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ON A CLASS OF PERIODIC FUNCTIONS IN $\mathbb{R}^n$

A.V. LUTSENKO, I.Kh. MUSIN, R.S. YULMUKHAMETOV

Abstract. By means of some family $\mathcal{H}$ of separately radially convex in $\mathbb{R}^n$ functions we define a space $G(\mathcal{H})$ of 2π-periodic in each variable infinitely differentiable in $\mathbb{R}^n$ functions with prescribed estimates on all partial derivatives. We describe the space $G(\mathcal{H})$ in terms of the Fourier coefficients. We find conditions on the family $\mathcal{H}$, under which the functions from $G(\mathcal{H})$ can be continued to functions holomorphic in a tubular domain in $\mathbb{C}^n$. We obtain an inner description of the space of such continuations. The considered problems are directly related with works by P.L. Ul'yanov in the end of 1980s, in which he succeeded to describe completely the classes of 2π-periodic functions of Gevrey type on the real axis not only by the decay rate of the Fourier coefficients but also in terms of the best trigonometric approximations. The obtained results are new both for the case of many variables and the case of a single variable. In particular, the novelty is owing to imposing condition i_4) on the family $\mathcal{H}$.

Keywords: Fourier series, Fourier coefficients, best possible approximation by trigonometric polynomials, entire functions, convex functions.

Mathematics Subject Classification: 42B05, 42A10

1. Introduction

Let $C_{2\pi}(\mathbb{R}^n)$ be a space of 2π-periodic in each variable continuous in $\mathbb{R}^n$ functions f with the norm $\|f\| = \max\limits_{x\in[0,2\pi]^n} |f(x)|$. Let $C_{2\pi}^\infty(\mathbb{R}^n) = C_{2\pi}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.

With each function $f \in C_{2\pi}(\mathbb{R}^n)$ we associate its Fourier series

$$f(x) \sim \sum_{\alpha\in\mathbb{Z}^n} \hat{f}_\alpha e^{i\langle\alpha,x\rangle}, \quad x \in \mathbb{R}^n,$$

where the Fourier coefficient $\hat{f}_\alpha$ is given by the formula

$$\hat{f}_\alpha = \frac{1}{(2\pi)^n} \int\limits_{[0,2\pi]^n} f(x) e^{-i\langle\alpha,x\rangle}\, dx.$$

Establishing the relations between the difference and differential properties of the functions from various subspaces of the space $C_{2\pi}(\mathbb{R}^n)$ and the properties of their Fourier coefficients is one of the main problems in the theory of the Fourier series. In this field we mention gentle results by P.L. Ul'yanov [1]–[3] obtained in the end of 1980s. In particular, he succeeded to characterize completely the classes of 2π-periodic functions of Gevrey type on the real axis not only by the decay rate of the Fourier coefficients but also in terms of the best trigonometric approximations, see, for instance, [3, Thms. 3, 4]. These studies by P.L. Ul'yanov served as a motivation for considering the following problem in the present note: to find subspaces of the functions in $C_{2\pi}^\infty(\mathbb{R}^n)$ with estimates for partial derivatives admitting the description in terms

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of the their Fourier series. In order to do this, we introduce the space $G(\mathcal{H})$ as follows. Let $\mathcal{H} = \{h_\nu\}_{\nu=1}^\infty$ be a family of convex functions $h_\nu : \mathbb{R}^n \to [0,\infty)$ with $h_\nu(0) = 0$ such that for each $\nu \in \mathbb{N}$

i_1) $h_\nu(x) = h_\nu(|x_1|,\dots,|x_n|)$, $x = (x_1,\dots,x_n) \in \mathbb{R}^n$;

i_2) $\lim\limits_{x\to\infty} \dfrac{h_\nu(x)}{\|x\|} = +\infty$;

i_3) $h_\nu(x) \geqslant h_{\nu+1}(x)$ for each $x \in \mathbb{R}^n$, and $\lim\limits_{x\to\infty}(h_\nu(x) - h_{\nu+1}(x)) = +\infty$;

i_4) the series

$$\sum_{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n} e^{h_\nu^*(\ln^+|\alpha_1|,\dots,\ln^+|\alpha_n|)-h_{\nu+1}^*(\ln^+|\alpha_1|,\dots,\ln^+|\alpha_n|)}$$

