left(k g_x(l,t) + D_{0t}^{\alpha}(\eta(l) g_x(l,\tau))\right)\\ &- D_{0t}^{\alpha} g(0,\tau)\left(k g_x(0,t) + D_{0t}^{\alpha}(\eta(0) g_x(0,\tau))\right)\\ =& D_{0t}^{\alpha} g(l,\tau)\left(\mu_2(t) - \beta_2(t) g(l,t) - D_{0t}^{\alpha} g(l,\tau)\right)\\ &- D_{0t}^{\alpha} g(0,\tau)\left(\beta_1(t) g(0,t) + D_{0t}^{\alpha} g(0,\tau) - \mu_1(t)\right)\\ &+ D_{0t}^{\alpha} g(l,\tau)\mu_2(t) - D_{0t}^{\alpha} g(l,\tau)\beta_2 g(l,t) - D_{0t}^{\alpha} g(l,\tau) D_{0t}^{\alpha} g(l,\tau)\\ &+ D_{0t}^{\alpha} g(0,\tau)\mu_1(t) - D_{0t}^{\alpha} g(0,\tau)\beta_1 g(0,t) - D_{0t}^{\alpha} g(0,\tau) D_{0t}^{\alpha} g(0,\tau).\end{aligned}$$

Let us estimate the sum $\mu_2(t) D_{ot}^{\alpha} g(l,t) + \mu_1(t) D_{ot}^{\alpha} g(0,t)$:

$$\mu_1 D_{ot}^{\alpha} g(0,t) + \mu_2 D_{ot}^{\alpha} g(l,t) \leqslant \frac{1}{2}(\mu_1^2 + \mu_2^2) + \left(\varepsilon \left\|(D_{0t}^{\alpha} g(x,t))_x\right\|_0^2 + \left(\frac{2}{\varepsilon} + \frac{1}{l}\right)\left\|D_{0t}^{\alpha} g(x,t)\right\|_0^2\right).$$

Taking into consideration the above obtained estimate, by (2.14) we have

$$\begin{aligned}\frac{A_1}{2}\frac{\partial}{\partial t}\left\|D_{0t}^{\alpha} g\right\|_0^2 + \left(1 - \varepsilon - \left(\frac{2}{\varepsilon} + \frac{1}{l}\right)\right)\left\|D_{0t}^{\alpha} g\right\|_0^2 + (k g_x, D_{0t}^{\alpha} g_x) + (c_1 - \varepsilon)\left\|D_{0t}^{\alpha} g_x\right\|_0^2&\\ + (qg, D_{0t}^{\alpha} g) + \beta_2(t) D_{0t}^{\alpha} g(l,\tau) g(l,t) + \beta_1(t) D_{0t}^{\alpha} g(0,\tau) g(0,t)&\\ + (D_{0t}^{\alpha} g(l,\tau))^2 + (D_{0t}^{\alpha} g(0,\tau))^2 \leqslant \frac{1}{4\varepsilon}\|F\|_0^2 + \frac{1}{2}(\mu_1^2 + \mu_2^2).&\end{aligned} \tag{3.4}$$

We integrate (3.4) over τ from 0 to t :

$$\begin{aligned}\frac{A_1}{2}\left\|D_{0t}^{\alpha} g\right\|_0^2 + \nu(\varepsilon)\int_0^t \left\|D_{0\tau}^{\alpha} g\right\|_0^2 d\tau + \int_0^t (k g_x, D_{0\tau}^{\alpha} g_x)\,d\tau + \nu_1(\varepsilon)\int_0^t \left\|D_{0\tau}^{\alpha} g_x\right\|_0^2 d\tau&\\ + \int_0^t (qg, D_{0\tau}^{\alpha} g)\,d\tau + \int_0^t \beta_2 g(l,\tau) D_{0\tau}^{\alpha}(l,\tau_1)d\tau&\\ + \int_0^t \beta_1 g(0,\tau) D_{0\tau}^{\alpha}(0,\tau_1)d\tau + \int_0^t (D_{0\tau}^{\alpha} g(l,\tau_1))^2 + (D_{0\tau}^{\alpha} g(0,\tau_1))^2 d\tau&\\ \leqslant \frac{1}{4\varepsilon}\int_0^t \|F\|_0^2 d\tau + \frac{1}{2}\int_0^t (\mu_1^2 + \mu_2^2)d\tau + \frac{A_1}{2}\left\|(D_{0t}^{\alpha} g)(x,0)\right\|_0^2,&\end{aligned}$$

where $\nu(\varepsilon) = 1 - \varepsilon - \left(\frac{2}{\varepsilon} + \frac{1}{l}\right)$, $\nu_1(\varepsilon) = c_1 - \varepsilon$.

