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LOWER BOUND FOR MINIMUM OF MODULUS OF ENTIRE FUNCTION OF GENUS ZERO WITH POSITIVE ROOTS IN TERMS OF DEGREE OF MAXIMAL MODULUS AT FREQUENT SEQUENCE OF POINTS

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Abstract. We consider entire function of genus zero, the roots of which are located at a single ray. On the class of all such functions, we obtain close to optimal lower bounds for the minimum of the modulus on a sequence of the circumferences in terms of a negative power of the maximum of the modulus on the same circumferences under a restriction on the quotient $a > 1$ of the radii of neighbouring circumferences. We introduce the notion of the optimal exponent $d(a)$ as an extremal exponent of the maximum of the modulus in this problem. We prove two-sided estimates for the optimal exponent for a "test" value $a = \frac{9}{4}$ and for $a \in (1, \frac{9}{8}]$. We find an asymptotics for $d(a)$ as $a \to 1$. The obtained result differs principally from the classical $\cos(\pi\rho)$-theorem containing no restrictions for the frequencies of the radii of the circumferences, on which the minimum of the modulus of an entire function of order $\rho \in [0, 1]$ is estimated by a power of the maximum of its modulus.

Keywords: entire function, minimum of modulus, maximum of modulus.

Mathematics Subject Classification: 30D15, 30D20

1. Introduction. Main results

We recall a classical $\cos(\pi\rho)$-theorem on lower bound for the modulus of an entire function of order $\rho \in [0, 1]$ on some sequence of circumference tending to infinity. As usually, we denote

$$M(f; r) = \max_{|z|=r} |f(z)| = \max_{|z|\leqslant r} |f(z)|, \qquad m(f; r) = \min_{|z|=r} |f(z)|,$$

where f is an entire function.

Let f be a not identically constant entire function of order $\rho \in [0, 1]$. Then for each $\varepsilon > 0$ there exists a sequence of positive numbers $r_n \to +\infty$ such that the inequality

$$m(f; r_n) > (M(f; r_n))^{\cos(\pi\rho)-\varepsilon} \tag{1.1}$$

holds. For the values $\rho \in [0, 1)$ this result was independently obtained by Valiron [1] and Wiman [2], while in the case $\rho = 1$ it was proved by Cartwright in [3], see also a fundamental work by Hayman [4]. We also mention that in [3], [4] there was considered a question on the vastness of the set $E \subset \mathbb{R}_+ = (0, +\infty)$ such that

$$m(f; r) > (M(f; r))^{\cos(\pi\rho)-\varepsilon}, \quad r \in E.$$

It was proved that E possesses a logarithmic density but this does not exclude the presence of wide gaps in this set, that is, the existence of a sequence $R_n \to +\infty$ and a number $p > 1$ such

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that the segments $[R_n, R_n^p]$ lie in $\mathbb{R}_+ \setminus E$. The mentioned results are called Valiron-Wiman type theorems, for more details we refere to monographs [5, Ch. 3], [6, Ch. V, Sect. 3], [7, Ch. 6] and surveys [8], [9].

We pose a problem on possibility of the power estimate

$$m(f;r_n) > M^{-d}(f;r_n) \tag{1.2}$$

with some exponent $d > 0$ on some sequence $r_n \to +\infty$ obeying the condition

$$\varlimsup_{n\to\infty} \frac{r_{n+1}}{r_n} < +\infty. \tag{1.3}$$

Of course, here we deal only with entire functions f of a finite order. In [4] there is an example of an entire function F of an infinite order for which

$$\lim_{r\to+\infty} \frac{\ln m(F;r)}{\ln M(F;r)} = -\infty.$$

However, even for an arbitrary non-constant entire function f of zero order the answer to the question on the existence of such sequence $\{r_n\}$ ensuring relations (1.2) and (1.3) for at least some value d is not known to us. This is why we restrict ourselves by considering a particular case: the function f is a canonical product of zero genus with roots lying on a single ray; for the sake of definiteness, this ray is positive. Namely, we consider all functions of form

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right), \tag{1.4}$$

where $\{\lambda_n\}_{n\in\mathbb{N}}$ is an arbitrary scalar sequence obeying the conditions

$$0 < \lambda_n \leqslant \lambda_{n+1} \quad \forall n \in \mathbb{N}, \qquad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < +\infty. \tag{1.5}$$

As it is known [5, Ch. 2], each entire function of order less than 1 different from a polynomial is obtained by multiplying an infinite product of form (1.4) with arbitrary complex roots, the series of reciprocals of their moduli converges, by az^m, where $a \in \mathbb{C} \setminus \{0\}$, $m \in \mathbb{Z}_+$. The belonging of all roots to a single ray (which is $\mathbb{R}_+$ in our case) is a rather strict restriction. For all functions (1.4), the roots of which satisfy condition (1.5), the identities hold:

$$m(f;r) = |f(r)|, \quad M(f;r) = f(-r), \quad r > 0. \tag{1.6}$$

But even for such narrow subclass of entire functions we known only two results on the above formulated problem. The first was obtained by A.M. Gaisin [10] for even canonical products

$$L(w) = \prod_{n=1}^{\infty} \left(1 - \frac{w^2}{\mu_n^2}\right) \tag{1.7}$$

with real roots $\{\pm\mu_n\}$, for which the series $\sum\limits_{n=1}^{\infty} \mu_n^{-2}$ converges. Gaisin proved that for each function of form (1.7) there exists an increasing sequence $R_n \to +\infty$ satisfying the condition $R_{n+1} \leqslant 4R_n$ for all $n \in \mathbb{N}$ such that the estimate $m(L;R_n) > M^{-20}(L;R_n)$ is true.

This theorem was improved in [11], see also [12]: it was proved that for each function (1.4) with roots obeying condition (1.5) there exists a sequence $r_n \uparrow +\infty$ such that the restriction $r_{n+1} \leqslant 3r_n + 1$ holds as well as the estimate $m(f;r_n) > M^{-9}(f;r_n)$. It is easy to confirm that if we take an arbitrary function L of form (1.7) and let $\lambda_n = \mu_n^2$, then by applying the formulated result from [11] to the function

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\mu_n^2}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right),$$

taking into consideration an obvious identity $f(w^2) = L(w)$, we obtain the existence of the sequence $R_n = \sqrt{r_n} \uparrow +\infty$ such that the restriction $R_{n+1} \leqslant \sqrt{3}R_n + 1$ holds as well as the estimate $m(L; R_n) > M^{-9}(L; R_n)$.

At the same time, the cited result of work [11] was obtained from the theorem, proven also in the same work, on the lower bound for $\max\limits_{qr\leqslant x\leqslant r} m(f; x)$ in terms of the negative power of $M(f; Ar)$, where a number $A > 1$ depends on the value of the parameter $q \in (0, 1)$ and the exponent of the maximum of the modulus; this lower bound held for an arbitrary non-constant entire function f. The features of products (1.4) with positive roots was used only while obtaining the asymptotic estimate $M(f; Ar) = o(M^A(f; r))$, $r \to +\infty$, $A > 1$, allowed us to estimate the maximum of the modulus of the function f on a larger circumference in terms of a power of the maximum of the modulus on a smaller circumference. This is why it is natural to expect that for the products of zero genus with the roots on a single ray one can get a stronger result. We prove the following theorem in Section 3.

Theorem 1.1. *Let f be an arbitrary function of form* (1.4)*, the roots of which obey condition* (1.5)*. Then for each $R > 0$ there exists a point $r \in (R, 9R/4)$, at which the inequality*

$$m(f; r) > M^{-3}(f; r) \tag{1.8}$$

holds. Moreover, the product

$$\Pi(t) \equiv m(f; t)M^3(f; t)$$

exceeds 1 *“in mean” on each segment $[R, 9R/4]$ in the sense that*

$$\int\limits_R^{9R/4} t^{-3} \ln \Pi(t)dt > 0 \qquad \text{for all} \quad R > 0. \tag{1.9}$$

Corollary 1.1. *Let L be an arbitrary function of form* (1.7) *with real roots $\{\pm\mu_n\}_{n\in\mathbb{N}}$. Then for each $R > 0$ there exists a point $r \in (R, 3R/2)$, at which the inequality*

$$|L(r)| = m(L; r) > M^{-3}(L; r) = L^{-3}(ir)$$

is satisfied.

The impossibility of an essential strengthening of inequality (1.8), namely, the change of the exponent in the power of maximum of the modulus in (1.8) by -2 is due to the following result proved in Section 2.