converges, where

$$h_\nu^*(x) = \sup_{\alpha\in\mathbb{Z}^n}(\langle\alpha,x\rangle - h_\nu(\alpha)), \quad x \in \mathbb{R}^n,$$

and, as usually, $\ln^+ t = \ln t$ for $t \geqslant 1$, $\ln^+ t = 0$ for $0 \leqslant t < 1$. For each $\nu \in \mathbb{N}$ we introduce a normed space

$$G(h_\nu) = \left\{ f \in C_{2\pi}^\infty(\mathbb{R}^n) : \|f\|_\nu = \sup_{x\in\mathbb{R}^n,\alpha\in\mathbb{Z}_+^n} \frac{|(D^\alpha f)(x)|}{e^{h_\nu(\alpha)}} < \infty \right\}.$$

By Condition i_3) the space $G(h_{\nu+1})$ is embedded into $G(h_\nu)$ completely continuous. We note that $G(h_{\nu+1})$ is a proper subspace of the space $G(h_\nu)$. Indeed, once we suppose that $G(h_{\nu+1}) = G(h_\nu)$, then for some $C_\nu > 0$ the inequality holds

$$\|f\|_{\nu+1} \leqslant C_\nu \|f\|_\nu, \quad f \in G(h_\nu).$$

In particular, for the functions $e^{i\langle m,x\rangle}$ with $m = (m_1,\dots,m_n) \in \mathbb{Z}^n$ we have

$$h_{\nu+1}^*(\ln^+|m_1|,\dots,\ln^+|m_n|) \leqslant h_\nu^*(\ln^+|m_1|,\dots,\ln^+|m_n|) + \ln C_\nu.$$

But this inequality is impossible due to Condition i_4). Now we let $G(\mathcal{H}) = \bigcap\limits_{\nu=1}^\infty G(h_\nu)$. We equip $G(\mathcal{H})$ with a locally convex topology by means of the family of the norms $\|\cdot\|_\nu$ ($\nu \in \mathbb{N}$). With this topology, $G(\mathcal{H})$ is a Fréchet space.

In Section 2 we show that the space of the functions $G(\mathcal{H})$ admits the description in terms of the estimates for the Fourier coefficients Theorem 2.1. It interesting to find the conditions for the family $\mathcal{H}$, under which the functions from $G(\mathcal{H})$ admit the continuation to the functions holomorphic in a tubular domain in $\mathbb{C}^n$, and to describe the space of such continuation. This problem is considered in the second section of this note, see Theorem 3.1.

2. Equivalent description of space $G(\mathcal{H})$

In the formulation of the main result, Theorem 2.1, the space $C(\mathcal{H})$ is involved. We introduce it as follows. For each $\nu \in \mathbb{N}$ let $C(h_\nu)$ be the space consisting of the functions $f \in C_{2\pi}(\mathbb{R}^n)$, the Fourier coefficients of which $\hat{f}_\alpha$ for some $a_\nu(f) > 0$ satisfy the estimate

$$|\hat{f}_\alpha| \leqslant a_\nu(f) e^{-h_\nu^*(\ln^+|\alpha_1|,\dots,\ln^+|\alpha_n|)}, \qquad \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n.$$

Since, owing to Condition i_2, for each $\nu \in \mathbb{N}$

$$\lim_{x\to\infty} \frac{h_\nu^*(x)}{\|x\|} = +\infty,$$

then the functions in $C(h_\nu)$ are infinitely differentiable. We equip $C(h_\nu)$ with the norm

$$p_\nu(f) = \sup_{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n} (|\hat{f}_\alpha| e^{h_\nu^*(\ln^+|\alpha_1|,\dots,\ln^+|\alpha_n|)}).$$