Under the conditions $k_t \leqslant 0$, $q_t \leqslant 0$, $\beta_{1t} \leqslant 0$, $\beta_{2t} \leqslant 0$ we can strengthen the latter inequality. As a result we get

$$\begin{aligned}\frac{A_1}{2}\left\|D_{0t}^{\alpha} g\right\|_0^2 + \nu(\varepsilon)\int_0^t \left\|D_{0\tau}^{\alpha} g\right\|_0^2 d\tau + \nu_1(\varepsilon)\int_0^t \left\|D_{0\tau}^{\alpha} g_x\right\|_0^2 d\tau + \int_0^t (D_{0\tau}^{\alpha} g(l,\tau_1))^2 + (D_{0\tau}^{\alpha} g(0,\tau_1))^2 d\tau&\\ \leqslant \frac{1}{4\varepsilon}\int_0^t \|F\|_0^2 d\tau + \frac{1}{2}\int_0^t (\mu_1^2 + \mu_2^2)d\tau + \frac{A_1}{2}\|u_1(x)\|_0^2.&\end{aligned}$$

This implies estimate (3.2), which proves the uniqueness of solution to problem (1.1), (3.1), (2.2). The proof is complete. □

Problem 3.2. *In problem (1.1), (3.1), (2.2) we replace boundary conditions (3.1) by ones of form*

$$\begin{cases} \Pi_4(0,t) = \beta_1(t)\left(u(0,t) + D_{0t}^{\alpha-1}u(0,\tau)\right) - \mu_1(t), \\ -\Pi_4(l,t) = \beta_2(t)\left(u(l,t) + D_{0t}^{\alpha-1}u(l,\tau)\right) - \mu_2(t). \end{cases} \tag{3.5}$$

Theorem 3.2. *If the assumptions of Theorem 2.2 are satisfied, then a solution of problem (1.1), (3.5), (2.2) satisfies an apriori estimate*

$$\begin{aligned} &\|D_{0t}^{\alpha}u\|_0^2 + \|D_{0t}^{\alpha}u\|_{2,\Omega_t}^2 + \|D_{0t}^{\alpha}u_x\|_{2,\Omega_t}^2 + \left(D_{0t}^{\alpha-1}u(l,\tau)\right)^2 + \left(D_{0t}^{\alpha-1}u(0,\tau)\right)^2 \\ &\leqslant M\left(\|f\|_{2,\Omega_t}^2 + \int_0^t (\mu_1^2+\mu_2^2)d\tau + \|u_1(x)\|_0^2 + u_0^2(0) + u_0^2(l)\right). \end{aligned} \tag{3.6}$$

Proof. We calculate the scalar product of equation (1.1) with $D_{0t}^{\alpha}u$:

$$\begin{aligned} &\left(A_1 D_{0t}^{\alpha+1}u, D_{0t}^{\alpha}u\right) + (D_{0t}^{\alpha}u, D_{0t}^{\alpha}u) - \left((ku_x)_x, D_{0t}^{\alpha}u\right) \\ &\quad - \left(D_{0t}^{\alpha}(\eta u_x)_x, D_{0t}^{\alpha}u\right) + (qu, D_{0t}^{\alpha}u) = (f, D_{0t}^{\alpha}u). \end{aligned} \tag{3.7}$$