Theorem 1.2. *For each value $\rho \in (0, 1)$ and an arbitrarily “fast” tending to $+\infty$ sequence of positive numbers $\{R_n\}_{n\in\mathbb{N}}$, that is, such that $\lim\limits_{n\to\infty}(R_n/R_{n+1}) = 0$, there exists an entire function F of order ρ being a canonical product of zero order with positive roots, for which the limiting relation holds:*

$$\lim_{n\to\infty} \max \left\{ m(F; r)M^2(F; r) \mid R_n \leqslant r \leqslant \frac{9}{4}R_n \right\} = 0.$$

Theorems 1.1 and 1.2 demonstrate an essential difference between the problems on obtaining lower bounds for the minimum of the modulus of an entire function of order $\rho \in (0, 1)$ in terms of the power of the maximum of its modulus on a sequence of the circumferences tending to infinity with no restrictions for the frequency of their radii, or with a weak condition for the quotient of the logarithms of the neighbouring radii, and under the presence of a corresponding estimate on each segment with a constant quotient of its ends. As we see by $\cos(\pi\rho)$-theorem, in the first case exactly the order of the function determines the best possible exponent in the power of the maximum of the modulus in estimate (1.1). In the second case, when we need to obtain a power estimate for $m(f; r)$ in terms of $M(f; r)$ at some appropriate point in the segment $R \leqslant r \leqslant aR$ (in Theorems 1.1, 1.2 we consider the value $a = \frac{9}{4}$), the order of the function f possibly plays some role (we failed trying to clarify it) but it is not principal: as we have established, the best exponent d in estimate (1.2) lies between the numbers 2 and 3.

Let us give a rigorous definition of the optimal exponent in power estimate (1.2) on the class of products (1.4) with conditions on roots (1.5), when the quotient r_{n+1}/r_n is bounded from above by a given number. We preliminary formulate one result.

In Section 2 we prove the following theorem. For an arbitrary $a > 1$ we denote

$$s(a) = \frac{\ln \frac{1}{1-1/a}}{\ln(1 + 1/a)}. \tag{1.10}$$

We note that $s(a) > 1$ for all $a > 1$.

Theorem 1.3. *For an arbitrary $\rho \in (0, 1)$ and an arbitrary scalar sequence $R_n \to +\infty$ such that $\lim\limits_{n\to\infty} (R_n/R_{n+1}) = 0$ there exists an entire function f of order ρ being canonical product* (1.4) *with condition for the roots* (1.5), *for which the limiting relation*

$$\lim_{n\to\infty} \max_{R_n \leqslant r \leqslant aR_n} \big(m(f; r)M^{s(a)}(f; r)\big) = 0$$

holds with the exponent $s(a)$ given by formula (1.10).

Definition 1.1. *An optimal exponent $d(a)$ is the supremum of all values s, for which a statement similar to Theorem 1.3 holds, namely, there exists a canonical product f of form* (1.4), (1.5) *and a sequence $R_n \to +\infty$, for which*

$$\lim_{n\to\infty} \max_{R_n \leqslant r \leqslant aR_n} (m(f; r)M^s(f; r)) = 0.$$

This definition yields that if we take an arbitrary exponent $d > d(a)$, then for each canonical product f of form (1.4), (1.5) the quantity

$$\max_{R \leqslant r \leqslant aR} \big(m(f; r)M^d(f; r)\big) \tag{1.11}$$

is separated from zero for all sufficiently large R. By the arbitrariness $d > d(a)$ in this statement we immediately obtain that the maximum in (1.11) tends to infinity $+\infty$ as $R \to +\infty$ if f differs from the identical constant.

Theorems 1.1, 1.2 show that a two-sided inequality holds:

$$2 \leqslant d\left(\frac{9}{4}\right) \leqslant 3; \tag{1.12}$$

as we shall find below, with strict signs, while Theorem 1.3 implies the lower bound

$$d(a) \geqslant s(a) = \frac{\ln \frac{1}{1-1/a}}{\ln(1 + 1/a)} > 1 \tag{1.13}$$

for each $a \in (1, +\infty)$. We find the asymptotics

$$d(a) = \frac{\ln \frac{1}{a-1}}{\ln 2} + O(1) = \log_2 \frac{1}{a-1} + O(1), \qquad a \to 1+. \tag{1.14}$$

A more detailed result is given by the following theorem, see Section 3.

Theorem 1.4. *For each $a \in (1, 9/8]$ the two-sided inequality*

$$\log_2 \frac{1}{a-1} < d(a) < \log_2 \frac{1}{a-1} + \frac{7}{2} \tag{1.15}$$

holds, which implies asymptotics (1.14).

2. Lower bound for optimal exponent

Here we construct canonical products of zero genus, for which there exists a sequence of segment tending to infinity in the real line with a constant quotient of the ends such that on each circumference in the complex plane of a radius belonging to this system of the segments, the minimum of the modulus of the product is less than some fixed negative power of the maximum of the modulus.

We shall employ a simple statement. Let $\{A_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive numbers tending to $+\infty$ fast enough so that the identity $\lim\limits_{n\to\infty}(A_n/A_{n+1})=0$ holds. Then the following asymptotic estimates hold:

$$\sum_{j=1}^{n-1} A_j = o\left(A_n\right), \qquad \sum_{j=n+1}^{\infty} A_j^{-1} = o\left(A_n^{-1}\right), \qquad n\to\infty. \tag{2.1}$$

Let P be a polynomial, $\deg P = p$, all roots $x_1,\dots,x_p$ are real and positive but not necessarily different. Supposing that $P(0)=1$, we write the polynomial P as the product

$$P(z)=\prod_{k=1}^{p}\left(1-\frac{z}{x_k}\right). \tag{2.2}$$

Lemma 2.1. *Let $a>1$, $d>0$ and the inequality*

$$\mu \equiv \max_{1\leqslant x\leqslant a}\left(|P(x)|\,P^d(-x)\right)<1 \tag{2.3}$$

be satisfied and $\{R_n\}_{n\in\mathbb{N}}$ be an arbitrary increasing and fast tending to $+\infty$ sequence of positive numbers obeying the identity

$$\lim_{n\to\infty}\frac{R_n}{R_{n+1}}=0. \tag{2.4}$$

Then for each $\rho\in(0,1)$ there exists an entire function F of normal type for order ρ being a canonical product of zero genus with roots located at the ray $(0,+\infty)$ of the real axis such that the limiting relation

$$\lim_{n\to\infty}\max\left\{m(F;r)M^d(F;r)\mid R_n\leqslant r\leqslant aR_n\right\}=0 \tag{2.5}$$

holds true.

Proof. We choose an arbitrary increasing sequence of natural numbers ν_n obeying the order relation

$$\nu_n \asymp R_n^{\rho}, \quad n\to\infty. \tag{2.6}$$

By (2.4), (2.6) we see that $\{\nu_n\}$, as well as $\{R_n\}$, very fast tends to infinity and is very lacunary:

$$\lim_{n\to\infty}\frac{\nu_n}{\nu_{n+1}}=0. \tag{2.7}$$

It is easy to confirm that by the Lindelöf theorem, see, for instance, [5, Ch. 2, Sect. 2.9], by (2.6) the infinite product

$$F(z)=\prod_{n=1}^{\infty}P^{\nu_n}\left(\frac{z}{R_n}\right) \tag{2.8}$$

is an entire function of normal type for order ρ and all its roots are located on the ray $(0,+\infty)$ of the real axis and $F(0)=1$. In other words, the function F is a canonical product of zero genus:

$$F(z)=\prod_{l=1}^{\infty}\left(1-\xi_l z\right), \qquad \xi_l>0, \qquad \sum_{l=1}^{\infty}\xi_l<+\infty. \tag{2.9}$$

This identity implies

$$m(F;r) = |F(r)|, \qquad M(F;r) = F(-r), \qquad r > 0, \tag{2.10}$$

and similar relations

$$m(P;r) = |P(r)|, \qquad M(P;r) = P(-r), \qquad r > 0, \tag{2.11}$$

hold true for polynomial (2.2). Formulae (2.8), (2.10), (2.11) show that the function $m(F;r)M^d(F;r)$ to be maximized in (2.5) is expanded into the following product

$$m(F;r)M^d(F;r) = \prod_{n=1}^{\infty} \left| P^{\nu_n}\left(\frac{r}{R_n}\right) \right| P^{d\nu_n}\left(-\frac{r}{R_n}\right), \qquad r > 0.$$