Since $h_\nu^*(x) \leqslant h_{\nu+1}^*(x)$ for each $x \in \mathbb{R}^n$, then $p_\nu(f) \leqslant p_{\nu+1}(f)$ for an arbitrary function $f \in C(h_{\nu+1})$. Hence, the space $C(h_{\nu+1})$ is continuously embedded into $C(h_\nu)$. At the same time, $C(h_{\nu+1})$ is a proper subspace of the space $C(h_\nu)$. Indeed, there are functions $C(h_\nu)$ not belonging to $C(h_{\nu+1})$. This is, for instance, the function

$$f_\nu(x) = \sum_{\alpha\in\mathbb{Z}^n} e^{-h_\nu^*(\ln^+|\alpha_1|,\dots,\ln^+|\alpha_n|)} e^{i\langle\alpha,x\rangle}, \quad x \in \mathbb{R}^n.$$

For this function $p_\nu(f_\nu) = 1$ and $p_{\nu+1}(f_\nu) = +\infty$ since owing to Conditions i_2) and i_3)

$$\lim_{x\to\infty} (h_{\nu+1}^*(x) - h_\nu^*(x)) = +\infty. \tag{2.1}$$

We define the space $C(\mathcal{H})$ as the intersection of the spaces $C(h_\nu)$. We equip $C(\mathcal{H})$ by a locally convex topology by means of the family of the norms p_ν.

We recall once again that the Young-Fenchel transform of a function $g : \mathbb{R}^n \to [-\infty, +\infty]$ is the function $\tilde{g} : \mathbb{R}^n \to [-\infty, +\infty]$ defined by the formula

$$\tilde{g}(x) = \sup_{y\in\mathbb{R}^n} (\langle x, y\rangle - g(y)), \quad x \in \mathbb{R}^n.$$

In the proof of Theorem 2.1 we shall need the following statement.

Proposition 2.1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that*

$$\lim_{x\to\infty} \frac{g(x)}{\|x\|} = +\infty.$$

Then $g(\alpha) = \widetilde{(g^*)}(\alpha)$, $\alpha \in \mathbb{Z}^n$.

Proof. By the assumptions on g, the convex in $\mathbb{R}^n$ functions g^* and $\widetilde{g}$ take finite values. Hence, g^* and $\widetilde{g}$ are continuous in $\mathbb{R}^n$. Since

$$g(\alpha) \geqslant \langle x, \alpha\rangle - g^*(x), \quad \alpha \in \mathbb{Z}^n, \quad x \in \mathbb{R}^n,$$

then $g(\alpha) \geqslant \widetilde{(g^*)}(\alpha)$ for each $\alpha \in \mathbb{Z}^n$. We recall that $g = \tilde{\tilde{g}}$ according to the formula of inverting the Young-Fenchel transform [4], that is,

$$g(x) = \sup_{\xi\in\mathbb{R}^n} (\langle x, \xi\rangle - \tilde{g}(\xi)), \quad x \in \mathbb{R}^n.$$

Using assumptions on the convex function g and it continuity, for each $\alpha \in \mathbb{Z}^n$ we find a point $\xi(\alpha) \in \mathbb{R}^n$ such that $g(\alpha) = \langle\alpha, \xi(\alpha)\rangle - \tilde{g}(\xi(\alpha))$. Now, using this identity and the inequality $g^*(x) \leqslant \widetilde{g}(x)$ for $x \in \mathbb{R}^n$, we have:

$$\widetilde{(g^*)}(\alpha) \leqslant g(\alpha) = \langle\alpha, \xi(\alpha)\rangle - \tilde{g}(\xi(\alpha)) \leqslant \langle\alpha, \xi(\alpha)\rangle - g^*(\xi(\alpha)) \leqslant \widetilde{(g^*)}(\alpha).$$

Therefore, $g(\alpha) = \widetilde{(g^*)}(\alpha)$ for each $\alpha \in \mathbb{Z}^n$. □

The following theorem holds true.