We reproduce the same arguing as in the proof of inequality (2.14), then we obtain

$$\begin{aligned} &\frac{A_1}{2}\frac{\partial}{\partial t}\left\|D_{0t}^{\alpha}u\right\|_0^2 + \left\|D_{0t}^{\alpha}u\right\|_0^2 + \frac{1}{\Gamma(1-\alpha)}\int_0^l ku_x(x,t)\frac{\partial}{\partial t}\int_0^t \frac{u_x(x,\tau)\,d\tau}{(t-\tau)^{\alpha}}dx \\ &\quad + c_1\left\|D_{0t}^{\alpha}u_x\right\|_0^2 + \frac{1}{\Gamma(1-\alpha)}\int_0^l qu(x,t)\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)d\tau}{(t-\tau)^{\alpha}}dx \\ &\quad \leqslant \Pi_4(x,t)D_{0t}^{\alpha}u(x,\tau)\Big|_0^l + \frac{1}{4\varepsilon}\|f\|_0^2 + \varepsilon\left\|D_{0t}^{\alpha}u\right\|_0^2. \end{aligned} \tag{3.8}$$

In order to obtain an apriori estimate, we use (3.5) in order to transform the first term in the right hand side of inequality (3.8):

$$\begin{aligned} \Pi_4 D_{0t}^{\alpha}u\Big|_0^l =& D_{0t}^{\alpha}u(l,\tau)\left(ku_x(l,t) + D_{0t}^{\alpha}\left(\eta(l)\,u_x(l,\tau)\right)\right) - D_{0t}^{\alpha}u(0,\tau)\big(ku_x(0,t) \\ &+ D_{0t}^{\alpha}\left(\eta(0)\,u_x(0,\tau)\right)\big) = D_{0t}^{\alpha}u(l,\tau)\left(\mu_2(t) - \beta_2(t)\left(u(l,t) + D_{0t}^{\alpha-1}u(l,\tau)\right)\right) \\ &- D_{0t}^{\alpha}u(0,\tau)\left(\beta_1(t)\left(u(0,t) + D_{0t}^{\alpha-1}u(0,\tau)\right) - \mu_1(t)\right) \\ =& D_{0t}^{\alpha}u(l,\tau)\mu_2(t) - D_{0t}^{\alpha}u(l,\tau)\beta_2(t)u(l,t) - D_{0t}^{\alpha}u(l,\tau)\beta_2(t)D_{0t}^{\alpha-1}u(l,\tau) \\ &+ D_{0t}^{\alpha}u(0,\tau)\mu_1(t) - D_{0t}^{\alpha}u(0,\tau)\beta_1(t)u(0,t) - D_{0t}^{\alpha}u(0,\tau)\beta_1(t)D_{0t}^{\alpha-1}u(0,\tau). \end{aligned}$$

Taking into consideration the estimates

$$\begin{aligned} D_{0t}^{\alpha}u(l,\tau)\beta_2(t)D_{0t}^{\alpha-1}u(l,\tau) &\geqslant c_0\frac{1}{2}\frac{\partial}{\partial t}\left(D_{0t}^{\alpha-1}u(l,\tau)\right)^2, \\ D_{0t}^{\alpha}u(0,\tau)\beta_1(t)D_{0t}^{\alpha-1}u(0,\tau) &\geqslant c_0\frac{1}{2}\frac{\partial}{\partial t}\left(D_{0t}^{\alpha-1}u(0,\tau)\right)^2, \end{aligned}$$

by (3.8) we find that

$$\begin{aligned} &\frac{A_1}{2}\frac{\partial}{\partial t}\|D_{0t}^{\alpha}u\|_0^2 + \left(1-\varepsilon-\left(\frac{2}{\varepsilon}+\frac{1}{l}\right)\right)\|D_{0t}^{\alpha}u\|_0^2 + (ku_x, D_{0t}^{\alpha}u_x) + (c_1-\varepsilon)\|D_{0t}^{\alpha}u_x\|_0^2 \\ &\quad + (qu, D_{0t}^{\alpha}u) + \beta_2(t)D_{0t}^{\alpha}u(l,\tau)u(l,t) + \beta_1(t)D_{0t}^{\alpha}u(0,\tau)u(0,t) \\ &\quad + \frac{c_0}{2}\frac{\partial}{\partial t}\left(D_{0t}^{\alpha-1}u(l,\tau)\right)^2 + \frac{c_0}{2}\frac{\partial}{\partial t}\left(D_{0t}^{\alpha-1}u(0,\tau)\right)^2 \leqslant \frac{1}{4\varepsilon}\|f\|_0^2 + \frac{1}{2}(\mu_1^2+\mu_2^2). \end{aligned} \tag{3.9}$$