This implies that for each $n \in \mathbb{N}$ the relation hold:

$$\begin{aligned} \gamma_n &\equiv \max_{R_n \leqslant r \leqslant aR_n} \left(m(F;r)M^d(F;r) \right) \\ &= \max_{1 \leqslant x \leqslant a} \prod_{j=1}^{\infty} \left| P^{\nu_j}\left(\frac{xR_n}{R_j}\right) \right| P^{d\nu_j}\left(-\frac{xR_n}{R_j}\right) \leqslant \gamma_{n,1}\gamma_{n,2}\gamma_{n,3}, \end{aligned} \tag{2.12}$$

where $\gamma_{1,1} \equiv 1$,

$$\gamma_{n,1} \equiv \max_{1 \leqslant x \leqslant a} \prod_{j=1}^{n-1} \left| P^{\nu_j}\left(\frac{xR_n}{R_j}\right) \right| P^{d\nu_j}\left(-\frac{xR_n}{R_j}\right) \leqslant \tilde{\gamma}_{n,1} \equiv \prod_{j=1}^{n-1} P^{(1+d)\nu_j}\left(-\frac{aR_n}{R_j}\right), \quad n \geqslant 2, \tag{2.13}$$

$$\gamma_{n,2} \equiv \max_{1 \leqslant x \leqslant a} |P^{\nu_n}(x)|\, P^{d\nu_n}(-x) = \max_{1 \leqslant x \leqslant a} \left(|P(x)|\, P^d(-x) \right)^{\nu_n}, \tag{2.14}$$

$$\gamma_{n,3} \equiv \max_{1 \leqslant x \leqslant a} \prod_{j=n+1}^{\infty} \left| P^{\nu_j}\left(\frac{xR_n}{R_j}\right) \right| P^{d\nu_j}\left(-\frac{xR_n}{R_j}\right) \leqslant \tilde{\gamma}_{n,3} \equiv \prod_{j=n+1}^{\infty} P^{(1+d)\nu_j}\left(-\frac{aR_n}{R_j}\right). \tag{2.15}$$

In view of (2.3), (2.14), the identity holds:

$$\gamma_{n,2} = \mu^{\nu_n}. \tag{2.16}$$

We need to show that the sequence γ_n tends to zero. By (2.12)–(2.16) we see that in order to do this, it is sufficient to obtain asymptotic estimates:

$$\ln \tilde{\gamma}_{n,1} = o(\nu_n), \qquad \ln \tilde{\gamma}_{n,3} = o(\nu_n), \qquad n \to \infty. \tag{2.17}$$

In (2.17) only upper bounds are to be proved since the quantities $\ln \tilde{\gamma}_{n,1}$, $\ln \tilde{\gamma}_{n,3}$ are positive: the inequality $\tilde{\gamma}_{n,1} > 1$, $\tilde{\gamma}_{n,3} > 1$ are implied immediately by the fact that $P(-t) > 1$ for each $t > 0$ according (2.2).

The inequality $\ln(1+u) < u$ for $u > 0$ allows to obtain from (2.2) the estimate

$$\ln P(-t) < c\,t, \quad t > 0, \quad \text{where} \quad c = \sum_{k=1}^{p} x_k^{-1}. \tag{2.18}$$

By (2.18), (2.15), (2.6) we find

$$\ln \tilde{\gamma}_{n,3} \leqslant O\left(R_n \sum_{j=n+1}^{\infty} \nu_j R_j^{-1} \right) = O\left(R_n \sum_{j=n+1}^{\infty} R_j^{\rho-1} \right), \qquad n \to \infty. \tag{2.19}$$

Due to restriction (2.4) the sequence $A_n \equiv R_n^{1-\rho}$ is very lacunary, that is, $\lim\limits_{n\to\infty} (A_n/A_{n+1}) = 0$. This is according to (2.1) we have:

$$\sum_{j=n+1}^{\infty} R_j^{\rho-1} = o\left(R_n^{\rho-1}\right), \qquad n \to \infty. \tag{2.20}$$

By (2.19), (2.20), (2.6) we obtain the needed upper bound:

$$\ln \tilde{\gamma}_{n,3} \leqslant o\left(R_n \cdot R_n^{\rho-1}\right) = o\left(R_n^{\rho}\right) = o(\nu_n), \qquad n \to \infty.$$

Assuming that the roots of the polynomial P are taken in the non-ascending order, apart of estimate (2.18), which is good for "small" values of t, we have another estimate

$$P(-t) \leqslant \left(\frac{2t}{x_1}\right)^p, \qquad t \geqslant x_1,$$

appropriate for "large" t. Therefore,

$$\ln P(-t) < 2p \ln t, \qquad t > x_1 + \frac{2}{x_1}.$$

Hence, in view of (2.13), for sufficiently large n we find:

$$\ln \tilde{\gamma}_{n,1} \leqslant O\left(\sum_{j=1}^{n-1} \nu_j \ln\left(\frac{aR_n}{R_j}\right)\right).$$

By order relation (2.6) we have

$$\ln\left(\frac{R_n}{R_j}\right) = \ln\left(\frac{\nu_n}{\nu_j}\right) + O(1) = O\left(\ln\left(\frac{\nu_n}{\nu_j}\right)\right), \qquad 1 \leqslant j \leqslant n-1.$$

Thus,

$$\ln \tilde{\gamma}_{n,1} \leqslant O\left(\sum_{j=1}^{n-1} \nu_j \ln\left(\frac{\nu_n}{\nu_j}\right)\right).$$

This shows that in order to obtain first asymptotic estimate (2.17), it remains to show that the sum

$$\sum_{j=1}^{n-1} \frac{\nu_j}{\nu_n} \ln\left(\frac{\nu_n}{\nu_j}\right) < \sum_{j=1}^{n-1} \left(\frac{\nu_j}{\nu_n}\right)^{1/2}$$

tends to zero. This is immediately implied by (2.1) since the sequence $A_n \equiv \nu_n^{1/2}$, due to (2.4) and (2.6), satisfy the condition $\lim\limits_{n\to\infty} (A_n/A_{n+1}) = 0$. The asymptotic estimates (2.17) have been obtained and this completes the proof of the lemma. □

Lemma 2.1 shows that to prove Theorem 1.3, it is sufficient for each $a > 1$ to provide a polynomial P_a with positive roots and $P_a(0) = 1$ such that

$$\max_{1 \leqslant x \leqslant a} \left(|P_a(x)| \left(P_a(-x)\right)^{s(a)}\right) < 1. \tag{2.21}$$

If we take $P_a(z) = 1 - z/a$, it turns out that the maximum in the left hand side of (2.21) is exactly 1. Let us show that relation (2.21) holds for the binomial $P_a(z) = 1 - z/c$ under the choice

$$c = c(a) = a\left(1 + 2^{-s(a)-1/2}\right)^{-1}, \qquad a > 1. \tag{2.22}$$

First we provide a general description of the behavior of the quantity $s(a)$ in (1.10).

Lemma 2.2. *The function $s(a)$ decays on the ray $1 < a < +\infty$. Moreover,*

$$\lim_{a \to 1+} s(a) = +\infty, \qquad \lim_{a \to +\infty} s(a) = 1, \tag{2.23}$$

$$\max\left\{1, \log_2 \frac{1}{a-1}\right\} < s(a) < \frac{a+1}{a-1}, \qquad a > 1. \tag{2.24}$$

Proof. A direct calculation based on the formula

$$s(a) = \frac{\ln a - \ln(a-1)}{\ln(a+1) - \ln a}, \qquad a > 1,$$

leads to the following relations

$$s'(a)a(a^2-1)\ln^2(1+1/a) = \psi(a) - \psi(a+1), \tag{2.25}$$

$$s'(a)a(a^2-1)\ln(1+1/a) = s(a)(a-1) - (a+1). \tag{2.26}$$

Here

$$\psi(a) \equiv a\ln a - (a-1)\ln(a-1), \quad a > 1. \tag{2.27}$$

Since auxiliary function (2.27) increases as $a > 1$, then for all such a, by (2.25) we have $s'(a) < 0$. This is why the function $s(a)$ decays on the ray $1 < a < +\infty$. Using then identity (2.26) and taking into consideration that its left hand side is negative as $a > 1$, we obtain the right hand side of (2.24).

Let us obtain the left hand side of inequality (2.24). The simplest estimate $s(a) > 1$ was mentioned in Section 1. In addition, it follows immediately from Definition (1.10) that

$$s(a) \equiv \frac{\ln \frac{1}{1-1/a}}{\ln(1+1/a)} > \frac{\ln \frac{1}{a-1}}{\ln 2} = \log_2 \frac{1}{a-1}, \qquad a > 1.$$

This completes the proof of two-sided estimate (2.24). Finally, both limiting relations (2.23) are easily obtained both from original formula (1.10) and (2.24). The proof is complete. □

We proceed to proving Theorem 1.3. For an arbitrary $a > 1$ we define a quantity $c = c(a)$ by the rule (2.22) and construct a binomial $P_a(z) = 1 - z/c$. We first confirm that

$$1 < c \equiv a\left(1 + 2^{-s(a)-1/2}\right)^{-1} < a. \tag{2.28}$$

The inequality $c < a$ is obvious. We rewrite inequality $c > 1$ in an equivalent form

$$2^{-s(a)-1/2} < a - 1 \Leftrightarrow s(a) > -\frac{1}{2} + \log_2 \frac{1}{a-1},$$

which is true due to (2.24).