Theorem 2.1. *The spaces $G(\mathcal{H})$ and $C(\mathcal{H})$ coincide.*

Proof. Let $f \in G(\mathcal{H})$. We are going to show that $f \in C(\mathcal{H})$. Since $f \in G(h_\nu)$ for each $\nu \in \mathbb{N}$, then

$$|(D^\beta f)(x)| \leqslant \|f\|_\nu e^{h_\nu(\beta)}, \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{Z}_+^n.$$

By the representations

$$\hat{f}_\alpha (i\alpha)^\beta = \frac{1}{(2\pi)^n} \int\limits_{[0,2\pi]^n} (D^\beta f)(x) e^{-i\langle\alpha,x\rangle}\, dx, \quad \alpha \in \mathbb{Z}^n, \quad \beta \in \mathbb{Z}_+^n,$$

we hence obtain

$$|\hat{f}_\alpha| \leqslant \|f\|_\nu \frac{e^{h_\nu(\beta)}}{(|\alpha_1|^+)^{\beta_1}\cdots(|\alpha_n|^+)^{\beta_n}}$$

for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_+$, where $t \geqslant 0$ $t^+ = \max(t, 1)$. Therefore,

$$|\hat{f}_\alpha| \leqslant \|f\|_\nu \inf_{\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{Z}^n_+} \frac{e^{h_\nu(\beta)}}{(|\alpha_1|^+)^{\beta_1}\cdots(|\alpha_n|^+)^{\beta_n}}, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n.$$

That is,

$$|\hat{f}_\alpha| \leqslant \|f\|_\nu e^{-h^*_\nu(\ln^+|\alpha_1|,\ldots,\ln^+|\alpha_n|)}, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n.$$

Thus, $p_\nu(f) \leqslant \|f\|_\nu$, $\quad f \in G(\mathcal{H})$. In view of the arbitrariness of ν we conclude that $f \in C(\mathcal{H})$ and the embedding $G(\mathcal{H})$ into $C(\mathcal{H})$ is continuous.

Now let $f \in C(\mathcal{H})$. Then for each $k \in \mathbb{N}$

$$|\hat{f}_\alpha| \leqslant p_k(f) e^{-h^*_k(\ln^+|\alpha_1|,\ldots,\ln^+|\alpha_n|)}, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n. \tag{2.2}$$

Therefore,

$$|\hat{f}_\alpha| \leqslant p_k(f) \frac{e^{h_k(\beta)}}{(|\alpha_1|^+)^{\beta_1}\cdots(|\alpha_n|^+)^{\beta_n}}$$

for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_+$. Hence, $f \in C^\infty_{2\pi}(\mathbb{R}^n)$. Let us show that $f \in G(\mathcal{H})$. Let $\nu \in \mathbb{N}$ be arbitrary. For $x \in \mathbb{R}^n, \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_+$ we estimate from above $|(D^\beta f)(x)|$ employing inequality (2.2) and Condition i_3). We have:

$$\begin{aligned}|(D^\beta f)(x)| &\leqslant \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n} |\hat{f}_\alpha|(|\alpha_1|^+)^{\beta_1}\cdots(|\alpha_n|^+)^{\beta_n} \\ &\leqslant p_{\nu+1}(f) \sum_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n} e^{-h^*_{\nu+1}(\ln^+|\alpha_1|,\ldots,\ln^+|\alpha_n|)}(|\alpha_1|^+)^{\beta_1}\cdots(|\alpha_n|^+)^{\beta_n}, \quad x \in \mathbb{R}^n.\end{aligned}$$

Hence, letting

$$\tau_\nu = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n} e^{h^*_\nu(\ln^+|\alpha_1|,\ldots,\ln^+|\alpha_n|)-h^*_{\nu+1}(\ln^+|\alpha_1|,\ldots,\ln^+|\alpha_n|)},$$

we obtain that

$$|(D^\beta f)(x)| \leqslant \tau_\nu p_{\nu+1}(f) e^{\sup\limits_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n}(\beta_1\ln^+|\alpha_1|+\cdots+\beta_n\ln^+|\alpha_n|-h^*_\nu(\ln^+|\alpha_1|,\ldots,\ln^+|\alpha_n|))}.$$

This yields the inequality

$$|(D^\beta f)(x)| \leqslant \tau_\nu p_{\nu+1}(f) e^{\sup\limits_{t\in\mathbb{R}^n}(\langle\beta,t\rangle-h^*_\nu(t))}.$$

Using now Proposition 2.1, we get:

$$|(D^\beta f)(x)| \leqslant \tau_\nu p_{\nu+1}(f) e^{h_\nu(\beta)}, \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{Z}^n_+.$$