We integrate (3.9) over τ from 0 to t :

$$\frac{A_1}{2}\left\|D_{0t}^{\alpha}u\right\|_0^2+\nu(\varepsilon)\int_0^t\left\|D_{0\tau}^{\alpha}u\right\|_0^2d\tau+\int_0^t\left(ku_x,D_{0\tau}^{\alpha}u_x\right)d\tau+\nu_1(\varepsilon)\int_0^t\left\|D_{0\tau}^{\alpha}u_x\right\|_0^2d\tau$$
$$+\int_0^t\left(qu,D_{0\tau}^{\alpha}u\right)d\tau+\int_0^t\beta_2u(l,\tau)D_{0\tau}^{\alpha}u(l,\tau_1)d\tau+\int_0^t\beta_1u(0,\tau)D_{0\tau}^{\alpha}u(0,\tau_1)d\tau$$
$$+\frac{c_0}{2}\left[\left(D_{0t}^{\alpha-1}u(l,\tau)\right)^2+\left(D_{0t}^{\alpha-1}u(0,\tau)\right)^2\right]\leqslant\frac{1}{4\varepsilon}\int_0^t\|f\|_0^2d\tau+\frac{1}{2}\int_0^t\left(\mu_1^2+\mu_2^2\right)d\tau$$
$$+\frac{A_1}{2}\left\|\left(D_{0t}^{\alpha}u\right)(x,0)\right\|_0^2+\frac{c_0}{2}\left(D_{0t}^{\alpha-1}u(l,0)\right)^2+\frac{c_0}{2}\left(D_{0t}^{\alpha-1}u(0,0)\right)^2,$$

where $\nu(\varepsilon)=1-\varepsilon-\left(\frac{2}{\varepsilon}+\frac{1}{l}\right)$, $\nu_1(\varepsilon)=c_1-\varepsilon$.

Under the conditions $k_t\leqslant 0$, $q_t\leqslant 0$, $\beta_{1t}\leqslant 0$, $\beta_{2t}\leqslant 0$ we strengthen the latter inequality. As a result we obtain the estimate

$$\frac{A_1}{2}\left\|D_{0t}^{\alpha}u\right\|_0^2+\nu(\varepsilon)\int_0^t\left\|D_{0\tau}^{\alpha}u\right\|_0^2d\tau+\nu_1(\varepsilon)\int_0^t\left\|D_{0\tau}^{\alpha}u_x\right\|_0^2d\tau+\frac{c_0}{2}\Big[\left(D_{0t}^{\alpha-1}u(l,\tau)\right)^2$$
$$+\left(D_{0t}^{\alpha-1}u(0,\tau)\right)^2\Big]\leqslant\frac{1}{4\varepsilon}\int_0^t\|f\|_0^2+\frac{1}{2}\int_0^t(\mu_1^2+\mu_2^2)d\tau+\frac{A_1}{2}\left\|\left(D_{0t}^{\alpha}u\right)(x,0)\right\|_0^2$$
$$+\frac{c_0}{2}\left(D_{0t}^{\alpha-1}u(l,0)\right)^2+\frac{c_0}{2}\left(D_{0t}^{\alpha-1}u(0,0)\right)^2,$$

which implies desired estimate (3.6) and it proves the uniqueness of solution to problem (1.1), (3.5), (2.2). The proof is complete. □