We are going to confirm that the function

$$h(x) \equiv |P_a(x)|\,(P_a(-x))^{s(a)} = \left|1 - \frac{x}{c}\right|\left(1 + \frac{x}{c}\right)^{s(a)}$$

decreases on the segment $1 \leqslant x \leqslant c$ and increases on the segment $c \leqslant x \leqslant a$. Indeed, for $x \in [1, c)$ we have:

$$\ln h(x) = \ln\left(1 - \frac{x}{c}\right) + s(a)\ln\left(1 + \frac{x}{c}\right),$$

and hence

$$h'(x) = h(x)\left(\frac{s(a)}{c+x} - \frac{1}{c-x}\right) = \frac{h(x)}{c^2 - x^2}\left(c\,(s(a)-1) - (s(a)+1)\,x\right).$$

Since

$$c\,(s(a)-1) - (s(a)+1)\,x < a\,(s(a)-1) - (s(a)+1) = (a-1)s(a) - (a+1) < 0$$

due to (2.28), (2.24), then $h'(x) < 0$ as $1 \leqslant x < c$ and the function $h(x)$ decreases on the segment $1 \leqslant x \leqslant c$. The increasing of the function $h(x)$ on the segment $c \leqslant x \leqslant a$ is obvious since here it becomes

$$h(x) = \left(\frac{x}{c} - 1\right)\left(1 + \frac{x}{c}\right)^{s(a)}$$

being the product of two positive and increasing as $x \in (c, a]$ functions.

As we see,

$$\max_{1\leqslant x\leqslant a} h(x)=\max\left\{h(1),h(a)\right\}. \tag{2.29}$$

It remains to show that maximum in (2.29) is less than 1. First,

$$h(1)=\left(1-\frac{1}{c}\right)\left(1+\frac{1}{c}\right)^{s(a)}<1.$$

Indeed, the inequality $h(1)<1$ is equivalent to the following one:

$$\ln\left(1-\frac{1}{c}\right)+s(a)\ln\left(1+\frac{1}{c}\right)<0,$$

or, which is the same,

$$s(a)<\frac{\ln\frac{1}{1-1/c}}{\ln(1+1/c)}\equiv s(c).$$

But inequality $s(a)<s(c)$ is true since $a>c>1$ and by Lemma 2.2 the function (1.10) decreases on the ray $(1,+\infty)$. Second, in view of (2.22) we write

$$h(a)=\left(\frac{a}{c}-1\right)\left(1+\frac{a}{c}\right)^{s(a)}=2^{-s(a)-1/2}\left(2+2^{-s(a)-1/2}\right)^{s(a)}=2^{-1/2}\left(1+2^{-s(a)-3/2}\right)^{s(a)}.$$

Then we successively employ elementary estimates:

$$1+u<e^u \quad\text{as}\quad u=2^{-s(a)-3/2}>0 \quad\text{and}\quad v2^{-v}\leqslant 1/(e\ln 2) \quad\text{as}\quad v=s(a)>1.$$

As a result we obtain that

$$h(a)<2^{-1/2}\exp\left(s(a)2^{-s(a)-3/2}\right)\leqslant\frac{1}{\sqrt{2}}\exp\left(\frac{1}{\sqrt{8}e\ln 2}\right)<0.86.$$

Thus, by (2.29), for each $a>1$, for the binomial $P_a(z)=1-z/c(a)$ we have:

$$\max_{1\leqslant x\leqslant a}\left(|P_a(x)|\left(P_a(-x)\right)^{s(a)}\right)=\max_{1\leqslant x\leqslant a} h(x)=\max\left\{h(1),h(a)\right\}<1$$

if we choose the coefficient $c(a)$ by rule (2.22), while the coefficient $s(a)$ is to be chosen by rule (1.10). In other words, relation (2.21) holds. Applying Lemma 2.1, we complete the proof of Theorem 1.3.

We proceed to proving Theorem 1.2. We are going to show that once we take

$$P(z)=\left(1-\frac{8}{11}z\right)\left(1-\frac{7}{15}z\right),$$

the inequality holds:

$$\max_{1\leqslant x\leqslant 9/4}\left\{|P(x)|\left(P(-x)\right)^{2.04}\right\}<0.996. \tag{2.30}$$

According to Lemma 2.1, this implies the validity of Theorem 1.2 in a strong version. Namely, the lower bound in (1.12) is specified: $2.04<d(9/4)$. We also mention that estimate (1.13) following Theorem 1.3, for $a=9/4$ gives

$$d(9/4)\geqslant s(9/4)=\frac{\ln(9/5)}{\ln(13/9)}=1.59\ldots,$$

but this is insufficient for our closest aim.

Let us estimate from above the maximums of the function

$$H(x)\equiv|P(x)|P^2(-x)=\left|1-\frac{8}{11}x\right|\left|1-\frac{7}{15}x\right|\left(1+\frac{8}{11}x\right)^2\left(1+\frac{7}{15}x\right)^2$$

on the segments $[1, 11/8]$ and $[11/8, 9/4]$. First we are going to prove the decreasing of the function H on the segment $[1, 11/8]$. Together with the relations

$$H(1) = \frac{3}{11} \cdot \frac{8}{15} \left(\frac{19}{11} \cdot \frac{22}{15}\right)^2 = \frac{8}{55} \left(\frac{38}{15}\right)^2 < 0.94$$

this will gives the estimate

$$\max\left\{H(x) \mid 1 \leqslant x \leqslant \frac{11}{8}\right\} = H(1) < 0.94. \tag{2.31}$$

For $x \in [1, 11/8)$ we have

$$l(x) \equiv \ln H(x) = \ln\left(1 - \frac{8x}{11}\right) + \ln\left(1 - \frac{7x}{15}\right) + 2\ln\left(1 + \frac{8x}{11}\right) + 2\ln\left(1 + \frac{7x}{15}\right),$$

$$l'(x) = \frac{14}{15 + 7x} + \frac{16}{11 + 8x} - \frac{8}{11 - 8x} - \frac{7}{15 - 7x}.$$

This immediately shows the decreasing of the derivative l' on the semi-interval $[1, 11/8)$. This implies the estimate

$$l'(x) \leqslant l'(1) = \frac{7}{11} + \frac{16}{19} - \frac{8}{3} - \frac{7}{8} < 0,$$

which shows the decreasing of the function l on the semi-interval $[1, 11/8)$, and hence, the decreasing of the function H on the segment $[1, 11/8]$.

The increasing of the function

$$H(x) = \left(\frac{8x}{11} - 1\right)\left(\frac{7x}{15} - 1\right)\left(1 + \frac{8x}{11}\right)^2\left(1 + \frac{7x}{15}\right)^2, \qquad x \geqslant \frac{15}{7},$$

on the ray $[15/7, +\infty)$ is obvious and this is why

$$\max_{15/7 \leqslant x \leqslant 9/4} H(x) = H\left(\frac{9}{4}\right) = \left(\frac{18}{11} - 1\right)\left(\frac{21}{20} - 1\right)\left(1 + \frac{18}{11}\right)^2\left(1 + \frac{21}{20}\right)^2 < 0.93. \tag{2.32}$$

Let us estimate the maximum of the function H on the segment $[11/8, 15/7]$. We have:

$$H\left(\frac{11}{8}\right) = H\left(\frac{15}{7}\right) = 0, \qquad H(x) = \left(\frac{8x}{11} - 1\right)\left(1 - \frac{7x}{15}\right)\left(1 + \frac{8x}{11}\right)^2\left(1 + \frac{7x}{15}\right)^2,$$

$$l(x) = \ln H(x) = \ln\left(\frac{8x}{11} - 1\right) + \ln\left(1 - \frac{7x}{15}\right) + 2\ln\left(1 + \frac{8x}{11}\right) + 2\ln\left(1 + \frac{7x}{15}\right),$$

$$l'(x) = \frac{14}{15 + 7x} + \frac{16}{11 + 8x} + \frac{8}{8x - 11} - \frac{7}{15 - 7x}, \qquad x \in \left(\frac{11}{8}, \frac{15}{7}\right).$$

Elementary arguing show the decreasing of l' on the interval $(11/8, 15/7)$. And since

$$\lim_{x \to \frac{11}{8}+} l'(x) = +\infty, \qquad \lim_{x \to \frac{15}{7}-} l'(x) = -\infty,$$

by the continuity of the l' there exists a point $x_0 \in (11/8, 15/7)$ such that

$$l'(x) > 0 \quad \text{as} \quad x \in \left(\frac{11}{8}, x_0\right), \qquad l'(x_0) = 0, \qquad l'(x) < 0 \quad \text{as} \quad x \in \left(x_0, \frac{15}{7}\right).$$

This is why the function H increases on the segment $[11/8, x_0]$ and decreases on the segment $[x_0, 15/7]$. At the point x_0, the function H attains the maximum on the segment $[11/8, 15/7]$ and we are going to show that

$$H(x_0) = \max\left\{H(x) \mid \frac{11}{8} \leqslant x \leqslant \frac{15}{7}\right\} < 0.91. \tag{2.33}$$