Therefore, for each $\nu \in \mathbb{N}$ the inequality holds:

$$\|f\|_\nu \leqslant \tau_\nu p_{\nu+1}(f).$$

Thus, $f \in G(\mathcal{H})$ and the embedding of $C(\mathcal{H})$ into $G(\mathcal{H})$ is continuous. The proven statements imply that the spaces $G(\mathcal{H})$ and $C(\mathcal{H})$ coincide as topological spaces. □

3. On continuing functions from $G(\mathcal{H})$ to holomorphic ones in convex tubular domain

For each $\nu \in \mathbb{N}$ we define a function u_ν in $\mathbb{R}^n$ letting

$$u_\nu(x) = h_\nu^*(\ln^+ |x_1|, \dots, \ln^+ |x_n|), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

It is clear that the function u_ν is continuous, negative and $u_\nu(0) = 0$, and its restriction to $[0, \infty)^n$ does not decrease in each variable. By (2.1) we have

$$\lim_{x\to\infty} (u_{\nu+1}(x) - u_\nu(x)) = +\infty. \tag{3.1}$$

Hereafter we suppose that the functions u_ν obey the condition

$$\underline{\lim}_{x\to\infty} \frac{u_\nu(x)}{\|x\|} > 0, \quad \nu = 1, 2, \dots. \tag{3.2}$$

We define a set $B_\nu = \{y \in \mathbb{R}^n : \widetilde{u_\nu}(y) < \infty\}$. It is obvious that if

$$\lim_{x\to\infty} \frac{u_\nu(x)}{\|x\|} = +\infty, \quad \nu \in \mathbb{N},$$

then $B_\nu = \mathbb{R}^n$. By (3.2) the interior B_ν° of the set B_ν is non-empty. Since the function $\widetilde{u_\nu}$ is convex in $\mathbb{R}^n$, then B_ν is a convex set. Since $\widetilde{u_{\nu+1}}(y) \leqslant \widetilde{u_\nu}(y)$ for each $y \in \mathbb{R}^n$, then $B_\nu \subseteq B_{\nu+1}$ $(\nu = 1, 2, \dots)$.

Let $B = \bigcup\limits_{\nu=1}^{\infty} B_\nu^\circ$; we see that B is a convex domain in $\mathbb{R}^n$.

We observe that each function $f \in G(\mathcal{H})$ admits a continuation for a 2π-periodic in each variable holomorphic in a tubular domain $T_B = \mathbb{R}^n + iB$ function F_f introduced by the rule

$$F_f(z) = \sum_{\alpha\in\mathbb{Z}^n} \hat{f}_\alpha e^{i\langle\alpha,z\rangle}, \quad z \in T_B. \tag{3.3}$$

Indeed, for each $\nu \in \mathbb{N}$, for each $z \in \mathbb{R}^n + iB_\nu^\circ$,

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^n} |\hat{f}_\alpha||e^{i\langle\alpha,z\rangle}| &\leqslant p_{\nu+1}(f) \sum_{\alpha\in\mathbb{Z}^n} e^{-u_\nu(\alpha)-\langle\alpha,\operatorname{Im} z\rangle} \\ &\leqslant \tau_\nu p_{\nu+1}(f) e^{\sup\limits_{\alpha\in\mathbb{Z}^n}(-u_\nu(\alpha)-\langle\alpha,\operatorname{Im} z\rangle)} = \tau_\nu p_{\nu+1}(f) e^{\widetilde{u_\nu}(-\operatorname{Im} z)} \\ &= \tau_\nu p_{\nu+1}(f) e^{\widetilde{u_\nu}(\operatorname{Im} z)} < \infty.\end{aligned}$$

Thus, the series in the right hand side in (3.3) converges absolutely and uniformly in the domain $T_{B_\nu^\circ} = \mathbb{R}^n + iB_\nu^\circ$ for each $\nu \in \mathbb{N}$. Hence, F_f is holomorphic in T_B and

$$|F_f(z)| \leqslant \tau_\nu p_{\nu+1}(f) e^{\widetilde{u_\nu}(\operatorname{Im} z)}, \quad z \in \mathbb{R}^n + iB_\nu^\circ. \tag{3.4}$$

This continuation is obviously unique.