BIBLIOGRAPHY

1. A.F. Chudnovskiy. *Thermophysics of soils.* Nauka, Moscow (1976). (in Russian).
2. A.M. Nakhushev. *Fractional calculus and its applications.* Fizmatlit, Moscow (2003). (in Russian).
3. S.M. Arkhestova, M.Kh. Shkhanukov-Lafishev. *Difference schemes for Aller-Lykov moisture transfer equation with nonlocal condition* // Izv. KBNTs RAN. **3**, 7–16 (2012). (in Russian).
4. M.M. Lafisheva, M.A. Kerefov, R.V. Dyshekova. *Difference schemes for the Aller-Lykov moisture transfer equations with a nonlocal condition* // Vladikavkaz. Matem. Zhurn. **19**:1, 50–58 (2017). (in Russian).
5. S.Kh.Gekkieva. *Nonlocal boundary-value problem for the generalized Aller-Lykov moisture transport equation* // Vestnik KRAUNC. Fiz.-mat. Nauki. **4 (24)**, 19–28 (2018). (in Russian).
6. M.A. Kerefov, F.M. Nakhusheva, S.Kh. Gekkieva. *Boundary value problem for the Aller-Lykov moisture transport generalized equation with concentrated heat capacity* // Vestnik Samar. Gosud. Univ. Estestvennonauch. ser. **24**:3, 23–29 (2018). (in Russian).
7. S.Kh. Gekkieva, M.A. Kerefov. *Boundary-value problem for the Aller-Lykov nonlocal moisture transfer equation* // Itogi Nauki Tekhn. Ser. Sovrem. Matem. Pril. Temat. Obz. **167**, 27–33 (2019). (in Russian).
8. S.Kh. Gekkieva, M.A. Kerefov. *Dirichlet boundary value problem for Aller-Lykov moisture transfer equation with fractional derivative in time* // Ufimskij Matem. Zhurn. **11**:2, 72–82 (2019). [Ufa Math. J. **11**:2, 71–81 (2019).]
9. M.A. Kerefov, S.Kh. Gekkieva. *Second boundary-value problem for the generalized Aller-Lykov equation* // Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki. **23**:4, 607–621 (2019). (in Russian).

10. M.A. Kerefov, S.Kh. Gekkieva. *Numerical-analytical method for solving boundary value problem for the generalized moisture transport equation* // Vestn. Udmurt. Univ. Matem. Mekh. Komp. Nauki. **31**:1, 19–34 (2021). (in Russian).
11. A.V. Pskhu. *Partial differential equations of fractional order.* Nauka, Moscow (2005). (in Russian).
12. A.A. Samarskii. *The theory of difference schemes.* Nauka, Moscow (1989). [Marcel Dekker, New York (2001).]
13. M.A. Kerefov. *Boundary value problems for a modified moisture transfer equation with afractional time derivative.* PhD thesis, Nalchik (2000). (in Russian).
14. St.G. Samko, A.A. Kilbas, O.I. Marichev. *Fractional integrals and derivatives: theory and applications.* Nauka i Tekhnika, Minsk (1987). [Gordon and Breach, New York (1993).]
15. O.A. Ladyzhenskaya. *The boundary value problems of mathematical physics.* Nauka, Moscow (1973). [Springer-Verlag, New York (1985).]
16. A.V. Bitsadze, A.A. Samarskii. *On some simple generalizations of linear elliptic boundary problems* // Dokl. AN SSSR **185**:4, 739–740 (1969). [Sov. Math. Dokl. **10**, 398–400 (1969).]
17. A.M. Nakhushev. *Equations of mathematical biology.* Fizmatlit, Moscow (1995). (in Russian).
18. A.I. Kozhanov. *On a nonlocal boundary value problem with variable coefficients for the heat equation and the Aller equation* // Differ. Uravn. **40**:6, 763–774 (2004). [Diff. Equat. **40**:6, 815–826 (2004).]
19. M.Kh. Beshtokov. *Local and nonlocal boundary value problems for degenerating and nondegenerating pseudoparabolic equations with a Riemann-Liouville fractional derivative* // Diff. Equat. **54**:6, 763–778 (2018).
20. M.Kh. Beshtokov. *Boundary value problems for degenerate and degenerate fractional order differential equations with non-local linear source and difference methods for their numerical implementation* // Ufimsk. Matem. Zhurn. **11**:2, 36–55 (2019). [Ufa Math. J. **11**:2, 34–55 (2019).]

Sakinat Khasanovna Gekkieva,
Institute of Applied Mathematics and Automation,
Kabardin-Balkar Scientific Center of RAS,
Shortanova str. 89 A,
360000, Nalchik, Russia
E-mail: gekkieva_s@mail.ru

Marat Aslanbievich Keferov,
Kabardino-Balkarian State University named after H.M. Berbekov,
Chernyshevsky str. 173,
360004, Nalchik, Russia
E-mail: kerefov@mail.ru

Fatima Mukhamedovna Nakhusheva,
Kabardino-Balkarian State University named after H.M. Berbekov,
Chernyshevsky str. 173,
360004, Nalchik, Russia
E-mail: fatima-nakhusheva@mail.ru