By straightforward calculations we confirm the validity of the inequalities

$$l'\left(\frac{24}{13}\right) < 0 < l'\left(\frac{11}{6}\right) < \frac{1}{11},$$

by which we find

$$\frac{11}{6} < x_0 < \frac{24}{13}, \qquad 0 < l'(x) < \frac{1}{11} \quad \forall x \in \left(\frac{11}{6}, x_0\right). \tag{2.34}$$

We also have

$$H\left(\frac{11}{6}\right) = \frac{1}{3} \cdot \frac{13}{90} \cdot \left(\frac{7 \cdot 167}{3 \cdot 90}\right)^2 < 0.903.$$

This is why

$$l\left(\frac{11}{6}\right) < \ln 0.903 < -0.1. \tag{2.35}$$

By (2.34), (2.35) we get the inequalities

$$l(x_0) - l\left(\frac{11}{6}\right) = \int\limits_{11/6}^{x_0} l'(x)dx < \frac{1}{11}\left(x_0 - \frac{11}{6}\right) < \frac{1}{11}\left(\frac{24}{13} - \frac{11}{6}\right) = \frac{1}{858},$$

$$l(x_0) < l\left(\frac{11}{6}\right) + \frac{1}{858} < -0.098 \quad \Longrightarrow \quad H(x_0) < \exp(-0.098) < 0.91.$$

This proves relation (2.33). By (2.32), (2.33) we get the following estimate for the maximum of the function H on the segment $[11/8, 9/4]$:

$$\max\left\{ H(x) \mid \frac{11}{8} \leqslant x \leqslant \frac{9}{4} \right\} < 0.93. \tag{2.36}$$

Now let us obtain inequality (2.30) from inequalities (2.31), (2.36). In view of the increasing of the square trinomial

$$P(-x) = \left(1 + \frac{8x}{11}\right)\left(1 + \frac{7x}{15}\right)$$

on the ray $(0, +\infty)$, we have:

$$\max\left\{ P(-x) \mid 1 \leqslant x \leqslant \frac{11}{8} \right\} = P\left(-\frac{11}{8}\right) = 2\left(1 + \frac{77}{120}\right) = \frac{197}{60} < \exp(1.2), \tag{2.37}$$

$$\max\left\{ P(-x) \mid \frac{11}{8} \leqslant x \leqslant \frac{9}{4} \right\} = P\left(-\frac{9}{4}\right) = \left(1 + \frac{18}{11}\right)\left(1 + \frac{21}{20}\right) = \frac{1189}{220} < \exp(1.69). \tag{2.38}$$

Then (2.31), (2.37) imply the estimate

$$\max_{1 \leqslant x \leqslant 11/8} \left\{ |P(x)| \left(P(-x)\right)^{2.04} \right\} < 0.94 \exp(0.048) < 0.987,$$

and by (2.36), (2.38) we find:

$$\max_{11/8 \leqslant x \leqslant 9/4} \left\{ |P(x)| \left(P(-x)\right)^{2.04} \right\} < 0.93 \exp(1.69 \cdot 0.04) < 0.996.$$

Thus, we have provided a polynomial P of second degree obeying condition (2.30). For such polynomial relation (2.3) is surely obeyed with the values $a = 9/4$ and $d = 2.04$. Applying Lemma 2.1, we complete the proof of Theorem 1.2.

3. Upper bound for optimal exponent

The method for estimating the quantity $d(a)$ from above is based on the following lemma.

Lemma 3.1. *If the numbers $a > 1$, $b > 1$, $\alpha \in \mathbb{R}$ are such that the function*

$$\Phi(x; a, b, \alpha) \equiv \int\limits_x^{ax} t^\alpha \left(\ln|1-t| + b\ln(1+t)\right) dt$$

is positive on the ray $0 < x < +\infty$, then for each $R > 0$ for arbitrary canonical product (1.4) with condition (1.5) for the roots, the inequality

$$\int\limits_R^{aR} t^\alpha \ln\left(m(f;t)M^b(f;t)\right) dt > 0 \tag{3.1}$$

holds true. In particular, for each $R > 0$ there exists a point $r \in (R, aR)$, for which

$$m(f;r) > M^{-b}(f;r).$$

Proof. According to (1.4), (1.6), on the ray $t > 0$ outside the roots of the function f we have

$$\ln\left(m(f;t)M^b(f;t)\right) = \sum_{n=1}^{\infty}\left(\ln\left|1-\frac{t}{\lambda_n}\right| + b\ln\left(1+\frac{t}{\lambda_n}\right)\right). \tag{3.2}$$

For a fixed $R > 0$ as $t \in [R, aR]$ all terms of the series in (3.2) are integrable in the improper sense. Moreover, except for a possibly finitely many terms depending on $R > 0$ and $a > 1$, this series consists of continuous on the segment $[R, aR]$ functions

$$u_n(t) \equiv \ln\left|1-\frac{t}{\lambda_n}\right| + b\ln\left(1+\frac{t}{\lambda_n}\right) \leqslant \frac{(1+b)aR}{\lambda_n}, \qquad n \geqslant n_0, \quad t \in [R, aR],$$

and the series $\sum\limits_{n=n_0}^{\infty} u_n(t)$ converges uniformly on such segment due to (1.5). We multiply both sides of the identity (3.2) by t^α and integrated this product over the segment $[R, aR]$. Since we can integrated the series in (3.2) term-by-term, then

$$\begin{aligned}\int\limits_R^{aR} t^\alpha \ln\left(m(f;t)M^b(f;t)\right) dt &= \sum_{n=1}^{\infty}\int\limits_R^{aR} t^\alpha\left(\ln\left|1-\frac{t}{\lambda_n}\right| + b\ln\left(1+\frac{t}{\lambda_n}\right)\right) dt \\ &= \sum_{n=1}^{\infty}\lambda_n^{\alpha+1}\int\limits_{R/\lambda_n}^{aR/\lambda_n} u^\alpha\left(\ln(1-u) + b\ln(1+u)\right) du \\ &= \sum_{n=1}^{\infty}\lambda_n^{\alpha+1}\Phi\left(\frac{R}{\lambda_n}; a, b, \alpha\right).\end{aligned}$$

We have obtained a representation for the integral in (3.1) as the sum of a converging scalar series

$$\int\limits_R^{aR} t^\alpha \ln\left(m(f;t)M^b(f;t)\right) dt = \sum_{n=1}^{\infty}\lambda_n^{\alpha+1}\Phi\left(\frac{R}{\lambda_n}; a, b, \alpha\right). \tag{3.3}$$

We mention that the definition of the function Φ implies easily the asymptotic estimate

$$\Phi(x; a, b, \alpha) = O\left(x^{\alpha+2}\right), \qquad x \to 0+,$$

which together with (1.5) confirms the convergence of the series in (3.3).

By the assumptions of the lemma, the function Φ is positive. This is why the sum of the series in (3.3) is positive and the same is true for the integral in (3.1). The proof is complete. $\square$

We proceed to proving Theorem 1.1. As Lemma 3.1 shows, it is sufficient to confirm the positivity of the function $\Phi(x; 9/4, 3, -3)$ for each $x > 0$. We move a needed fact into a separate statement.

Lemma 3.2. *The function*

$$\Phi(x) = \Phi\left(x; \frac{9}{4}, 3, -3\right) \equiv \int\limits_x^{9x/4} t^{-3}\left(\ln|1-t| + 3\ln(1+t)\right) dt \tag{3.4}$$

is positive on the ray $x > 0$.

Proof. By definition (3.4) we see that $\Phi(x)$ is continuous as $x > 0$. We denote

$$\varphi(x) \equiv \ln|1-x| + 3\ln(1+x), \qquad x > 0, \quad x \neq 1, \tag{3.5}$$

and integrate in (3.4) by parts. After elementary calculations we arrive at the representation

$$\begin{aligned} 2\Phi(x) &\equiv 2\int\limits_x^{9x/4} t^{-3}\varphi(t)dt \\ &= \left(1 - \frac{16}{81x^2}\right)\varphi\left(\frac{9x}{4}\right) - \left(1 - \frac{1}{x^2}\right)\varphi(x) - 8\ln\frac{3}{2} + \frac{10}{9x}, \qquad x > 0. \end{aligned} \tag{3.6}$$

We mention that despite the identity $\lim\limits_{x\to 1}\varphi(x) = -\infty$, expression (3.6) at the points $x = 4/9$ and $x = 1$ has no jumps and take finite values

$$2\Phi\left(\frac{4}{9}\right) = \frac{65}{16}\varphi\left(\frac{4}{9}\right) - 8\ln\frac{3}{2} + \frac{5}{2} = 1.35\ldots, \qquad 2\Phi(1) = \frac{65}{81}\varphi\left(\frac{9}{4}\right) - 8\ln\frac{3}{2} + \frac{10}{9} = 0.88\ldots.$$

Let us confirm the validity of the limiting relations

$$\lim_{x\to 0+}\Phi(x) = +\infty, \qquad \lim_{x\to+\infty}\Phi(x) = 0. \tag{3.7}$$

On the other hand, as $x \in (0, 4/9)$, according to (3.5), (3.6) the function $2\Phi(x)$ coincides with the quantity

$$\left(1 - \frac{16}{81x^2}\right)\left(\ln\left(1 - \frac{9x}{4}\right) + 3\ln\left(1 + \frac{9x}{4}\right)\right) - \left(1 - \frac{1}{x^2}\right)\left(\ln(1-x) + 3\ln(1+x)\right) - 8\ln\frac{3}{2} + \frac{10}{9x}.$$

This is why, taking the asymptotics of the logarithms,

$$\ln\left(1 \pm \frac{9x}{4}\right) = \pm\frac{9x}{4} + O\left(x^2\right), \qquad \ln(1 \pm x) = \pm x + O\left(x^2\right), \qquad x \to 0+,$$

we arrive at the relation

$$2\Phi(x) = \left(1 - \frac{16}{81x^2}\right)\left(\frac{9x}{2} + O\left(x^2\right)\right) - \left(1 - \frac{1}{x^2}\right)\left(2x + O\left(x^2\right)\right) - 8\ln\frac{3}{2} + \frac{10}{9x} = \frac{20}{9x} + O(1),$$

where $x \to 0+$.