Below we suppose that for each $\nu \in \mathbb{N}$ the function u_ν is convex in $\mathbb{R}^n$. This condition and the fact the function u_ν takes finite values in $\mathbb{R}^n$ imply that it is continuous in $\mathbb{R}^n$. Now we define the space $H_{2\pi}(T_{B_\nu^\circ}, \tilde{u}_\nu)$ consisting of 2π-periodic in each variable holomorphic in $T_{B_\nu^\circ}$ functions F, for which

$$|F(z)| \leqslant c_\nu(F) e^{\tilde{u}_\nu(\operatorname{Im} z)}, \quad z \in T_{B_\nu^\circ},$$

for some $c_\nu(F) > 0$. We equip $H_{2\pi}(T_{B_\nu^\circ}, \tilde{u}_\nu)$ with the norm

$$n_\nu(F) = \sup_{z\in T_{B_\nu^\circ}} \frac{|F(z)|}{e^{\tilde{u}_\nu(\operatorname{Im} z)}}, \quad F \in H_{2\pi}(T_{B_\nu^\circ}, \tilde{u}_\nu).$$

Since $\widetilde{u_{\nu+1}}(y) \leqslant \widetilde{u_\nu}(y)$ for each $y \in \mathbb{R}^n$, then

$$n_\nu(F) \leqslant n_{\nu+1}(F), \quad F \in H_{2\pi}(T_{B_{\nu+1}^\circ}, \tilde{u}_{\nu+1}).$$

Hence, the space $H_{2\pi}(T_{B^\circ_{\nu+1}},\tilde u_{\nu+1})$ is continuously embedded into $H_{2\pi}(T_{B^\circ_\nu},\tilde u_\nu)$.

We note that the space $H_{2\pi}(T_{B^\circ_{\nu+1}},\tilde u_{\nu+1})$ is a proper subspace of the space $H_{2\pi}(T_{B^\circ_\nu},\tilde u_\nu)$. Indeed, if we suppose that $H_{2\pi}(T_{B^\circ_{\nu+1}},\tilde u_{\nu+1}) = H_{2\pi}(T_{B^\circ_\nu},\tilde u_\nu)$, then for some $c_\nu > 0$ the inequality should hold:

$$n_{\nu+1}(F) \leqslant c_\nu n_\nu(F), \quad F \in H_{2\pi}(T_{B^\circ_\nu},\tilde u_\nu).$$

In particular, due to this inequality for the function $e^{-i\langle\alpha,z\rangle}$ with $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n$, we have

$$\sup_{y\in B^\circ_{\nu+1}} (\langle\alpha,y\rangle - \tilde u_{\nu+1}(y)) \leqslant \ln c_\nu + \sup_{y\in B^\circ_\nu} (\langle\alpha,y\rangle - \tilde u_\nu(y)).$$

This inequality can be written as follows:

$$\sup_{y\in \mathbb{R}^n} (\langle\alpha,y\rangle - \tilde u_{\nu+1}(y)) \leqslant \ln c_\nu + \sup_{y\in \mathbb{R}^n} (\langle\alpha,y\rangle - \tilde u_\nu(y)).$$

Now we take into consideration that for each $\nu \in \mathbb{N}$

$$\sup_{y\in B^\circ_\nu} (\langle\alpha,y\rangle - \widetilde u_\nu(y)) = \sup_{y\in B_\nu} (\langle\alpha,y\rangle - \widetilde u_\nu(y)) = \sup_{y\in \mathbb{R}^n} (\langle\alpha,y\rangle - \widetilde u_\nu(y)) = u_\nu(\alpha). \tag{3.5}$$

Here at the final step we employed the formula for inverting the Young-Fenchel transform [4]. In view of this identity, by the previous inequality we obtain that $u_{\nu+1}(\alpha) \leqslant \ln c_\nu + u_\nu(\alpha)$ for each $\alpha \in \mathbb{Z}^n$ and this contradicts (3.1).

We introduce the space $H_{2\pi}(T_B,\mathcal{H})$ as the intersection of the spaces $H_{2\pi}(T_{B^\circ_\nu},\tilde u_\nu)$. We equip $H_{2\pi}(T_B,\mathcal{H})$ by a locally convex topology defined by the system of the norms n_ν.