On the other hand, as $x \in (1, +\infty)$, for the function $2\Phi(x)$ we obtain the expression

$$\left(1 - \frac{16}{81x^2}\right)\left(\ln\left(\frac{9x}{4} - 1\right) + 3\ln\left(\frac{9x}{4} + 1\right)\right) - \left(1 - \frac{1}{x^2}\right)\left(\ln(x-1) + 3\ln(x+1)\right) - 8\ln\frac{3}{2} + \frac{10}{9x}.$$

This is why, taking the asymptotics of the logarithms,

$$\ln\left(\frac{9x}{4}\pm 1\right) = \ln\frac{9x}{4} + o(1), \qquad \ln(x\pm 1) = \ln x + o(1), \qquad x\to+\infty,$$

we arrive at the relation

$$2\Phi(x) = 4\ln\frac{9x}{4} - 4\ln x - 8\ln\frac{3}{2} + o(1) = o(1), \qquad x\to+\infty.$$

This justifies identities (3.7).

Differentiating (3.4), we obtain

$$x^3\Phi'(x) = \frac{16}{81}\varphi\left(\frac{9x}{4}\right) - \varphi(x), \qquad x>0, \tag{3.8}$$

with the specification $\Phi'(4/9) = -\infty$, $\Phi'(1) = +\infty$. At the other points of the semi-axis $0<x<+\infty$ the derivative Φ' is finite, and its sign, according to (3.8), coincides with the sign of the function

$$\psi(x) \equiv \frac{16}{81}\varphi\left(\frac{9x}{4}\right) - \varphi(x), \qquad x>0, \quad x\neq\frac{4}{9}, \quad x\neq 1. \tag{3.9}$$

To determine the latter, we take into consideration the relations

$$\lim_{x\to 0+}\psi(x) = 0, \qquad \lim_{x\to 4/9}\psi(x) = -\infty, \qquad \lim_{x\to 1}\psi(x) = +\infty, \qquad \lim_{x\to+\infty}\psi(x) = -\infty \tag{3.10}$$

easily implied by definitions (3.5), (3.9). Straightforward calculations, also based on formulae (3.5), (3.9), show that

$$\psi'(x) = -\frac{10(234x^3 - 133x^2 + 16)}{9(81x^2-16)(x^2-1)} \tag{3.11}$$

for all $x\in(0,4/9)\cup(4/9,1)\cup(1,+\infty)$. Since $234x^3-133x^2+16>0$ as $x>0$, in view of (3.10), (3.11), the function ψ on the interval $(0,4/9)$ decreases from 0 to $-\infty$ and then, on the interval $(4/9,1)$, it increases from $-\infty$ to $+\infty$, and finally, on the ray $(1,+\infty)$ it decreases from $+\infty$ to $-\infty$. Therefore, there exist points $x_1\in(4/9,1)$, $x_2\in(1,+\infty)$ such that $\psi(x)<0$ as $x\in(0,4/9)\cup(4/9,x_1)\cup(x_2,+\infty)$ as $\psi(x)>0$ as $x\in(x_1,1)\cup(1,x_2)$.

Combining (3.7)–(3.9), we make the following conclusion on the behavior of integral (3.4) on the positive semi-axis: on the interval $(0,x_1)$ the function Φ decreases from $+\infty$ to the value $\Phi(x_1)$; on the segment $[x_1,x_2]$ the function Φ increases from $\Phi(x_1)$ to $\Phi(x_2)$; on the ray $(x_2,+\infty)$ the function Φ decreases from $\Phi(x_2)$ to 0. As an additional information we say that the graph of the function Φ has four inflection points and vertical tangentials at two of these points with the abscissas 4/9 and 1.

Thus, to prove inequality $\Phi(x)>0$ for $x>0$, it is sufficient to check the positivity of the function Φ only at the point of the minimum $x_1\in(4/9,1)$. Since $\Phi'(x_1)=0$, then, see (3.8), the identity

$$\varphi\left(\frac{9x_1}{4}\right) = \frac{81}{16}\,\varphi(x_1)$$

holds. Employing it in (3.6), we obtain

$$2\Phi(x_1) = \left(1-\frac{16}{81x_1^2}\right)\frac{81}{16}\,\varphi(x_1) - \left(1-\frac{1}{x_1^2}\right)\varphi(x_1) - 8\ln\frac{3}{2} + \frac{10}{9x_1} = \frac{65}{16}\,\varphi(x_1) - 8\ln\frac{3}{2} + \frac{10}{9x_1}.$$

We introduce an auxiliary function

$$\chi(x) \equiv \frac{65}{16}\,\varphi(x) + \frac{10}{9x} = \frac{65}{16}\left(\ln(1-x) + 3\ln(1+x)\right) + \frac{10}{9x}, \qquad x\in(4/9,1),$$

in order to write the latter result in a compact form:

$$2\Phi(x_1) = \chi(x_1) - 8\ln\frac{3}{2}. \tag{3.12}$$

It remains to estimate from below quantity (3.12). In order to do this, let us confirm that $x_1 < 0.68 = 17/25$. Indeed, since

$$\frac{16}{81}\psi\left(\frac{153}{100}\right) - \psi\left(\frac{17}{25}\right) = \frac{16}{81}\left(\ln\frac{53}{100} + 3\ln\frac{253}{100}\right) - \left(\ln\frac{8}{25} + 3\ln\frac{42}{25}\right) > 0.0077,$$

then $\Phi'(17/25) > 0$ due to (3.8). But $17/25 \in (4/9, 1)$ and this gives a specified localization of the minimum point: $x_1 \in (4/9, 17/25)$. The decreasing of the function χ on the entire interval $(4/9, 1)$ implies the inequality $\chi(x_1) > \chi(17/25)$. Applying this inequality in (3.12), we write

$$2\Phi(x_1) > \chi(17/25) - 8\ln\frac{3}{2} = \frac{65}{16}\left(\ln\frac{8}{25} + 3\ln\frac{42}{25}\right) + \frac{250}{153} - 8\ln\frac{3}{2} > 0.0841.$$

Thus, for all $x > 0$ we have $\Phi(x) \geqslant \Phi(x_1) > 0$. The proof is complete. □

According to Lemmata 3.1, 3.2 the choice of the parameters $a = 9/4$, $b = 3$, $\alpha = -3$ ensure inequality (3.1). In other words, (1.9) holds with obvious corollary (1.8). This completes the proof Theorem 1.1. The upper estimate in (1.12) implied by this theorem allows a little strengthening: $d(9/4) < 2.95$. As computer-assisted calculations show, the function $\Phi(x; 9/4, b, -3)$ with the parameters $b \in (2.91, 2.95)$ is positive for all $x > 0$, while the function $\Phi(x; 9/4, 2.9, -3)$ already does not possess such property.

Finally, in the proof of Theorem 1.4 we shall need the following auxiliary fact.

Lemma 3.3. *The function*

$$\Phi(x) = \Phi\left(x; a, s(a)+3, 0\right) \equiv \int_x^{ax} \left(\ln|1-t| + (s(a)+3)\ln(1+t)\right) dt \tag{3.13}$$

is positive everywhere on the ray $0 < x < +\infty$ *for each value of the parameter* $a \in (1, 9/8]$.

Proof. We argue for a fixed $a \in (1, 9/8]$. The positivity of integral (3.13) on the interval $0 < x < a^{-2}$ follows from the positivity of the integrand on the segments $[x, ax] \subset (0, a^{-1})$. Indeed, the function

$$\mathcal{L}(t) \equiv \ln(1-t) + s(a)\ln(1+t)$$

is continuous and concave on the semi-interval $[0, 1)$. The identities $\mathcal{L}(0) = \mathcal{L}(a^{-1}) = 0$ are also true. This implies immediately the positivity of $\mathcal{L}(t)$ as $t \in (0, a^{-1})$. Thus, the integrand in (3.13) on all segments $[x, ax]$ for $0 < x < a^{-2}$ exceed the quantity $3\ln(1+t) > 0$.

We further have

$$\int_{1/a}^{1} \ln(1-t)dt = \left(1-\frac{1}{a}\right)\ln\left(1-\frac{1}{a}\right) - \left(1-\frac{1}{a}\right), \qquad \int_{1/a}^{1} \ln(1+t)dt > \left(1-\frac{1}{a}\right)\ln\left(1+\frac{1}{a}\right).$$

By (3.13) this implies

$$\begin{aligned}\Phi\left(\frac{1}{a}\right) &> \left(1-\frac{1}{a}\right)\ln\left(1-\frac{1}{a}\right) - \left(1-\frac{1}{a}\right) + (s(a)+3)\left(1-\frac{1}{a}\right)\ln\left(1+\frac{1}{a}\right)\\ &= \left(1-\frac{1}{a}\right)\left[\ln\left(1-\frac{1}{a}\right) - 1 + \ln\frac{1}{1-1/a} + 3\ln\left(1+\frac{1}{a}\right)\right]\\ &= \left(1-\frac{1}{a}\right)\left(3\ln\left(1+\frac{1}{a}\right) - 1\right) \geqslant \left(1-\frac{1}{a}\right)\left(3\ln\frac{17}{9} - 1\right).\end{aligned}$$

This is why

$$\Phi\left(\frac{1}{a}\right) > 0.9\left(1-\frac{1}{a}\right). \tag{3.14}$$

Since the function Φ, being continuous on $\mathbb{R}$, is differentiable on the interval $(-1/a, 1/a)$ and its derivative is absolutely integrable on this interval, then

$$\Phi\left(\frac{1}{a}\right) - \Phi(x) = \int\limits_{x}^{1/a} \Phi'(y)dy, \qquad x \in \left(-\frac{1}{a}, \frac{1}{a}\right).$$

And if we succeed to get an upper bound

$$\Phi'(y) \leqslant C \qquad \forall y \in \left(a^{-2}, a^{-1}\right) \tag{3.15}$$

(we observe that on the interval (a^{-2}, a^{-1}) the derivative is not bounded from below), then for each value $x \in [a^{-2}, a^{-1})$ we obtain the inequality

$$\Phi(x) = \Phi\left(\frac{1}{a}\right) - \int\limits_{x}^{1/a} \Phi'(y)dy \geqslant \Phi\left(\frac{1}{a}\right) - C\left(\frac{1}{a} - x\right) \geqslant \Phi\left(\frac{1}{a}\right) - C\left(\frac{1}{a} - \frac{1}{a^2}\right). \tag{3.16}$$

By (3.14), (3.16) for all $x \in [a^{-2}, a^{-1})$ we have

$$\Phi(x) \geqslant 0.9\left(1-\frac{1}{a}\right) - \frac{C}{a}\left(1-\frac{1}{a}\right) = \left(1-\frac{1}{a}\right)\left(0.9 - \frac{C}{a}\right) > \left(1-\frac{1}{a}\right)(0.9 - C). \tag{3.17}$$

Let us show that estimate (3.15) is true once we take $C = 0.7$. Then according to (3.14), (3.17), this will imply the positivity $\Phi(x)$ for $a^{-2} \leqslant x \leqslant a^{-1}$. For $x \in (a^{-2}, a^{-1})$ we write

$$\begin{aligned}\Phi'(x) &= a\left(\ln(1-ax) + (s(a)+3)\ln(1+ax)\right) - \left(\ln(1-x) + (s(a)+3)\ln(1+x)\right)\\ &= (a-1)\left(\ln(1-ax) + (s(a)+3)\ln(1+ax)\right) + \ln\frac{1-ax}{1-x} + (s(a)+3)\ln\frac{1+ax}{1+x}.\end{aligned}$$

In view of the identities

$$\begin{aligned}\max\left\{\frac{1-ax}{1-x} \mid \frac{1}{a^2} \leqslant x \leqslant \frac{1}{a}\right\} &= \frac{1-1/a}{1-1/a^2} = \frac{a}{a+1},\\ \max\left\{\frac{1+ax}{1+x} \mid \frac{1}{a^2} \leqslant x \leqslant \frac{1}{a}\right\} &= \frac{2}{1+1/a} = \frac{2a}{a+1}\end{aligned}$$

and the negativity of the function $\mathcal{L}$ on the interval $(a^{-1}, 1)$, the estimate holds:

$$\Phi'(x) < \frac{a-1}{2}\left(s(a) + 3 + 6\ln 2\right) + \ln\frac{a}{a+1}, \qquad x \in \left(a^{-2}, a^{-1}\right). \tag{3.18}$$

Indeed, for such x we have

$$\begin{aligned}\Phi'(x) &< (a-1)\left(\ln(1-ax) + (s(a)+3)\ln(1+ax)\right) + \ln\frac{a}{a+1} + (s(a)+3)\ln\frac{2a}{a+1}\\ &= (a-1)\mathcal{L}(ax) + 3(a-1)\ln(1+ax) + \ln\frac{a}{a+1} + (s(a)+3)\ln\left(1+\frac{a-1}{a+1}\right)\\ &< 3(a-1)\ln 2 + \ln\frac{a}{a+1} + \frac{a-1}{a+1}(s(a)+3) < \frac{a-1}{2}\left(s(a)+3+6\ln 2\right) + \ln\frac{a}{a+1}.\end{aligned}$$

We denote $\delta = a-1$ and estimate from above the function

$$\Phi_1(a) \equiv (a-1)s(a) = \frac{\delta\left(\ln(1+\delta) - \ln\delta\right)}{\ln(1+1/a)} \leqslant \frac{\delta\left(\ln(1+\delta) - \ln\delta\right)}{\ln(17/9)} < \frac{\delta\left(\delta - \ln\delta\right)}{\ln(17/9)} < 1.6\,\delta\left(\delta - \ln\delta\right).$$

It is easy to confirm that the function $\Phi_2(\delta) \equiv \delta\,(\delta - \ln\delta)$ increases on the ray $0 < \delta < +\infty$. This is why as $a \in (1, 9/8]$ the estimate holds:

$$\Phi_1(a) < 1.6\,\Phi_2\left(\frac{1}{8}\right) = \frac{1.6}{8}\left(\frac{1}{8} + \ln 8\right) < \frac{1}{2}.$$

Together with (3.18) this gives the desired estimate for the derivative

$$\Phi'(x) < \frac{1}{4} + \frac{(a-1)(3+6\ln 2)}{2} \leqslant \frac{1}{4} + \frac{3+6\ln 2}{16} < \frac{1}{4} + \frac{7.2}{16} = 0.7, \qquad x \in \left(a^{-2}, a^{-1}\right).$$

This justifies the positivity of the function Φ on the semi-interval $(0, a^{-1}]$.

Similar to obtaining (3.14) we have

$$\Phi(1) > (a-1)\left(\ln(a-1) - 1 + 3\ln 2 + s(a)\ln 2\right), \tag{3.19}$$

since

$$\begin{aligned}\Phi(1) &= \int_1^a \left(\ln(t-1) + (s(a)+3)\ln(t+1)\right)dt \\ &= (a-1)\ln(a-1) - (a-1) + (s(a)+3)\int_1^a \ln(t+1)dt \\ &> (a-1)\ln(a-1) - (a-1) + (a-1)\,(s(a)+3)\ln 2 \\ &= (a-1)\left(\ln(a-1) - 1 + 3\ln 2 + s(a)\ln 2\right).\end{aligned}$$

The definition of the quantity $s(a)$ immediately implies the lower bound

$$s(a)\ln 2 > \ln\frac{1}{a-1} \qquad \forall a > 1,$$

which together with (3.19) gives the inequality

$$\Phi(1) > (a-1)(3\ln 2 - 1) > a - 1 > 0.$$

Now we see that to prove the positivity of the function Φ on the ray $[1, +\infty)$, it is sufficient to prove the positivity of its derivative on $(1, +\infty)$. According to (3.13), for $x > 1$ we have

$$\begin{aligned}\Phi'(x) &= a\left(\ln(ax-1) + (s(a)+3)\ln(ax+1)\right) - \left(\ln(x-1) + (s(a)+3)\ln(x+1)\right) \\ &= (a-1)\left(\ln(ax-1) + (s(a)+3)\ln(ax+1)\right) + \ln\frac{ax-1}{x-1} + (s(a)+3)\ln\frac{ax+1}{x+1} \\ &> (a-1)\left(\ln(ax-1) + (s(a)+3)\ln(ax+1)\right) > (a-1)\left(\ln(a-1) + (s(a)+3)\ln(a+1)\right);\end{aligned}$$

here we have employed the increasing of the functions $\ln(t-1)$ and $\ln(t+1)$ on the ray $(1, +\infty)$. We straightforwardly confirm the positivity of the quantity $\ln(a-1) + s(a)\ln(a+1)$ for each $a > 1$. This gives inequality $\Phi'(x) > 3(a-1)\ln(a+1)$ for all $x > 1$ and $a > 1$ and this proves the needed fact.