Theorem 3.1. *The spaces $G(\mathcal{H})$ and $H_{2\pi}(T_B,\mathcal{H})$ are isomorphic.*

Proof. Employing estimate (3.4), we have: $n_\nu(F_f) \leqslant \tau_\nu p_{\nu+1}(f)$ for each $f \in G(\mathcal{H})$. This means that the linear mapping A acts from $G(\mathcal{H})$ into $H_{2\pi}(T_B,\mathcal{H})$ and is continuous. It is clear that the mapping A is injective.

We are going to show that the mapping A is surjective. Let $F \in H_{2\pi}(T_B,\mathcal{H})$. Then, in particular, $F \in C^\infty_{2\pi}(\mathbb{R}^n)$. Therefore,

$$F(x) = \sum_{\alpha\in\mathbb{Z}^n} \hat F_\alpha e^{i\langle\alpha,x\rangle}, \quad x \in \mathbb{R}^n.$$

Employing the analyticity and periodicity of F, we can write a representation for the Fourier coefficient $\hat F_\alpha$ of the function F:

$$\hat F_\alpha = \frac{1}{(2\pi)^n} \int\limits_{[0,2\pi]^n} F(x+iy) e^{-i\langle\alpha,x+iy\rangle}\,dx, \quad y \in B^\circ_\nu.$$

Then for each $\alpha \in \mathbb{Z}^n$

$$|\hat F_\alpha| \leqslant \frac{1}{2\pi} \int\limits_{[0,2\pi]^n} |F(x+iy)| e^{\langle\alpha,y\rangle}\,dx, \quad y \in B^\circ_\nu.$$

Since $F \in H_{2\pi}(T_{B^\circ_\nu},\tilde u_\nu)$ for each $\nu \in \mathbb{N}$, by this inequality we obtain that

$$|\hat F_\alpha| \leqslant n_\nu(F) e^{\widetilde u_\nu(y)} e^{\langle\alpha,y\rangle}, \quad y \in B^\circ_\nu.$$

Therefore,

$$|\hat F_\alpha| \leqslant n_\nu(F) e^{\inf\limits_{y\in B^\circ_\nu}(\widetilde u_\nu(y)+\langle\alpha,y\rangle)}.$$

In view of (3.5) we have:

$$\inf_{y\in B^\circ_\nu} (\widetilde u_\nu(y) + \langle\alpha,y\rangle) = \inf_{y\in B^\circ_\nu} (\widetilde u_\nu(-y) + \langle\alpha,y\rangle) = -\sup_{y\in B^\circ_\nu} (\langle\alpha,y\rangle - \widetilde u_\nu(y)) = -u_\nu(\alpha).$$

By this and the previous inequality we find:

$$|\hat{F}_\alpha| \leqslant n_\nu(F) e^{-u_\nu(\alpha)}. \tag{3.6}$$

Hence, $F_{|\mathbb{R}^n} \in C(\mathcal{H})$. But then by Theorem 2.1 we have $F_{|\mathbb{R}^n} \in G(\mathcal{H})$. It is obvious that $A(F_{|\mathbb{R}^n}) = F$. Thus, the mapping A is surjective. We also note that by estimate (3.6) and Theorem 2.1 the linear mapping $A^{-1} : F \in H_{2\pi}(T_B, \mathcal{H}) \to F_{|\mathbb{R}^n}$ acts continuously from $H_{2\pi}(T_B, \mathcal{H})$ into $G(\mathcal{H})$. The proven statements imply that the mapping A makes an isomorphism of the spaces $G(\mathcal{H})$ and $H_{2\pi}(T_B, \mathcal{H})$. □

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Anastasia Vladimirovna Lutsenko,
Bashkir State University,
Zaki Validi str. 32,
450076, Ufa, Russia
E-mail: Lutsenko.AV@yandex.ru

Ildar Khamitovich Musin,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: musin_ildar@mail.ru

Rinad Salavatovich Yulmukhametov,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia,
Bashkir State University,
Zaki Validi str. 32,
450076, Ufa, Russia
E-mail: Yulmukhametov@mail.ru