It remains to show the positivity of $\Phi(x)$ on the interval $a^{-1} < x < 1$. For such values of the variable x, the identity holds:

$$J(x;a) \equiv \int_x^{ax} \ln|1-t|dt = (ax-1)\ln(ax-1) + (1-x)\ln(1-x) - (ax-x).$$

Denoting $u = ax - 1$, $v = 1 - x$, we obtain

$$J(x;a) = u\ln u + v\ln v - (u+v), \qquad u + v = x(a-1) < a - 1.$$

It is easy to confirm that the function of two variables $H(u,v) \equiv u\ln u + v\ln v$ in the triangle

$$T_h \equiv \left\{(u,v) \in \mathbb{R}^2 \mid u > 0, v > 0, u + v \leqslant h\right\},$$

although it is not closed, takes its minimum at the point $u = v = h/2$ for an arbitrary $h \in (0, 1/e)$. This is the upper bound holds:

$$J(x;a) \geqslant (a-1)\ln\frac{a-1}{2} - (a-1) \qquad \forall x \in (a^{-1}, 1) \quad \forall a \in (1, 1+1/e). \tag{3.20}$$

Let us estimate the integral

$$I(x;a) \equiv \int\limits_x^{ax} \ln(1+t)dt$$

from below. Since the function $\ln(1+t)$ is concave on the ray $0 < t < +\infty$, the integral of this function over each segment on $(0, +\infty)$ is greater that the length of this segment multiplied by the half of the sum of the values of $\ln(1+t)$ at its ends. Therefore, for each $x > a^{-1}$, the inequality holds

$$I(x;a) > \int\limits_{a^{-1}}^{1} \ln(1+t)dt > \left(1-\frac{1}{a}\right)\frac{1}{2}\left(\ln\left(1+\frac{1}{a}\right)+\ln 2\right), \tag{3.21}$$

by which we find

$$s(a)I(x;a) > \frac{a-1}{a}\ln\left(\frac{a}{a-1}\right)\left(\frac{1}{2}+\frac{1}{2}\frac{\ln 2}{\ln(1+1/a)}\right) \qquad \forall x \in (a^{-1}, 1). \tag{3.22}$$

And since according to (3.13) the representation holds

$$\Phi(x) = \Phi\left(x; a, s(a)+3, 0\right) = J(x;a) + \left(s(a)+3\right) I(x;a),$$

then by (3.20)–(3.22) for $x \in (a^{-1}, 1)$ we obtain the estimate

$$\begin{aligned}
\Phi(x) >&(a-1)\ln\frac{a-1}{2} - (a-1)\\
&+\frac{a-1}{a}\left(\left(\frac{1}{2}+\frac{1}{2}\frac{\ln 2}{\ln(1+1/a)}\right)(\ln a - \ln(a-1)) + \frac{3}{2}\ln\left(1+\frac{1}{a}\right)+\frac{3}{2}\ln 2\right)\\
=&\frac{a-1}{a}\left(\left(\frac{1}{2}+\frac{1}{2}\frac{\ln 2}{\ln(1+1/a)}-a\right)\ln\frac{1}{a-1} + \left(\frac{1}{2}+\frac{1}{2}\frac{\ln 2}{\ln(1+1/a)}\right)\ln a\right.\\
&\left.+\frac{3}{2}\ln\left(1+\frac{1}{a}\right)+\left(\frac{3}{2}-a\right)\ln 2 - a\right)\\
>&\frac{a-1}{a}\left(\left(\frac{1}{2}+\frac{1}{2}\frac{\ln 2}{\ln(1+1/a)}-a\right)\ln\frac{1}{a-1} + \ln a + \frac{3}{2}\ln\left(1+\frac{1}{a}\right)+\left(\frac{3}{2}-a\right)\ln 2 - a\right).
\end{aligned}$$

In view of this, to complete the proof of the positivity of $\Phi(x)$ on the ray $x > 0$ for each $a \in (1, 9/8]$ we need to check the positivity of the function

$$\xi(a) = \xi_1(a) + \xi_2(a)$$

on the semi-interval $(1, 9/8]$, where

$$\xi_1(a) \equiv \left(\frac{1}{2}+\frac{1}{2}\frac{\ln 2}{\ln(1+1/a)}-a\right)\ln\frac{1}{a-1}, \qquad \xi_2(a) \equiv \ln a + \frac{3}{2}\ln\left(1+\frac{1}{a}\right)+\left(\frac{3}{2}-a\right)\ln 2 - a.$$

It is easy to confirm that the function $\xi_2(a)$ decreases on the ray $[1, +\infty)$ and the inequality holds:

$$\xi_2\left(\frac{9}{8}\right) = \ln\frac{9}{8} + \frac{3}{2}\ln\frac{17}{9} + \frac{3}{8}\ln 2 - \frac{9}{8} > 0.2.$$

Therefore,

$$\xi_2(a) > 0.2 \qquad \forall a \in (1, 9/8]. \tag{3.23}$$

We then have:

$$\xi_1(a) = \left(1 - a + \frac{1}{2}\left(\frac{\ln 2}{\ln(1+1/a)} - 1\right)\right)\ln\frac{1}{a-1}$$
$$= \left(1 - a + \frac{1}{2}\frac{\ln\frac{2a}{a+1}}{\ln(1+1/a)}\right)\ln\frac{1}{a-1} > \left(1 - a + \frac{\ln\frac{2a}{a+1}}{2\ln 2}\right)\ln\frac{1}{a-1}.$$

Employing the well-known estimate for the logarithm $\ln t > (t-1)/t$ for all $t > 1$, taking $t = 2a/(a+1) > 1$ for the same $a \in (1, 9/8]$ and using then the decreasing of the function $t\ln t$ on the interval $0 < t < 1/e$, we write

$$\xi_1(a) > \left(1 - a + \frac{a-1}{4a\ln 2} - 1\right)\ln\frac{1}{a-1} = (a-1)\left(1 - \frac{1}{4a\ln 2}\right)\ln(a-1)$$
$$\geqslant \left(1 - \frac{2}{9\ln 2}\right)(a-1)\ln(a-1) \geqslant \left(1 - \frac{2}{9\ln 2}\right)\frac{1}{8}\ln\frac{1}{8} = -\frac{3}{8}\left(\ln 2 - \frac{2}{9}\right).$$

Thus,

$$\xi_1(a) > -0.18 \qquad \forall a \in (1, 9/8]. \tag{3.24}$$

By (3.23), (3.24) for each $a \in (1, 9/8]$ we find

$$\xi(a) = \xi_1(a) + \xi_2(a) > 0.02,$$

and this completes the checking of the positivity of the function Φ on the interval $(a^{-1}, 1)$.

Thus, integral (3.13) is positive for all $x > 0$ and all value of the parameter $a \in (1, 9/8]$. The proof is complete. □

The proof of Theorem 1.4, which is obtaining two-sided inequality (1.15), is simple. On one hand, combining estimates (1.13) and (2.24), for each $a > 1$ we obtain that

$$d(a) \geqslant s(a) > \log_2\frac{1}{a-1}.$$

On the other hand, by Lemmata 3.1 and 3.3 for all $a \in (1, 9/8]$ the estimate

$$d(a) \leqslant s(a) + 3$$

is true and we just need to check the inequality

$$s(a) < \log_2\frac{1}{a-1} + \frac{1}{2} \qquad \forall a \in (1, 9/8].$$

For such a we have

$$s(a) - \log_2\frac{1}{a-1} = \frac{1}{\ln(1+1/a)}\left(\ln a + \ln\frac{2a}{a+1}\cdot\log_2\frac{1}{a-1}\right)$$
$$< \frac{1}{\ln\frac{17}{9}}\left(\ln\frac{9}{8} + \frac{a-1}{a+1}\log_2\frac{1}{a-1}\right)$$
$$< \frac{1}{\ln\frac{17}{9}}\left(\ln\frac{9}{8} - \frac{1}{2\ln 2}(a-1)\ln(a-1)\right) < \frac{\ln\frac{9}{8} + \frac{3}{16}}{\ln\frac{17}{9}} < 0.49,$$

and this is the desired fact. The proof of inequality (1.15) is complete. Its obvious implication is asymptotics (1.14). The proof of Theorem 1.4 is complete.

In conclusion we mention that the problem on studying the behavior of the optimal exponent $d(a)$ for large values of a still waits for its solution. It should involve an upper bound for $d(a)$ and a possible refining of inequality (1.13).

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